1.2 Problems NS-2

Topic of this homework:

Prime numbers, greatest common divisors, the continued fraction algorithm

Prime numbers

Problem # *1: Every integer may be written as a product of primes.*

– *1.1: Write the numbers* $1,000,000$*,* $1,000,004$*, and* $999,999$ *in the form* $N = \prod_k \pi_k^{\beta_k}$. Hint: Use Matlab/Octave to find the prime factors.

Ans:

– *1.2: Give a generalized formula for the natural logarithm of a number* $\ln(N)$ *in terms of its primes* π_k *and their multiplicities* β_k*. Express your answer as a* sum *of terms.*

Ans:

Problem # 2: *Using the computer*

– 2.1: Explain why the following brief Matlab/Octave program returns the prime numbers π_k between 1 and 100.

```
n=2:100;
k = isprime(n);
n(k)
```

Ans:

– 2.2: How many primes are there between 2 *and* $N = 100$*?*

Ans:

Problem # 3: *Prime numbers may be identified using a sieve (See Figure).*

– 3.1: By hand, complete the sieve of Eratosthenes for $n = 1, \ldots, 49$*. Circle each prime* p*, then cross out each number that is a multiple of* p*.*

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	

Note: 1 should not be circled as it is not a prime.

– 3.2: What is the largest number you need to consider before only primes remain?

Ans:

– 3.3: Generalize: For $n = 1, \ldots, N$*, what is the largest number you need to consider before only the primes remain?*

Ans:

– *3.4: Write each of these numbers as a product of primes: 22, 30, 34, 43, 44, 48, 49.* **Ans:**

– *3.5: Find the largest prime* $\pi_k \le 100$. *Do not use Matlab/Octave other than to check your answer.* Hint: Write the numbers starting with 100 and count backward: 100, 99, 98, 97, ... Cross off the even numbers, leaving $99, 97, 95, \ldots$. Pull out a factor (only one is necessary to show that it is not prime).
Ans:

– *3.6: Find the largest prime* $\pi_k \le 1000$. *Do not use Matlab/Octave other than to check your answer.*
Ans:

– *3.7: Explain why* $\pi_k^{-s} = e^{-s \ln \pi_k}$.
Ans:

Greatest common divisors

Consider using the *Euclidean algorithm* to find the *greatest common divisor* (i.e., GCD; the largest common prime factor) of two numbers. Note that this algorithm may be performed using one of two methods:

Method	Division	Subtraction
On each iteration...	$a_{i+1} = b_i$ $b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$	$a_{i+1} = \text{max}(a_i, b_i) - \text{min}(a_i, b_i)$ $b_{i+1} = \text{min}(a_i, b_i)$
Terminates when...	$b = 0$ (GCD $= a$)	$b = 0$ (GCD $= a$)

The division method (Eq. 2.1, Sec. 2.1.2, Ch. 2) is preferred because the subtraction method is much slower.

Problem # *4: Understanding the Euclidean algorithm (GCD)*

– *4.1: Use the Octave/Matlab command* `factor` *to find the prime factors of* $a = 85$ *and* $b = 15$.

Ans:

– *4.2: What is the greatest common prime factor of* $a = 85$ *and* $b = 15$*?*

Ans:

– *4.3: By hand, perform the Euclidean algorithm for* $a = 85$ *and* $b = 15$.

Ans:

– *4.4: By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?*
Ans:

– *4.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g., $a = 25$ and $b = 75$). What happens to a and b in the first step? Does it matter if you begin the algorithm with $a < b$ rather than $b < a$?*
Ans:

– *4.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers that have already been divided into factors (e.g., $a = 5 \cdot 3$ and $b = 7 \cdot 3$).*
Ans:

– *4.7: Find the GCD of $2 \cdot \pi_{25}$ and $3 \cdot \pi_{25}$.*
Ans: .

Problem # *5: Coprimes*

– *5.1: Define the term* coprime.
Ans:

– 5.2: How can the Euclidean algorithm be used to identify coprimes?
Ans:

– 5.3: Give at least one application of the Euclidean algorithm.
Ans:

– 5.4: Write a Matlab function, `function x = my_gcd(a,b)`, that uses the Euclidean algorithm to find the GCD of any two inputs `a` and `b`. Test your function on the (a, b) combinations from the previous problem. Include a printout (or hand-write) your algorithm to turn in.
Hints and advice:

- Don't give your variables the same names as Matlab functions! Since `gcd` is an existing Matlab/Octave function, if you use it as a variable or function name, you won't be able to use `gcd` to check your `gcd()` function. Try `clear all` to recover from this problem.
- Try using a "while" loop for this exercise (see Matlab documentation for help).
- You may need to use some temporary variables for `a` and `b` in order to perform the algorithm.

Ans:

Continued fractions

Problem # *6: Here we explore the continued fraction algorithm (CFA), discussed in Sec. 2.4.4.*
In its simplest form, the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \mathbb{I}$, as an example because its CFA representation will be infinitely long. We can represent the CFA coefficients α as a vector of integers n_k, $k = 1, 2, \ldots, \infty$:

$$\begin{aligned}\alpha &= [n_1; n_2, n_3, n_4, \ldots] \\ &= n_1 + \cfrac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \cdots}}}.\end{aligned}$$

As discussed in Sec. 2.4.3 (p. 27), the CFA is recursive, with three steps per iteration. For $\alpha_1 = \pi$, $n_1 = 3$, $r_1 = \pi - 3$, and $\alpha_2 \equiv 1/r_1$.

$$\alpha_2 = 1/0.1416 = 7.0625\ldots$$
$$\alpha_1 = n_1 + \frac{1}{\alpha_2} = n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \cdots.$$

In terms of a Matlab/Octave script,

```
alpha0 = pi;
K=10;
n=zeros(1,K); alpha=zeros(1,K);
alpha(1)=alpha0;

for k=2:K   %k=1 to K
n(k)=round(alpha(k-1));
%n(k)=fix(alpha(k-1));
alpha(k)= 1/(alpha(k-1)-n(k));
%disp([fix(k), round(n(k)), alpha(k)]); pause(1)
end
disp([n; alpha]);
%Now compare this to matlab's rat() function
rat(alpha0,1e-20)
```

– 6.1: By hand (you may use Matlab/Octave as a calculator), find the first three values of n_k for $\alpha = e^\pi$.

Ans:

– *6.2: For the proceeding question, what is the error (remainder) when you truncate the continued fraction after $n_1, \dots, n_3$? Give the absolute value of the error and the percentage error relative to the original α.*
Ans:

– *6.3: Use the Matlab/Octave program provided to find the first 10 values of n_k for $\alpha = e^\pi$, and verify your result using the Matlab/Octave command `rat()`.*
Ans:

– *6.4: Discuss the similarities and differences between the Euclidean algorithm and the CFA.*
Ans:

Added Aug

Problem # *7:CFA of ratios of large primes*

– *7.1:* ***(4pts)*** *Expand 23/7 as a continued fraction. Express your answer in bracket notation* (e.g., $\pi = [3., 7, 16, \cdots]$). Show your work. **Ans:**

– *7.2:Starting from the primes below 10^6, form the CFA of π_j/π_k with $j = 78498$ and $k < j$.*
Ans:

– 7.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? **Ans:**

*– 7.4: **(1pts)** Try the Matlab/Octave functions `rats(23/7)`, `rats(3.2857)`, and `rats(3.2856)`. What an you conclude?*
Ans:

*– 7.5: **(2pts)** Can $\sqrt{2}$ be represented as a finite continued fraction? Why or why not?*
Ans:

*– 7.6: **(2pts)** What is the CFA for $\sqrt{2}-1$?*

$$\text{Hint:} \qquad \sqrt{2}+1 = \frac{1}{\sqrt{2}-1} = [2; 2, 2, 2, \cdots].$$

Ans:

– 7.7: Show that

$$\frac{1}{1-\sqrt{a}} = a^{\frac{11}{2}} + a^{\frac{9}{2}} + a^{\frac{7}{2}} + a^{\frac{5}{2}} + a^{\frac{3}{2}} + \sqrt{a} + a^5 + a^4 + a^3 + a^2 + a + 1 = 1 - a^6$$

```
syms a,b
b= taylor(1/( 1-sqrt(a) ))
simplify((1-sqrt(a))*b) = 1-a^6
```

Use symbolic analysis to show this, then explain. **Ans:**