

1.2 Problems NS-2

Topic of this homework:

Prime numbers, greatest common divisors, the continued fraction algorithm

Prime numbers

Problem # 1: Every integer may be written as a product of primes.

– 1.1: Write the numbers 1,000,000, 1,000,004, and 999,999 in the form $N = \prod_k \pi_k^{\beta_k}$. Hint: Use

Matlab/Octave to find the prime factors.

Sol: $1,000,000 = 2^6 \cdot 5^6$

$1,000,004 = 2^2 \cdot 53^2 \cdot 89$

$999,999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$

■

– 1.2: Give a generalized formula for the natural logarithm of a number $\ln(N)$ in terms of its primes π_k and their multiplicities β_k . Express your answer as a sum of terms.

Sol: $\ln N = \sum_k \beta_k \ln(\pi_k)$ ■

Problem # 2: Using the computer

– 2.1: Explain why the following brief Matlab/Octave program returns the prime numbers π_k between 1 and 100.

```
n=2:100;
k = isprime(n);
n(k)
```

Sol: The first line $n = 2 : 100$ defines the row vector $n = [2, 3, 4, \dots, 100]$. The second line creates a row vector the same length as n , with entries of 1 if the element is prime and zero if the element is not prime. The third line $n(k)$ prints out $n()$ if $k = 1$, namely it is a list of all the primes from 2 to 100. Run this program without the ';' at the end of each line, and to see what it is doing. ■

– 2.2: How many primes are there between 2 and $N = 100$?

Sol: `length(n(k))` returns 25. Thus there are 25 primes less than 100 ($N/4$, on average). ■

Problem # 3: Prime numbers may be identified using a sieve (See Figure).

– 3.1: By hand, complete the sieve of Eratosthenes for $n = 1, \dots, 49$. Circle each prime p , then cross out each number that is a multiple of p .

①	②	③	4	⑤	6	⑦	8	9	10
⑪	12	⑬	14	15	16	⑰	18	⑱	20
21	22	⑳	24	25	26	27	28	㉑	30
⑳	32	33	34	35	36	㉓	38	39	40
41	42	43	44	45	46	47	48	49	

Note: 1 should not be circled as it is not a prime.

– 3.2: What is the largest number you need to consider before only primes remain?

Sol: $\lfloor \sqrt{50} \rfloor = \lfloor 7.0711 \rfloor = 7$. ■

– 3.3: Generalize: For $n = 1, \dots, N$, what is the largest number you need to consider before only the primes remain?

Sol: $\text{floor}(\sqrt{N})$ ■

– 3.4: Write each of these numbers as a product of primes: 22, 30, 34, 43, 44, 48, 49. **Sol:** $22 = 2 \cdot 11 = \pi_1 \pi_5$

$$30 = 2 \cdot 3 \cdot 5 = \pi_1 \pi_2 \pi_3$$

$$34 = 2 \cdot 17 = \pi_1 \pi_7$$

$$43 = \pi_{14}$$

$$44 = 4 \cdot 11 = \pi_1^2 \pi_5$$

$$48 = 4 \cdot 12 = 4^2 \cdot 3 = \pi_1^4 \pi_2$$

$$49 = 7^2 = \pi_4^2$$

■

– 3.5: Find the largest prime $\pi_k \leq 100$. Do not use Matlab/Octave other than to check your answer. Hint: Write the numbers starting with 100 and count backward: 100, 99, 98, 97, Cross off the even numbers, leaving 99, 97, 95, Pull out a factor (only one is necessary to show that it is not prime).

Sol: $99 = 11 \cdot 9$, $\pi_{25} = 97$. ■

– 3.6: Find the largest prime $\pi_k \leq 1000$. Do not use Matlab/Octave other than to check your answer.

Sol: Write out the numbers starting with 1000 and counting backwards: 1000, 999, 998, 997, Cross off the even numbers, leaving 999, 997, 995, Pull out a factor (only one is necessary to show that it is not prime). $9 \cdot 111$, $997 = \pi_{168}$, $5 \cdot 199 = \pi_3 \cdot \pi_{46}$. ■

– 3.7: Explain why $\pi_k^{-s} = e^{-s \ln \pi_k}$.

Sol: This follows from the identity $z^a = e^{a \ln z}$ with $a, z \in \mathbb{C}$. ■

Greatest common divisors

Consider using the *Euclidean algorithm* to find the *greatest common divisor* (i.e., GCD; the largest common prime factor) of two numbers. Note that this algorithm may be performed using one of two methods:

Method	Division	Subtraction
On each iteration...	$a_{i+1} = b_i$ $b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$	$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$ $b_{i+1} = \min(a_i, b_i)$
Terminates when...	$b = 0$ (GCD = a)	$b = 0$ (GCD = a)

The division method (Eq. 2.1, Sec. 2.1.2, Ch. 2) is preferred because the subtraction method is much slower.

Problem # 4: Understanding the Euclidean algorithm (GCD)

– 4.1: Use the Octave/Matlab command `factor` to find the prime factors of $a = 85$ and $b = 15$.

Sol: From Octave's `factor()` we find $85 = 17 \cdot 5$, $15 = 3 \cdot 5$. ■

– 4.2: What is the greatest common prime factor of $a = 85$ and $b = 15$?

Sol: The largest common factor `gcd(85, 15)` is 5. ■

– 4.3: By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.

Sol: Division method:

$$\begin{array}{ll}
 a_1 = 15 & b_1 = 85 - 15 \left\lfloor \frac{85}{15} \right\rfloor = 10 \\
 a_2 = 10 & b_2 = 15 - 10 \left\lfloor \frac{15}{10} \right\rfloor = 5 \\
 a_3 = 5 & b_3 = 10 - 5 \left\lfloor \frac{10}{5} \right\rfloor = 0
 \end{array}$$

$\therefore \text{gcd} = 5$

Subtraction method:

$$a_1 = 85 - 15 = 70$$

$$a_2 = 70 - 15 = 55$$

$$a_3 = 55 - 15 = 40$$

$$a_4 = 40 - 15 = 25$$

$$a_5 = 25 - 15 = 10$$

swap

$$a_6 = 15 - 10 = 5$$

$$a_7 = 10 - 5 = 5$$

terminate

$$b_1 = 15$$

$$b_2 = 15$$

$$b_3 = 15$$

$$b_4 = 15$$

$$b_5 = 15$$

$$b_6 = 10$$

$$b_7 = 5$$

$\therefore \text{gcd} = 5$ ■

– 4.4: By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?

Sol: Division method:

$$a_1 = 25 \qquad b_1 = 75 - 25 \left\lfloor \frac{75}{25} \right\rfloor = 0$$

Subtraction method:

$$\begin{aligned} a_1 &= 75 - 25 = 50 & b_1 &= 25 \\ a_2 &= 50 - 25 = 25 & b_2 &= 25 \end{aligned}$$

$\therefore \text{gcd} = 25$

The result is $25 = 5^2$, the square of a prime number. ■

– 4.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g., $a = 25$ and $b = 75$). What happens to a and b in the first step? Does it matter if you begin the algorithm with $a < b$ rather than $b < a$?

Sol: If $a < b$, the first step of the division algorithm swaps the terms ($a \rightarrow b$ and $b \rightarrow a$). ■

– 4.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers that have already been divided into factors (e.g., $a = 5 \cdot 3$ and $b = 7 \cdot 3$).

Sol: Division method:

$$\begin{aligned} a_1 &= 5 \cdot 3 & b_1 &= 7 \cdot 3 - 5 \cdot 3 \left\lfloor \frac{7 \cdot 3}{5 \cdot 3} \right\rfloor = 2 \cdot 3 \\ a_2 &= 2 \cdot 3 & b_2 &= 5 \cdot 3 - 2 \cdot 3 \left\lfloor \frac{5 \cdot 3}{2 \cdot 3} \right\rfloor = 1 \cdot 3 \\ a_3 &= 1 \cdot 3 & b_3 &= 2 \cdot 3 - 1 \cdot 3 \left\lfloor \frac{2 \cdot 3}{1 \cdot 3} \right\rfloor = 0 \end{aligned}$$

Subtraction method:

$$\begin{aligned} a_1 &= 7 \cdot 3 - 5 \cdot 3 = 2 \cdot 3 & b_1 &= 5 \cdot 3 \\ a_2 &= 5 \cdot 3 - 2 \cdot 3 = 3 \cdot 3 & b_2 &= 2 \cdot 3 \\ a_3 &= 3 \cdot 3 - 2 \cdot 3 = 1 \cdot 3 & b_3 &= 2 \cdot 3 \\ a_4 &= 2 \cdot 3 - 1 \cdot 3 = 1 \cdot 3 & b_4 &= 1 \cdot 3 \end{aligned}$$

The algorithm iteratively converges on the GCD by subtracting out multiples of the GCD until only the GCD is left. ■

– 4.7: Find the GCD of $2 \cdot \pi_{25}$ and $3 \cdot \pi_{25}$.

Sol: π_{25} ■

Problem # 5: Coprimes

– 5.1: Define the term coprime.

Sol: when two integers have no common factors they are said to be coprime ■

– 5.2: How can the Euclidean algorithm be used to identify coprimes?

Sol: If $\text{gcd}(a, b) = 1$ they only have 1 as a common factor, thus they are coprime. ■

– 5.3: Give at least one application of the Euclidean algorithm.

Sol: Given two integers $n, d \in \mathbb{Z}$, if we wish to reduce the fraction n/d , we must cancel the common factors. Example: If $n = 9, d = 6$ then $9/6 = (3 \cdot 3)/(2 \cdot 3) = 3/2$, where the GCD, 3, may be identified using the Euclidean algorithm. While this fraction may be easily simplified via inspection, the GCD algorithm could be very helpful for larger numbers n, d . ■

– 5.4: Write a Matlab function, `function x = my_gcd(a,b)`, that uses the Euclidean algorithm to find the GCD of any two inputs a and b . Test your function on the (a, b) combinations from the previous problem. Include a printout (or hand-write) your algorithm to turn in.

Hints and advice:

- Don't give your variables the same names as Matlab functions! Since `gcd` is an existing Matlab/Octave function, if you use it as a variable or function name, you won't be able to use `gcd` to check your `gcd()` function. Try `clear all` to recover from this problem.
- Try using a “while” loop for this exercise (see Matlab documentation for help).
- You may need to use some temporary variables for a and b in order to perform the algorithm.

Sol: Division method:

```
function x = my_gcd(a,b)
while b>0
atmp= a; btmp = b;
a = btmp; b = atmp-btmp*floor(atmp/btmp);
end
```

Subtraction method:

```
function x = my_gcd(a,b)
while a~=b
atmp= a; btmp = b;
a = max(atmp,btmp) - min(atmp,btmp); b = min(atmp,btmp);
end ■
```

Continued fractions

Problem # 6: Here we explore the continued fraction algorithm (CFA), discussed in Sec. 2.4.4.

In its simplest form, the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \mathbb{I}$, as an example because its CFA representation will be infinitely long. We can represent the CFA coefficients α as a vector of integers n_k , $k = 1, 2, \dots, \infty$:

$$\begin{aligned}\alpha &= [n_1; n_2, n_3, n_4, \dots] \\ &= n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \dots}}}\end{aligned}$$

As discussed in Sec. 2.4.3 (p. 27), the CFA is recursive, with three steps per iteration. For $\alpha_1 = \pi$, $n_1 = 3$, $r_1 = \pi - 3$, and $\alpha_2 \equiv 1/r_1$.

$$\begin{aligned}\alpha_2 &= 1/0.1416 = 7.0625 \dots \\ \alpha_1 &= n_1 + \frac{1}{\alpha_2} = n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \dots\end{aligned}$$

In terms of a Matlab/Octave script,

```
alpha0 = pi;
K=10;
n=zeros(1,K); alpha=zeros(1,K);
alpha(1)=alpha0;

for k=2:K %k=1 to K
n(k)=round(alpha(k-1));
%n(k)=fix(alpha(k-1));
alpha(k)= 1/(alpha(k-1)-n(k));
%disp([fix(k), round(n(k)), alpha(k)]); pause(1)
end
disp([n; alpha]);
%Now compare this to matlab's rat() function
rat(alpha0,1e-20)
```

– 6.1: By hand (you may use Matlab/Octave as a calculator), find the first three values of n_k for $\alpha = e^\pi$.

Sol: The CFA for this is: $e^\pi = 23.1407\dots = [23; 7, 9, 4, \dots]$. ■

– 6.2: For the preceding question, what is the error (remainder) when you truncate the continued fraction after n_1, \dots, n_3 ? Give the absolute value of the error and the percentage error relative to the original α .

Sol: The remainder is $e^\pi - (23 + 1/(7 + (1/9)))$ which gives an error of $\epsilon = |e^\pi - (23 + 1/(7 + (1/9)))|/e^\pi = 2.92 \cdot 10^{-6} = 0.0003\%$ ■

– 6.3: Use the Matlab/Octave program provided to find the first 10 values of n_k for $\alpha = e^\pi$, and verify your result using the Matlab/Octave command `rat()`.

Sol: $e^\pi = 23.1407\dots = [23; 7, 9, 4, -2, -591, -2, -10, 3, -2, \dots]$. ■

– 6.4: Discuss the similarities and differences between the Euclidean algorithm and the CFA.

Sol:

1. Both are recursive, meaning that the steps are repeated one after another.
2. The EA starts from two numbers (a,b). The output of the `gcd(a,b)` is the GCD. The CFA starts with a single number and the output is a sequence of integers. If the sequence terminates the number was rational. If the sequence does not terminate, the number is irrational.
3. The EA works with the difference between the minimum and maximum of the two numbers whereas the CFA works with the rounding function and the reciprocal of the error.
4. It would seem that the goals of the two algorithms, the starting point, and the results are totally different. Both are very useful and powerful. Both generalize to more difficult situations than working with simple numbers.

■

Added Aug

Problem # 7: CFA of ratios of large primes

– 7.1: (4pts) Expand $23/7$ as a continued fraction. Express your answer in bracket notation (e.g., $\pi = [3., 7, 16, \dots]$). Show your work. **Sol:** $23/7 = (21 + 2)/7 = 3 + 2/7 = 3 + 1/(6 + 1)/2 = 3 + 1/(6 + 1/2)$. In bracket notation $23/7 = [3., 6, 2]$. Matlab gives `rat(23/7) = 3 + 1/(4 + 1/(-2))`, or `[1., 4, -2]` because rounding $7/2$ can be taken as either $3 + 1/2$ or $4 - 1/2$. ■

– 7.2: Starting from the primes below 10^6 , form the CFA of π_j/π_k with $j = 78498$ and $k < j$.

Sol: First generate 10^6 primes with the matlab command `$\pi = \text{primes}(1e6)$` .

The length of π is $j = 78499$, $\pi(j) = 1,000,003$, $\pi(j-1) = 999,983$ and $\pi(j-2) = 999,979$.

Let the target fraction be

$$T = \frac{\pi(\text{end} - 1)}{\pi(\text{end} - 2)} = \frac{999983}{999979} = 1.000004000084002.$$

Finding the CFA of T gives

$$\text{rat}(T) = 1 + 1/249995 = [1; 249995].$$

Factoring this integer gives `factor(249995) = 5 * 49999`. ■

– 7.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? **Sol:** The CFA terminates in only one term, as in the above example. ■

– 7.4: (1pts) Try the Matlab/Octave functions `rats(23/7)`, `rats(3.2857)`, and `rats(3.2856)`. What do you conclude?

Sol: This function is similar to the CFA but uses rounding rather than truncation arithmetic. `rats(3.2857) = 32857/10000` but `rats(23/7) = 23/7` because it rounds to $23/7$, whereas `rats(3.2856) = 4107/1250` because it does not. ■

– 7.5: (2pts) Can $\sqrt{2}$ be represented as a finite continued fraction? Why or why not?

Sol: No, because it is irrational. ■

– 7.6: (2pts) What is the CFA for $\sqrt{2} - 1$?

$$\text{Hint: } \quad \sqrt{2} + 1 = \frac{1}{\sqrt{2} - 1} = [2; 2, 2, 2, \dots].$$

Sol: $1 + \sqrt{2} = 2 + 1/(2 + 1/(2 + \dots))$ or $[2., 2, 2, 2, \dots]$, thus

$$\sqrt{2} - 1 = [2., 2, 2, 2, \dots] - 2 = 0 + 1/(2 + 1/(2 + 1/(2 + \dots))).$$

■

– 7.7: Show that

$$\frac{1}{1 - \sqrt{a}} = a^{\frac{11}{2}} + a^{\frac{9}{2}} + a^{\frac{7}{2}} + a^{\frac{5}{2}} + a^{\frac{3}{2}} + \sqrt{a} + a^5 + a^4 + a^3 + a^2 + a + 1 = 1 - a^6$$

syms a, b

b= taylor(1/(1-sqrt(a)))

simplify((1-sqrt(a))*b) = 1-a^6

Use symbolic analysis to show this, then explain. **Sol:** This is a Taylor expansion of 1 expressed in terms of removable singularities. See Cotes Theorem (1716) (Stillwell, 2010, p. 289). ■