Since the signals we encounter in engineering, science, and everyday life are as varied as the applications in which we engage them, it is often helpful to first study these applications in the presence of simplified versions of these signals. Much like a child learning to play an instrument for the first time, it is easier to start by attempting to play a single note before an entire musical score. Then, after learning many notes, the child becomes a musician and can synthesize a much broader class of music, building up from many notes. This approach of building-up our understanding of complex concepts by first understanding their basic building blocks is a fundamental precept of engineering and one that we will use frequently throughout this book.

In this chapter, we will explore signals in both continuous time and discrete time, together with a number of ways in which these signals can be built-up from simpler signals. Simplicity is in the eye of the beholder and what makes a signal appear simple in one context may not shed much light in another context. Many of the concepts we will develop throughout this text arise from studying large classes of signals, one building block at a time, and extrapolating system (or application) level behavior by considering the whole as a sum of its parts. In this chapter, we will focus specifically on sinusoidal signals as our basic building blocks as we consider both periodic and aperiodic signals in continuous and discrete time. Along this path, we will encounter the Fourier series representations of periodic signals as well as Fourier transform representations of aperiodic, infinite-length signals. In later chapters, we will find that so-called "time-domain" representations of signals sometimes prove more fruitful, and for discrete-time signals there is a natural way to construct signals one sample at a time.

2.1 Fourier Series representation of finite-length and periodic CT signals

In many applications in science and engineering, we often work with signals that are periodic in time. That is, the signal repeats itself over and over again with a given period of repetition. Examples of periodic signals might include the acoustic signal that emenates from a musical instrument, such as a trumpet when a single sustained note is played, or the vertical displacement of a mass in a frictionless spring-mass oscillator set into motion, or the horizontal displacement of a pendulum swaying to and fro in the absence of friction.

Mathematically, we represent a periodic signal, $x(t)$, as one whose value repeats at a fixed interval of time from the present. This interval, denoted T below, is called the "period" of the signal, and we express this relationship

$$x(t) = x(t+T), \text{ for all } t. \tag{2.1}$$

Equation (2.1) will, in general, be satisfied for a countably infinite number of possible values of T when $x(t)$ is periodic. The smallest, positive value of T for which Eq. (2.1) is satisfied, is called the "fundamental period" of the signal $x(t)$. For sinusoidal signals, such as

$$x(t) = \sin(\omega_0 t + \phi), \tag{2.2}$$

we can relate the frequency of oscillation, ω_0 to the fundamental period, T. This can be computed by noting that sinusoidal functions are equal when their arguments are either equal or differ only through a multiple of 2π, i.e.

$$\begin{aligned} x(t) &= x(t+T) \\ \sin(\omega_0 t + \phi) &= \sin(\omega_0(t+T) + \phi) \\ \sin(\omega_0 t + \phi + 2k\pi) &= \sin(\omega_0(t+T) + \phi) \\ \sin(\omega_0(t + 2k\pi/\omega_0) + \phi) &= \sin(\omega_0(t+T) + \phi) \end{aligned} \tag{2.3}$$

which, for $k = 1$, yields the relationship

$$T = 2\pi/\omega_0, \tag{2.4}$$

between the fundamental period, T, and the "fundamental frequency" ω_0. By analogy to sinusoidal signals, we refer to the value of $\omega_0 = 2\pi/T$ as the fundamental frequency of any signal that is periodic with a fundamental period T.

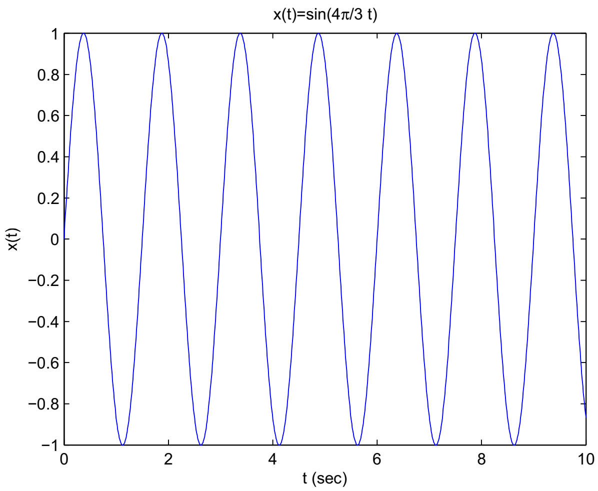

Figure 2.1: The periodic sinusoidal signal $x(t) = \sin((4\pi/3)t)$.

Here, we will provide a number of examples of periodic signals in continuous-time, including sinusoidal, square wave, traingular wave and complex exponential signals. By noting that any two periodic signals, $x(t)$ and $y(t)$ with the same period T can be added together to produce a new periodic signal of the same period, i.e.,

$$\begin{aligned} s(t) =& x(t) + y(t) \\ s(t+T) =& x(t+T) + y(t+T) = s(t+T), \end{aligned}$$

in 1807 Jean Baptiste Fourier (1807) considered the notion of building a large set of periodic signals from sinusoidal signals sharing the same period. Ignoring the phase,ϕ, for now, note that from (2.4), sinusoidal signals that share the same period must have fundamental frequencies given by $k\omega_0 = 2k\pi/T$ for different values of k. If two sinusoidal signals shared the same fundamental frequency, then they would be the same sinusoidal signal (recall that, for now, we are neglecting the phase, ϕ). We call such sinusoidal signals whose fundamental frequencies $k\omega_0$ are integer multiples of one fundamental frequency, harmonically-related sinusoids. Such harmonically-related sinusoids could indeed share the period, $2\pi/\omega_0$ while they would have different fundamental freqeuncies and hence different "fundamental periods."

We now consider how we might build-up a larger class of periodic signals from the basic building blocks of harmonically-related sinusoids. To extend our discussion to include complex-valued signals, we will employ Euler's relation to construct complex exponential signals of the form

$$\begin{aligned} x(t) =& e^{j(\omega_0 t + \phi)} \\ =& \cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi) \end{aligned} \tag{2.5}$$

and in doing so, we can push the phase out of the picture so that it can be absorbed in a complex scalar constant out front, i.e.

$$x(t) = ce^{j\omega_0 t},$$

where, $c = e^{j\phi}$ is simply a complex constant whose effects on the sinusoidal nature of the signal have been conveniently parked outside the discussion. Complex-exponential signals of the form (2.5) are periodic with fundamental frequency $\omega_0 = 2\pi/T$ since they are simply constructed by pairing the real-valued periodic signal $\cos(\omega_0 t)$ with the purely imaginary signal $j\sin(\omega_0 t)$.

By simply adding together harmonically-related sinusoidal signals, we can construct a large class of periodic waveforms of amazing variety. For example, in Figure 2.2, note how by taking odd-valued harmonics (sinusoids with harmonically-related fundamental frequencies that are odd multiples of a single frequency, $\omega_0 = 2\pi$), we obtain an increasingly improving approximation to a square wave with unit period.

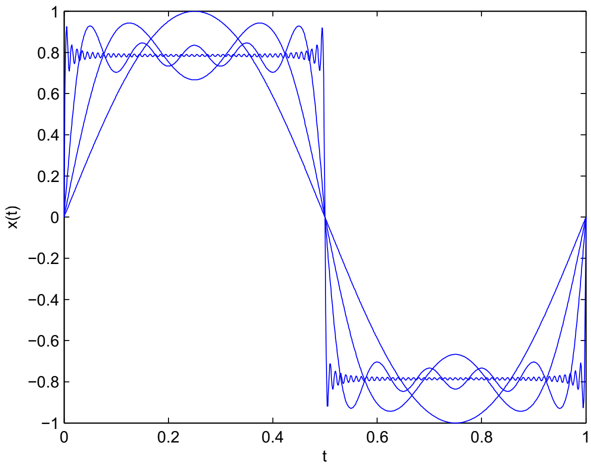

Figure 2.2: The periodic sinusoidal signal $x(t) = \sum_{k=1}^{N} \frac{1}{k} \sin(2k\pi t)$, for $k = 1, 3, 9$ and 99.

Generalizing this idea, we can explore the class of signals that can be constructed by such harmonically-related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}. \tag{2.6}$$

To bring the period of the periodic signal $x(t)$ into the equation, (2.6) is often written

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T}, \tag{2.7}$$

where $T = 2\pi/\omega_0$ is the fundamental period and ω_0 is the fundamental frequency of the periodic signal $x(t)$. The construction in (2.7) is referred to as the continuous-time Fourier series (CTFS) representation of $x(t)$ and (2.7) is often called the continuous-time Fourier series synthesis equation.

The Fourier series coefficients $X[k]$ can be obtained by multiplying (2.7) by $e^{-j2\pi kt/T}$ and integrating over a period of duration T to obtain

$$\int_0^T x(t) e^{-j2\pi kt/T} dt = \int_0^T \left(\sum_{m=-\infty}^{\infty} X[m] e^{j2\pi(m-k)t/T} \right) dt,$$

where the limits of integration indicate that the we have chosen to evaluate the integral over the period $0 \leq t \leq T$. Note the use of the dummy variable m in the summation for the CTFS, since the variable k was already in use. To use k again would invite disaster into our derivation. Interchanging the order of integration and summation (which can be done under suitable conditions on the summation), we obtain,

$$\int_0^T x(t) e^{-j2\pi kt/T} dt = \sum_{m=-\infty}^{\infty} \int_0^T X[m] e^{j2\pi(m-k)t/T} dt. \tag{2.8}$$

To proceed, we need to evaluate the integral

$$\begin{aligned}\int_0^T e^{j2\pi(m-k)t/T}dt =& \frac{T}{j2\pi(m-k)}e^{j2\pi(m-k)t/T}\Big|_0^T \\ =& \frac{T}{j2\pi(m-k)}[e^{j2\pi(m-k)}-1] \\ =& T\delta[m-k],\end{aligned}$$

where the second line arises from simple integration of an exponential function. The second line is readily seen to be equal to zero when $m \neq k$ and though one might be tempted to evaluate this line for $m = k$ (using a formula bearing the name of a famous 17th-century French mathematician), our efforts will be better spent setting $m = k$ into the integrand on the left hand side of the first line, from which we obtain

$$\int_0^T 1dt = T.$$

An interpretation of this result is that integration of a periodic complex exponential over an integer multiple, $(m-k)$, of its fundamental period, in this case $T/2\pi(m-k)$, is zero. The only periodic complex exponential that survives integration over the period T is the DC, i.e. $m = k$, term.

We can now return to (2.8) and apply this result, to obtain

$$\begin{aligned}\int_0^T x(t)e^{-j2\pi kt/T}dt =& \sum_{m=-\infty}^{\infty} X[m]T\delta[m-k] \\ =& TX[k], \end{aligned} \tag{2.9}$$

by the sifting property of the Kronocker delta function. We can now turn (2.9) around to obtain the continuous-time Fourier series analysis equation,

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-j2\pi kt/T}dt. \tag{2.10}$$

Putting the synthesis and analysis equations together, we have the continuous-time Fourier series representation of a periodic signal $x(t)$ as

CT Fourier Series Representation of a Periodic Signal

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi kt}{T}}dt \tag{2.11}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}} \tag{2.12}$$

Example: CTFS of a Square Wave

Let us return to the square wave signal that we visited in Figure 2.2. In the figure, we appeared to have a method for constructing the periodic signal that, in the interval $[0,1]$, satisfies

$$x(t) = \begin{cases} 1, & 0 \le t \le 0.5 \\ -1 & \text{else.} \end{cases} \tag{2.13}$$

Using (2.10), we obtain,

$$
\begin{aligned}
X[k] &= \int_0^1 x(t)e^{-j2\pi kt}dt \\
&= \int_0^{0.5} e^{-j2\pi kt}dt - \int_{0.5}^1 e^{-j2\pi kt}dt \\
&= \frac{-1}{j2\pi k}\left([e^{-j\pi k}-1]-[1-e^{-j\pi k}]\right) \\
&= \frac{-1}{j2\pi k}2[(-1)^k-1] \\
&= \begin{cases} 0, & k \text{ even} \\ \frac{2}{j\pi k} & k \text{ odd.} \end{cases}
\end{aligned}
\tag{2.14}
$$

Note that the $k = 0$ case can be readily evaluated by considering the integral in (2.14) for which the integral can be easily seen to vanish by the antisymmetry of $x(t)$ over the unit interval.

2.1.1 CT Fourier Series Properties

We have now been properly introduced to a method for building-up continuous-time periodic signals from a class of simple sinusoidal signals in (2.11)and a method for analysing the make-up of such periodic signals in terms of their constituent sinusoidal components in (2.12). Now that introductions are out of the way, we can explore some of the many useful properties of the CTFS representation. As we shall see, it is often helpful to consider the properties of a whole signal by virtue of the properties of its parts, and the relations we develop next will often prove useful in this process.

2.1.1.1 Linearity

The CTFS can be viewed as a linear operation, in the following manner. When two signals $x(t)$ and $y(t)$ are each constructed from their constituent sinusoidal signals according to the CTFS synthesis equation (2.12), the linear combination of these signals, $z(t) = ax(t) + by(t)$, for a, b real-valued constants, can be readily obtained by combining the constituent sinusoidal signals through the same linear combination. More specifically, when $x(t)$ is a periodic signal with CTFS coefficients $X[k]$ and $y(t)$ is a periodic signal with CTFS coefficients $Y[k]$ then the signal $z(t) = ax(t) + by(t)$ has CTFS coefficients given by $Z[k] = aX[k] + bY[k]$. The linearity property of the CTFS can be compactly represented as follows

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k], y(t) \overset{CTFS}{\longleftrightarrow} Y[k] \implies z(t) = ax(t) + by(t) \overset{CTFS}{\longleftrightarrow} aX[k] + bY[k].$$

This result can be readily shown by substituting $z(t) = ax(t)+by(t)$ into the integral in (2.11) and expanding the integral into the two separate terms, one for $X[k]$ and one for $Y[k]$.

2.1.1.2 Time Shift

When a sinusoidal signal $x(t) = \sin(\omega_0 t)$ is shifted in time, the resulting signal $x(t - t_0)$ can be represented in terms of a simple phase shift of the origional sinusoidal signal, i.e. $x(t - t_0) = \sin(\omega_0(t - t_0)) = \sin(\omega_0 t - \phi)$,where $\phi = \omega_0 t_0 = 2\pi t_0/T$. Periodic signals that can be represented using the CTFS contain many, possibly infinitely many, sinusoidal (or complex exponential) signals. When such periodic signals are delayed in time, each of the constituent sinusoidal components of the signal are delayed by the same amount, however this translates into a different phase shift for each component. This can be readily seen from the CTFS analysis equation (2.11), as follows. For the signal $y(t) = x(t - t_0)$, we have

$$\begin{aligned}
Y[k] &= \frac{1}{T}\int_{t=0}^{T} x(t-t_0)e^{-j\frac{2\pi k}{T}t}dt \\
&= \frac{1}{T}\int_{s=-t_0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}dt \\
&= \frac{1}{T}\int_{s=-t_0}^{0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds + \frac{1}{T}\int_{s=0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds \\
&= \frac{1}{T}\int_{s=-t_0}^{0} x(s+T)e^{-j\frac{2\pi k}{T}(s+T+t_0)}ds + \frac{1}{T}\int_{s=0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds \\
&= \frac{1}{T}\int_{\tau=T-t_0}^{T} x(\tau)e^{-j\frac{2\pi k}{T}(\tau+t_0)}d\tau + \frac{1}{T}\int_{s=0}^{T-t_0} x(s)e^{-j\frac{2\pi k}{T}(s+t_0)}ds \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)e^{-j\frac{2\pi k t_0}{T}}e^{-j\frac{2\pi k}{T}t}dt \\
&= X[k]e^{-j\frac{2\pi k t_0}{T}},
\end{aligned}$$

where, the second line follows from the change of variable, $s = t - t_0$, the fourth line follows from the periodicity of both the signal $x(t)$ and the signal $e^{-j2\pi kt/T}$with period T, the fifth line follows from the change of variable $\tau = s + T$, and the last line follows from the definition of $X[k]$ after factoring the linear phase term $e^{-j2\pi kt_0/T}$ out of the integral. The time shift property of the CTFS can be compactly represented as follows

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(t-t_0) \overset{CTFS}{\longleftrightarrow} X[k]e^{-j\frac{2\pi k}{T}t_0}.$$

We see that a shift in time of a periodic signal corresponds to a modulation in frequency by a phase term that is linear with frequency with a slope that is proportional to the delay. This can be made easier if we adopt the convenient, but conceptually more challenging concept of integration over a period for the definition of the CTFS.

2.1.1.3 Frequency Shift

When a periodic signal $x(t)$ has a CTFS representation given by $X[k]$, a natural question that might arise is the what happens when the shifting that was discussed in section 2.1.1.2 is applied to the CTFS representation, $X[k]$. Specifically, if a periodic signal $y(t)$ were known to have a CTFS representation given by $Y[k] = X[k - k_0]$, it is interesting to understand the relationship in the time-domain between $y(t)$ and $x(t)$. This can be readily seen through examination of the CTFS analysis equation,

$$\begin{aligned}
Y[k] &= X[k-k_0] \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)e^{-j\frac{2\pi}{T}(k-k_0)t}dt \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)e^{j\left(\frac{2\pi k_0}{T}\right)t}e^{-j\frac{2\pi}{T}kt}dt \\
&= \frac{1}{T}\int_{t=0}^{T} \left(x(t)e^{j\left(\frac{2\pi k_0}{T}\right)t}\right)e^{-j\frac{2\pi}{T}kt}dt,
\end{aligned}$$

which leads to the relation

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(t)e^{jk_0\omega_0 t} \overset{CTFS}{\longleftrightarrow} X[k-k_0],$$

where $\omega_0 = \frac{2\pi}{T}$. We observe that a shift in the continuous time Fourier series coefficients by an integer amount k_0 corresponds to a modulation in the time domain signal $x(t)$ by a term whose frequency is proportional to the shift amount.

2.1.1.4 Time Reversal

When a periodic signal $x(t) = e^{j2\pi t/T}$ is time-reversed, i.e. $y(t) = x(-t)$, the effect on its CTFS representation can be simply observed

$$\begin{aligned} X[k] &= \frac{1}{T}\int_{t=0}^{T} e^{j\frac{2\pi}{T}t}e^{-j\frac{2\pi k}{T}t}dt \\ &= \begin{cases} 1, & \text{for } k = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} Y[k] &= \frac{1}{T}\int_{t=0}^{T} e^{-j\frac{2\pi}{T}t}e^{-j\frac{2\pi k}{T}t}dt \\ &= \begin{cases} 1, & \text{for } k = -1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

More generally, from the CTFS synthesis equation,

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi k}{T}t},$$

we see that by simply changing the sign of the time variable t, we obtain the general relation

$$\begin{aligned} y(t) &= x(-t) = \sum_{k=-\infty}^{\infty} X[k]e^{-j\frac{2\pi k}{T}t} \\ &= \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi(-k)}{T}t} \\ &= \sum_{m=-\infty}^{\infty} X[-m]e^{j\frac{2\pi k}{T}t}, \end{aligned}$$

yielding the relation

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(-t) \overset{CTFS}{\longleftrightarrow} X[-k],$$

i.e., changing the sign of the time axis corresponds to changing the sign of the CTFS frequency index.

2.1.1.5 Time Scaling

When a periodic signal undergoes a time-scale change, such as one that compresses the time axes, $y(t) = x(at)$, where $a > 1$ is a real-valued constant, the resulting signal $y(t)$ would remain periodic, however the period would change correspondingly, such that $y(t+T_y) = y(t)$ would be satisfied for a different period T_y. This can be easily seen by substituting in for $x(t)$ in the relation, $y(t) = x(at) = y(t+T_y) = x(a(t+T_y))$ and the noting that $x(at) = x(at+T)$, due to the periodicity of $x(t)$ with period T. This leads to the relation $x(a(t+T_y)) = x(at+T)$ or $T_y = T/a$. This makes intuitive sense, since the time-axis in the signal $y(t)$ has been compressed by a factor of a, therefore the time at which it will repeat must also have compressed by the same factor. Now, even though the period of the signal $y(t)$ has changed, we also are interested in the full CTFS representaiton of $y(t)$. This is given by

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T} at} \\ &= \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T_y} t}, \end{aligned}$$

where the second line follows from the definition of T_y. Note that although we have that

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = x(at) \overset{CTFS}{\longleftrightarrow} X[k],$$

that is the sequence of CTFS coefficients $Y[k]$ is identical to $X[k]$, the CTFS representation for $x(t)$ and $y(t)$ differ substantially, since they are defined for completely different periods, $T \neq T_y$. As a result, the fundamental frequency for the periodic signal $x(t)$ is $2\pi/T$, which is different from that of $y(t)$, which is $2\pi a/T$. Hence, the frequency content of the signals differ substantially.

2.1.1.6 Conjugate Symmetry

The effect of conjugating a complex-valued signal on its CTFS representation can be seen by simply conjugating the CTFS synthesis relation,

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T} t} \\ x^*(t) &= \left(\sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T} t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} X^*[k] e^{-j\frac{2\pi k}{T} t} \\ &= \sum_{k=-\infty}^{\infty} X^*[k] e^{j\frac{2\pi(-k)}{T} t} \\ &= \sum_{m=-\infty}^{\infty} X^*[-m] e^{j\frac{2\pi m}{T} t} \end{aligned}$$

yielding that

$$x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies x^*(t) \overset{CTFS}{\longleftrightarrow} X^*[-k].$$

When the periodic signal $x(t)$ is real valued, i.e. $x(t)$ only takes on values that are real numbers, then the CTFS exhibits a symmetry property. This arises directly from the definintion of the CTFS, and that real numbers equal their conjugates, i.e. $x(t) = x^*(t)$, such that

$$x(t) = x^*(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies X[k] = X^*[-k].$$

Note that when the signal is real-valued and is an even function of time, such that $x(t) = x(-t)$, then its CTFS is also real-valued and even, i.e. $X[k] = X^*[k] = X[-k]$. It can be shown by similar reasoning that when the signal is periodic, real-valued, and an odd function of time, that the CTFS coefficients are purely imaginary and odd, i.e. $X[k] = -X^*[k] = -X[-k]$.

2.1.1.7 Products of Signals

When two periodic signals of the same period are multiplied in time, such that $z(t) = x(t)y(t)$, the resulting signal remains periodic with the same period, such that $z(t) = x(t)y(t) = x(t+T)y(t+T) = z(t+T)$. Hence, each of the three signals admit CTFS representations using the same set of harmonically related signals. We can observe the effect on the resulting CTFS representation through the analysis equation,

$$\begin{aligned}
Z[k] &= \frac{1}{T}\int_{t=0}^{T}(x(t)y(t))e^{-j\frac{2\pi k}{T}t}dt \\
&= \frac{1}{T}\int_{t=0}^{T}(y(t)\left(\sum_{m=-\infty}^{\infty}X[m]e^{j\frac{2\pi m}{T}t}\right))e^{-j\frac{2\pi k}{T}t}dt \\
&= \sum_{m=-\infty}^{\infty}X[m]\frac{1}{T}\int_{t=0}^{T}y(t)e^{-j\frac{2\pi(k-m)}{T}t}dt \\
&= \sum_{m=-\infty}^{\infty}X[m]Y[k-m].
\end{aligned}$$

The relationship between the CTFS coefficients for $z(t)$ and those of $x(t)$ and $y(t)$ is called a discrete convolution between the two sequences $X[k]$ and $Y[k]$,

$$x(t)\stackrel{CTFS}{\longleftrightarrow}X[k], y(t)\stackrel{CTFS}{\longleftrightarrow}Y[k] \implies z(t)=x(t)y(t)\stackrel{CTFS}{\longleftrightarrow}\sum_{m=-\infty}^{\infty}X[m]Y[k-m].$$

2.1.1.8 Convolution

A dual relationship to that of multiplication in time, is multiplication of CTFS coefficients. Specifically, when the two signals $x(t)$ and $y(t)$ are each periodic with period T, the periodic signal $z(t)$ of period T, whose CTFS representation is given by $Z[k]=X[k]Y[k]$ corresponds to a periodic convolution of the signals $x(t)$ and $y(t)$. This can be seen as follows,

$$\begin{aligned}
z(t) &= \sum_{k=-\infty}^{\infty}\left(X[k]Y[k]\right)e^{j\frac{2\pi k}{T}t} \\
&= \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\int_{\tau=0}^{T}x(\tau)e^{-j\frac{2\pi k}{T}\tau}d\tau\right)Y[k]e^{j\frac{2\pi k}{T}t} \\
&= \frac{1}{T}\int_{\tau=0}^{T}x(\tau)\left(\sum_{k=-\infty}^{\infty}Y[k]e^{j\frac{2\pi k}{T}(t-\tau)}\right)d\tau \\
&= \frac{1}{T}\int_{\tau=0}^{T}x(\tau)y(t-\tau)d\tau
\end{aligned}$$

where the integral relationship in the last line is called periodic convolultion. This leads to the following property of the CTFS,

$$x(t)\stackrel{CTFS}{\longleftrightarrow}X[k], y(t)\stackrel{CTFS}{\longleftrightarrow}Y[k] \implies z(t)=\frac{1}{T}\int_{\tau=0}^{T}x(\tau)y(t-\tau)d\tau\stackrel{CTFS}{\longleftrightarrow}Z[k]=X[k]Y[k].$$

2.1.1.9 Integration

When the signal $y(t)$ and $x(t)$ are related through a running integral, i.e. $y(t)=\int_{\tau=0}^{t}x(\tau)d\tau$, we can relate their CTFS as follows,

$$
\begin{aligned}
x(t) = \frac{d}{dt}\int_{\tau=0}^{t} x(\tau)d\tau &= \frac{d}{dt}\int_{\tau=0}^{t}\Big(\sum_{k=-\infty,k\neq 0}^{\infty} X[k]e^{j\frac{2\pi k}{T}\tau} + X[0]\Big)d\tau \\
&= \frac{d}{dt}\Big(\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\int_{\tau=0}^{t} e^{j\frac{2\pi k}{T}\tau}d\tau + X[0]t\Big) \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\left[\frac{T}{j2\pi k}e^{j\frac{2\pi k}{T}\tau}\right]_{\tau=0}^{t} + X[0] \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\left[\frac{T}{j2\pi k}\left(e^{j\frac{2\pi k}{T}t} - 1\right)\right] + X[0] \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\frac{T}{j2\pi k}e^{j\frac{2\pi k}{T}t} - \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty} X[k]\frac{T}{j2\pi k} + X[0] \\
&= \frac{d}{dt}\sum_{k=-\infty,k\neq 0}^{\infty}\left(X[k]\frac{T}{j2\pi k}\right)e^{j\frac{2\pi k}{T}t} + X[0] \\
&= \frac{d}{dt}\Big(\sum_{k=-\infty,k\neq 0}^{\infty} Y[k]e^{j\frac{2\pi k}{T}t} + X[0]t\Big) \\
\text{From this, if we let } y(t) &= \sum_{k=-\infty,k\neq 0}^{\infty}\left(X[k]\frac{T}{j2\pi k}\right)e^{j\frac{2\pi k}{T}t} + X[0]t \\
\frac{d}{dt}y(t) &= \sum_{k=-\infty,k\neq 0}^{\infty} X[k]e^{j\frac{2\pi k}{T}t} + X[0] \\
&= x(t).
\end{aligned}
$$

This yields the property,

$$
x(t) \overset{CTFS}{\longleftrightarrow} X[k] \implies y(t) = \int_{\tau=0}^{t} x(\tau)d\tau \overset{CTFS}{\longleftrightarrow} \begin{cases} \frac{T}{j2\pi k}X[k] & k \neq 0 \\ 0 & k = 0 \end{cases},
$$

where we must only consider $x(t)$ such that $X[0] = 0$, or else $y(t)$ would not be periodic.

2.1.1.10 Differentiation

Similarly, we can consider the relationship between $y(t) = \frac{d}{dt}x(t)$ and their corresponding CTFT representations. From the definition of the CTFS, we have

$$
\begin{aligned}
y(t) &= \frac{d}{dt}x(t) \\
&= \frac{d}{dt}\sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi k}{T}t} \\
&= \sum_{k=-\infty}^{\infty} X[k]\frac{d}{dt}e^{j\frac{2\pi k}{T}t} \\
&= \sum_{k=-\infty}^{\infty}\left(X[k]\frac{j2\pi k}{T}\right)e^{j\frac{2\pi k}{T}t}
\end{aligned}
$$

from which we obtain the relation

$$x(t) \stackrel{CTFS}{\longleftrightarrow} X[k] \implies y(t) = \frac{d}{dt}x(t) \stackrel{CTFS}{\longleftrightarrow} \left(\frac{j2\pi k}{T}\right) X[k].$$

2.1.1.11 Parseval's relation

The energy containted within a period of a periodic signal can also be computed in terms of its CTFS representation using Parseval's relation,

$$x(t) \stackrel{CTFS}{\longleftrightarrow} X[k] \implies \frac{1}{T}\int_{t=0}^{T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2.$$

This relation can be derived using the definition of the CTFS as follows,

$$\begin{aligned}
\frac{1}{T}\int_{t=0}^{T} |x(t)|^2 dt &= \frac{1}{T}\int_{t=0}^{T} x(t)x^*(t)dt \\
&= \frac{1}{T}\int_{t=0}^{T} x(t)\left(\sum_{k=-\infty}^{\infty} X^*[-k]e^{j\frac{2\pi k}{T}t}\right)dt \\
&= \sum_{k=-\infty}^{\infty} X^*[-k]\left(\frac{1}{T}\int_{t=0}^{T} x(t)e^{j\frac{2\pi k}{T}t}dt\right) \\
&= \sum_{k=-\infty}^{\infty} X^*[-k]\left(\frac{1}{T}\int_{t=0}^{T} x(t)e^{-j\frac{2\pi(-k)}{T}t}dt\right) \\
&= \sum_{k=-\infty}^{\infty} X^*[-k]X[-k] \\
&= \sum_{m=-\infty}^{\infty} |X[m]|^2.
\end{aligned}$$

Parseval's relation shows that the energy in a period of a periodic signal is equal to the sum of the energies contained within each of the harmonic components that make up the signal through the CTFS representation.

2.2 Fourier transform representation of CT signals

Now that we have seen how we may build-up a large class of continuous-time periodic signals from the set of simpler complex exponential periodic signals, we return to apply this line of thinking to the more general class of continuous-time aperiodic (not periodic) signals. Just as was the case for periodic signals, a remakably rich class of aperiodic signals can also be constructed from linear combinations of complex exponentials. In the case of periodic continuous-time signals, since the signals of interest were periodic, the CTFS was restricted to contruct such signals through combinations of harmonically related exponentials. However for more general aperiodic signals, we may consider building an even larger class of signals by removing this restriction on the ingredients used to makeup a given signal. Since harmonically related complex exponentials can be enumerated, the CTFS took the form of a summation over the countably infinite set of all harmoically related exponentials of a given fundamental frequency. However, removing the restriction to only using harmonically related terms, the class of all possible complex exponentials arises from a continuum of possible frequenecy components and the form used with which to contruct linear combinations will take the form of an integral, rather than an infinite summation. Just as with the continuous-time Fourier series, where the CTFS analysis equation provided a method for calculating the frequency components that makeup a given periodic signal, the continuous-time Fourier transform provides a method for calculating the spectrum of frequency components that makup an aperiodic signal from this class. The resulting integral used to contruct this large class of signals using this specific spectrum of frequency components is called the Fourier integral, or the continuous-time Fourier synthesis equation.

One method for introducing the continous-time Fourier transform is through the CTFS. By considering continuous-time aperiodic signals as the result of taking continous-time periodic signals to the limit of an infinite period, we may observe how the CTFS transitions from a countable sum of harmonically-related complex exponentials, into a continuous integral over the continuum of possible frequencies. Let us return to the square wave signal that we visited in Figure 2.2. In this case, however, we will alter the signal to take the form

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

over the unit interval, $t \in [0, 1]$. Using (2.10), we once again obtain its CTFS representation, however this time, we consider the period of repetition of the "on" period of the square wave to be given by the variable T, i.e. we have

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

for $t \in [0, T]$, and then repeating every T seconds. This yields the following CTFS representation

$$\begin{aligned} X[k] &= \int_0^T x(t) e^{-j\frac{2\pi k}{T}t} dt \\ &= \int_0^1 e^{-j\frac{2\pi k}{T}t} dt \\ &= \frac{-T}{j2\pi k}\left(e^{-j\frac{2\pi k}{T}} - 1\right) \\ &= \frac{-T}{j2\pi k} e^{-j\frac{\pi k}{T}}\left(e^{-j\frac{\pi k}{T}} - e^{j\frac{\pi k}{T}}\right) \\ &= \frac{T}{j2\pi k} e^{-j\frac{\pi k}{T}} 2j \sin\left(\frac{\pi k}{T}\right) \\ &= \begin{cases} \frac{\sin\left(\frac{\pi k}{T}\right)}{\frac{\pi k}{T}} e^{-j\frac{\pi k}{T}} & k \neq 0 \\ 1 & k = 0, \end{cases} \end{aligned} \tag{2.15}$$

where the $k = 0$ term is once again determined by closer examination of the first line of the derivation, rather than attempting further analysis on the expression at containing vanishing terms in the numerator and demoninator. We consider the expression in (2.15) for various values of T in Figure 2.3. By plotting the magnitude of the CTFS coefficients $|X[k]|$ versus the harmonicaly related frequency components $\frac{2\pi k}{T}$ for various values of T, ranging from $T = 4$, up to $T = 32$, we see that the envelope containing the CTFS coefficients remains constant, while the CTFS coefficients move closer and closer to one another in absolute frequency.

The envelope that is observed in the figure, can be viewed as the value that the CTFS representation would take on as the period of the signal is made larger and larger. Recognizing this process, Fourier defined this envelope as

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt, \tag{2.16}$$

where the frequency variable ω takes on all values on the real line, and for which (2.16) is known as the continuous-time Fourier transform (CTFT). For this example, the continuous-time Fourier transform would evaluate to

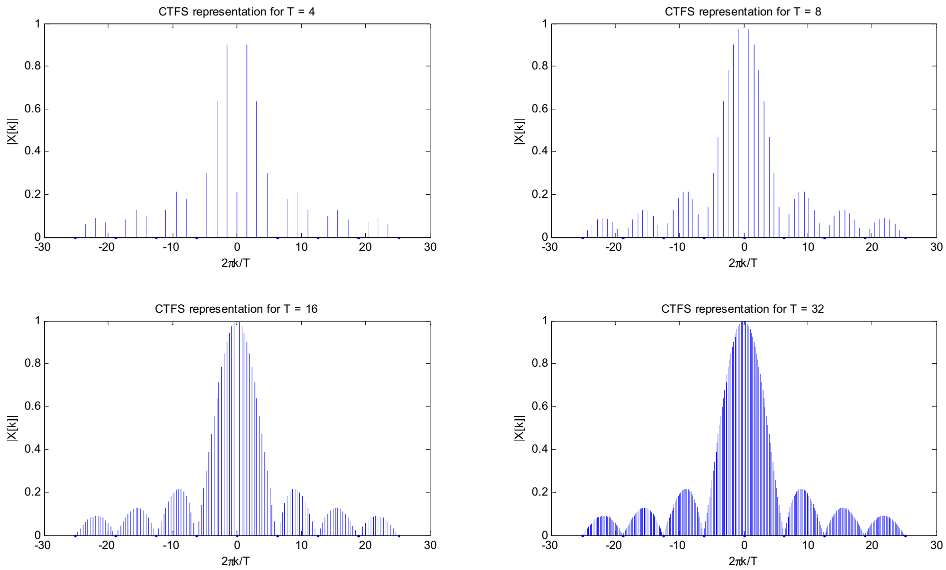

Figure 2.3: CTFS representation of the periodic signal in 2.17for $T = 4, 8, 16, 32$.

$$
\begin{aligned}
X(\omega) = & \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
= & \int_{0}^{1} e^{-j\omega t}dt \\
= & \frac{-1}{j\omega}\left(e^{-j\omega} - 1\right) \\
= & \frac{-1}{j\omega}e^{-j\omega/2}\left(e^{-j\omega/2} - e^{j\omega/2}\right) \\
= & \frac{1}{j\omega}e^{-j\omega/2}2j\sin\left(\omega/2\right) \\
= & \begin{cases} \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}e^{-j\frac{\omega}{2}} & \omega \neq 0 \\ 1 & \omega = 0. \end{cases}
\end{aligned} \tag{2.17}
$$

While the CTFT analysis equation (2.16) provides the composition of any of a large class of signals through a linear superposition of complex exponential signals of the form $e^{j\omega t}$, the CTFT synthesis equation provides the recipe for constructing such signals from their constituent set, as

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Together, the two expressions make up the CTFT representation for aperiodic signals,

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\
X(\omega) &= \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt
\end{aligned}
$$

CT Fourier Transform Representation of Aperiodic Signals

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \quad (2.18)$$

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad (2.19)$$

2.2.1 CT Fourier Transform Properties

We have now been properly introduced to a method for building-up continuous-time aperiodic signals from a class of complex exponential signals in (2.18) and a method for analysing the make-up of such periodic signals in terms of their constituent sinusoidal components in (2.19). Once again, now that introductions are out of the way, we can explore some of the many useful properties of the CTFT representation. Many of the properties of the CTFT follow directly, or along similar lines, of those of the CTFS.

2.2.1.1 Linearity

The CTFT can be viewed as a linear operation, in the following manner. When two signals $x(t)$ and $y(t)$ are each constructed from their constituent complex exponential signals according to the CTFT synthesis equation, the linear combination of these signals, $z(t) = ax(t) + by(t)$, for a, b real-valued constants, can be readily obtained by combining the constituent complex exponential signals through the same linear combination. More specifically, when $x(t)$ is an aperiodic signal with CTFT coefficients $X(\omega)$ and $y(t)$ is an aperiodic signal with CTFT $Y(\omega)$ then the signal $z(t) = ax(t) + by(t)$ has a CTFT representation given by $Z(\omega) = aX(\omega) + bY(\omega)$. The linearity property of the CTFT can be compactly represented as follows

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega), y(t) \overset{CTFT}{\longleftrightarrow} Y(\omega) \implies z(t) = ax(t) + by(t) \overset{CTFT}{\longleftrightarrow} aX(\omega) + bY(\omega).$$

2.2.1.2 Time Shift

For the signal $y(t) = x(t - t_0)$, we have

$$\begin{aligned} Y(\omega) &= \int_{t=-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt \\ &= \int_{s=-\infty}^{\infty} x(s)e^{-j\omega(s+t_0)}ds \\ &= \int_{s=-\infty}^{\infty} x(s)e^{-j\omega t_0}e^{-j\omega s}ds \\ &= X(\omega)e^{-j\omega t_0}, \end{aligned}$$

where, the second line follows from the change of variable, $s = t - t_0$. The time shift property of the CTFT can be compactly represented as follows

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(t-t_0) \overset{CTFT}{\longleftrightarrow} X(\omega)e^{-j\omega t_0}.$$

We see that a shift in time of an aperiodic signal corresponds to a modulation in frequency by a phase term that is linear with frequency with a slope that is proportional to the delay.

2.2.1.3 Frequency Shift

When a signal $x(t)$ has a CTFT representation given by $X(\omega)$, a natural question that might arise is the what happens when the shifting that was discussed in section 2.2.1.2 is applied to the CTFT representation, $X(\omega)$. Specifically, if a signal $y(t)$ were known to have a CTFT representation given by $Y(\omega) = X(\omega - \omega_0)$, it is interesting to understand the relationship in the time-domain between $y(t)$ and $x(t)$. This can be readily seen through examination of the CTFT analysis equation,

$$\begin{aligned} Y(\omega) &= X(\omega-\omega_0) \\ &= \int_{t=-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\ &= \int_{t=-\infty}^{\infty} \left(x(t)e^{j\omega_0 t}\right) e^{-j\omega t}dt, \end{aligned}$$

which leads to the relation

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(t)e^{j\omega_0 t} \overset{CTFT}{\longleftrightarrow} X(\omega-\omega_0).$$

We observe that a shift in the frequency of the continuous time Fourier transform by an amount ω_0 corresponds to a modulation in the time domain signal $x(t)$ by a term whose frequency is proportional to the shift amount.

2.2.1.4 Time Reversal

Analogous to the result for the CTFS, we have from the CTFT synthesis equation,

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega,$$

we see that by simply changing the sign of the time variable t, we obtain the general relation

$$\begin{aligned} y(t) &= x(-t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j(-\omega)t}d\omega \\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(-\omega)e^{j\omega t}d\omega, \end{aligned}$$

yielding the relation

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(-t) \overset{CTFT}{\longleftrightarrow} X(-\omega),$$

i.e., changing the sign of the time axis corresponds to changing the sign of the CTFT frequency index.

2.2.1.5 Time Scaling

When signal undergoes a time-scale change, such as one that compresses the time axes, $y(t) = x(at)$, where $a > 1$ is a real-valued constant, the resulting signal $y(t)$ is given by

$$\begin{aligned} y(t) &= \int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega at}d\omega \\ &= \int_{\nu=-\infty}^{\infty} \frac{1}{|a|}X(\nu/a)e^{j\nu t}d\nu, \end{aligned}$$

where the second line follows from the substitution $\nu = a\omega$. This yields the following relation for $y(t) = x(at)$,

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies y(t) = x(at) \overset{CTFT}{\longleftrightarrow} \frac{1}{|a|}X(\omega/a).$$

2.2.1.6 Conjugate Symmetry

The effect of conjugating a complex-valued signal on its CTFT representation can be seen by simply conjugating the CTFT synthesis relation,

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ x^*(t) &= \left(\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right)^* \\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)e^{j(-\omega)t}d\omega \\ &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(-\omega)e^{j\omega t}d\omega \end{aligned}$$

yielding that

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies x^*(t) \overset{CTFT}{\longleftrightarrow} X^*(-\omega).$$

When the signal $x(t)$ is real valued, then the CTFT exhibits a symmetry property. This arises directly from the definintion of the CTFT, and that real numbers equal their conjugates, i.e. $x(t) = x^*(t)$, such that

$$x(t) = x^*(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \implies X(\omega) = X^*(-\omega).$$

Note that when the signal is real-valued and is an even function of time, such that $x(t) = x(-t)$, then its CTFT is also real-valued and even, i.e. $X(\omega) = X^*(\omega) = X(-\omega)$. It can be shown by similar reasoning that when the signal real-valued, and an odd function of time, that the CTFT is purely imaginary and odd, i.e. $X(\omega) = -X^*(\omega) = -X(-\omega)$.

2.2.1.7 Products of Signals

When signals are multiplied in time, such that $z(t) = x(t)y(t)$, the resulting signal has a CTFS representation that can be obtained through the analysis equation,

$$\begin{aligned} Z(\omega) &= \int_{t=-\infty}^{\infty} (x(t)y(t))e^{-j\omega t}dt \\ &= \int_{t=-\infty}^{\infty} (y(t)\left(\frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)e^{j\nu t}d\nu\right))e^{-j\omega t}dt \\ &= \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)\left(\int_{t=-\infty}^{\infty} y(t)e^{-j(\omega-\nu)t}dt\right)d\nu \\ &= \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)Y(\omega-\nu)d\nu. \end{aligned}$$

The relationship between the CTFT representation for $z(t)$ and those of $x(t)$ and $y(t)$ is seen to be a convolution between the two CTFTs $X(\omega)$ and $Y(\omega)$,

$$x(t) \overset{CTFT}{\longleftrightarrow} X(\omega), y(t) \overset{CTFT}{\longleftrightarrow} Y(\omega) \implies z(t) = x(t)y(t) \overset{CTFT}{\longleftrightarrow} \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)Y(\omega-\nu)d\nu.$$

2.2.1.8 Convolution

A dual relationship to that of multiplication in time, is multiplication of CTFT representations. Specifically, the signal whose CTFT representation is given by $Z(\omega) = X(\omega)Y(\omega)$ corresponds to a convolution of the signals $x(t)$ and $y(t)$. This can be seen as follows,

$$\begin{aligned}
z(t) &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left(X(\omega)Y(\omega)\right)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left(\int_{\tau=-\infty}^{\infty}x(\tau)e^{-j\omega\tau}d\tau\right)Y(\omega)e^{j\omega t}d\omega \\
&= \int_{\tau=-\infty}^{\infty}x(\tau)\left(\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}Y(\omega)e^{j\omega(t-\tau)}\right)d\tau \\
&= \int_{\tau=-\infty}^{\infty}x(\tau)y(t-\tau)d\tau
\end{aligned}$$

where the integral relationship in the last line is recognized as a convolultion. This leads to the following property of the CTFT,

$$x(t)\overset{CTFT}{\longleftrightarrow}X(\omega), y(t)\overset{CTFT}{\longleftrightarrow}Y(\omega)\implies z(t)=\int_{\tau=-\infty}^{\infty}x(\tau)y(t-\tau)d\tau\overset{CTFT}{\longleftrightarrow}Z(\omega)=X(\omega)Y(\omega).$$

2.2.1.9 Integration

When the signal $y(t)$ and $x(t)$ are related through a running integral, i.e. $y(t)=\int_{\tau=-\infty}^{t}x(\tau)d\tau$, we can relate their CTFTs as follows,

$$x(t)\overset{CTFT}{\longleftrightarrow}X(\omega)\implies y(t)=\int_{\tau=-\infty}^{t}x(\tau)d\tau\overset{CTFT}{\longleftrightarrow}\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega),$$

where the relation is easiest shown using the differentiation property derived next together with the following observation. When $\omega=0$, $Y(\omega)$ is unbounded if $X(0)$ is nonzero.

2.2.1.10 Differentiation

Similarly, we can consider the relationship between $y(t)=\frac{d}{dt}x(t)$ and their corresponding CTFT representations. From the definition of the CTFT, we have

$$\begin{aligned}
y(t) &= \frac{d}{dt}x(t) \\
&= \frac{d}{dt}\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}X(\omega)e^{j\omega t}dt \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}X(\omega)\frac{d}{dt}e^{j\omega t}dt \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left(j\omega X(\omega)\right)e^{j\omega t}dt
\end{aligned}$$

from which we obtain the relation

$$x(t)\overset{CTFT}{\longleftrightarrow}X(\omega)\implies y(t)=\frac{d}{dt}x(t)\overset{CTFT}{\longleftrightarrow}j\omega X(\omega).$$

2.2.1.11 Parseval's relation

The energy containted in a finite-energy signal (note that the CTFT exists in the case of finite energy signals, i.e. signals that can be square integrated) can also be computed in terms of its CTFT representation using Parseval's relation,

$$x(t)\overset{CTFT}{\longleftrightarrow}X(\omega)\implies\int_{t=-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}|X(\omega)|^2d\omega.$$

Section	CTFT Property	Continuous Time Signal	Continuous Time Fourier Transform
	Definition	$x(t)$	$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$
2.2.1.1	Linearity	$z(t) = ax(t) + by(t)$	$Z(\omega) = aX(\omega) + bY(\omega)$
2.2.1.2	Time Shift	$y(t) = x(t-T)$	$Y(\omega) = X(\omega)e^{-j\omega T}$
2.2.1.3	Modulation	$y(t) = x(t)e^{j\omega_0 t}$	$Y(\omega) = X(\omega - \omega_0)$
2.2.1.4	Time Reversal	$y(t) = x(-t)$	$Y(\omega) = X(-\omega)$
2.2.1.5	Time Scaling	$y(t) = x(at)$	$Y(\omega) = \frac{1}{\lvert a\rvert}X(\omega/a)$
2.2.1.6	Conjugate Symmetry	$x(t) = x^*(t)$	$X(\omega) = X^*(-\omega)$
2.2.1.7	Products of Signals	$z(t) = x(t)y(t)$	$Z(\omega) = \frac{1}{2\pi}\int_{\nu=-\infty}^{\infty} X(\nu)Y(\omega-\nu)d\nu$
2.2.1.8	Convolution	$z(t) = \int_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$	$Z(\omega) = X(\omega)Y(\omega)$
2.2.1.9	Integration	$y(t) = \int_{\tau=-\infty}^{t} x(\tau)d\tau$	$Y(\omega) = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$
2.2.1.10	Differentiation	$y(t) = \frac{d}{dt}x(t)$	$Y(\omega) = j\omega X(\omega)$
2.2.1.11	Parseval's Relation	$x(t)$	$\int_{t=-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega.$
	Other properties?	tx(t), even part, odd part	
		conjsym part, conjasym part	

Table 2.1: Properties of the Continuous Time Fourier Transform

This relation can be derived using the definition of the CTFS as follows.

$$\begin{aligned}
\int_{t=-\infty}^{\infty} |x(t)|^2 dt &= \int_{t=-\infty}^{\infty} x(t)x^*(t)dt \\
&= \int_{t=-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega\right)dt \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)\left(\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)\left(X(\omega)\right)d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X^*(\omega)X(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega.
\end{aligned}$$

Parseval's relation shows that the energy measured in the time-domain of a finite-energy signal is equal to the energy measured in the frequency domain through its CTFT representation.

2.2.2 CTFT Examples

Derivations of some of the signals in the Table 2.2.

2.3 Discrete-Fourier Series representation of DT periodic signals

In Section 2.1 we discussed the Fourier series representation as a means of building a large class of continuous time periodic signals from a set of simpler, harmonically related complex exponential signals. In this section, we consider the analogous notion of building a large class of periodic signals in discrete time from a set of simpler, harminically related complex exponential discrete time signals. An important difference between the continuous time Fourier series and what we will develop in this section as the discrete time Fourier series (DTFS), is that while the series used to construct periodic signals in continuous time is infinite, the series used to construct discrete time periodic signals is in fact a finite sum. This difference simplifies a number

Continuous Time Signal	Continuous Time Fourier Transform
$x(t)$	$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$e^{-at}u(t),\ Real\{a\} > 0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t),\ Real\{a\} > 0$	$\frac{1}{(j\omega+a)^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
1	$2\pi\delta(\omega)$
$\delta(t - T_0)$	$e^{-j\omega T_0}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin(\omega_0 t)$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$\frac{W}{\pi}\text{sinc}(\frac{Wt}{\pi}) = \begin{cases} \frac{sin(Wt)}{\pi t} & t \neq 0 \\ \frac{W}{\pi} & t = 0 \end{cases}$	$\begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$\begin{cases} 1, & \lvert t\rvert < T \\ 0, & \lvert t\rvert > T \end{cases}$	$2T\text{sinc}(\frac{\omega T}{\pi}) = \begin{cases} \frac{2sin(\omega T)}{\omega} & \omega \neq 0 \\ 2T & \omega = 0 \end{cases}$
more	more
more	more
more	more
more	more
more	more

Table 2.2: Continuous Time Fourier Transform Pairs

of issues that were delicate in the continuous case, such as notions of convergence, and existence of certain limits.

Mathematically, we represent a periodic discrete time signal, $x[n]$, as a signal whose value repeats at a fixed number of samples from the present. This interval, denoted N below, is called the "period" of the signal, and we express this relationship

$$x[n] = x[n + N], \text{ for all } n. \tag{2.20}$$

Equation (2.20) will, in general, be satisfied for a countably infinite number of possible values of N. The smallest, positive value of N for which Eq. (2.20) is satisfied, is called the "fundamental period" of the signal $x[n]$. Discrete time sinusoidal signals, such as

$$x[n] = \sin(\omega_0 n + \phi), \tag{2.21}$$

often enable us to relate the frequency of oscillation, ω_0 to a fundamental period, N. While analogous to their continuous time cousins, discrete time sinusoids need not always be periodic. While this may require a more careful notion of what is meant by discrete time "frequency," we will place this issue aside for the moment and consider how the period of a periodic sinusoid relates to the arguments of the sinusoidal function. This can again be computed by noting that sinusoidal functions are equal when their arguments are either equal or differ only through a multiple of 2π, i.e.

$$\begin{aligned} x[n] &= x[n + N] \\ \sin(\omega_0 n + \phi) &= \sin(\omega_0(n + N) + \phi) \\ \sin(\omega_0 n + \phi + 2k\pi) &= \sin(\omega_0(n + N) + \phi) \\ \sin(\omega_0(n + 2k\pi/\omega_0) + \phi) &= \sin(\omega_0(n + N) + \phi) \end{aligned} \tag{2.22}$$

which yields the relationship

$$N = 2\pi k/\omega_0. \tag{2.23}$$

Depending on the value of ω_0, (2.23) may not provide an integer solution for N for any value of k. Note that only if ω_0/π is rational, will there be an integral solution to (2.23), for which the smallest integer value of N is the fundamental period associated with the discrete time frequency ω_0. In Figure (2.4), the two