Complex linear algebra for undergraduate engineers

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1 Engineering math for 3 year electrical engineers

Mathematics is a key subject to be mastered by engineers, and this is particularly true for Electrical engineers. By the third year of their undergraduate program they are learning how to analyze electrical circuits, sharpening their programming skills, and learning signal processing. To perform these tasks they need to know complex analysis and the simple applications of linear algebra. We know from many years of experience these are topics not taught in almost all high schools, yet engineering students have a need for these basic skills.

Around 2018 it was decided that the University of IL Electrical Engineering curriculum include a half semester course in *complex linear algebra*. It is assumed that the students will take this course in the second half of the semester when they are in the process of learning Laplace transforms, electrical circuit theory and signal processing.

The instructional materials were taken from my book *An introduction to mathematical physics and its history*, which has extensive material on both linear algebra and complex analysis. The course text consisted of about one third of the full text. Books were printed on demand for $10.

The course ran from March 11 to May 1, for a total of 20 50min lectures. Lectures 1-5 were on 2x2 matrix methods, including Pythagorean triples, Pell's equation, Fibonacci's equation, 2x2 matrix eigen-analysis and transmission line matrix theory. The details are at:
`https://auditorymodels.org/index.php?n=Courses.ECE298-ComplexLinearAlg-S19`.
The class notes ($\approx$170 pages) may be downloaded from:
`https://jontalle.web.engr.illinois.edu/uploads///298.S19/ClassNotes-298.pdf`

There were 8 homework sets, a midterm and a final exam. The students were expected to use simple Matlab/Octave exercises, which included plotting simple functions and evaluating mathematical expressions. Commands that were especially important were required for manipulating prime numbers, forming Taylor series expression of simple functions such as $\sin(s)$, e^s, $J_0(s)$, $\Gamma(s)$, and to find Laplace and inverse Laplace transforms using symbolic analysis.

A 130 page book was provided, with 100 pages of exercises, which was available in hard-copy for $10. Initially 40 students signed up for the class, but 24 took the final exam. The feedback from the students was positive, and it was noted that everyone came to class for every lecture, and if they were not available, they provided an excuse as to why they could not attend, without my asking.

1.1 Midterm exam: Complex 2x2 linear algebra

The course began with an introduction to simple number theory concepts:

1. **Greatest common divisor:** (aka: GCD; coprimes; Euclidian algorithm),

Coprimes (aka GCD): *Coprimes* $m \perp n$ are integers with no common factors

Examples:

(a) *Primes* $11, 13$ *have no common factors, thus:* $11 \perp 13$, *and* $gcd(11, 13) = 1$

(b) $\gcd(13 \cdot 5, 11 \cdot 5) = 5$, *common factor 5.*

(c) $\gcd(13 \cdot 10, 11 \cdot 10)$ *= 10* ($\gcd(130, 110) = 10 = 2 \cdot 5$, *is not prime)*

Important property of the GCD:

(a) $k = \gcd(m, n)$*: Then* k *cancels in the fraction* m/n.

The Euclidean algorithm for finding the GCD of two numbers is one of the oldest algorithms in mathematics, and used by the Chinese during the Han dynasty for simplifying fractions.

2. **Pythagorean triplets:** (Euclid's formula)

Given $p, q \in \mathbb{N}$ with $p > q$, the three lengths $[a, b, c] \in \mathbb{N}$, then $c^2 = a^2 + b^2$

$$a = p^2 - q^2, \qquad b = 2pq, \qquad c = p^2 + q^2.$$

Example: $p = 2, q = 1$, then that $5^2 = 4^2 + 3^2$ or $25 = 16 + 9$.

For any integers $p > q$ this will give a right triangle with integer sides $\{a, b, c\}$. Try it.

3. **Continued fraction algorithm:**

Example: Let $x_o \equiv \pi \approx 3.14159\ldots = 1 + 0.14159 = 3 + \frac{1}{7 + \cancel{0.0625}}$.
The integer part of π is 3. The remainder $r_o = 0.14159 \approx 1/7.065 \approx 1/7$.
If we stop here we have

$$\widehat{\pi}_2 = 3 + \frac{1}{7 + \cancel{0.0625}\ldots} = 3 + \frac{1}{7} = \frac{22}{7}.$$

This approximation of $\widehat{\pi}_2 = 22/7$ has a relative error of 0.04%

$$\frac{22/7 - \pi}{\pi} \approx 4 \times 10^{-4}.$$

4. **Pell's equation:** Find integer solutions ($x_n, y_n, N \in \mathbb{N}$) to

$$x_n^2 - N y_n^2 = 1$$

A 2×2 matrix recursion solution was used by the Pythagoreans:

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \jmath \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n$$

Solution examples ($N = 2$): $[x, y]_n = [0, 1]_0, \jmath[1, 1]_1, -[3, 2]_2, \cdots, \jmath[41, 29]_5, \cdots$.

5. **Fibonacci's equation:** Another classic problem is the Fibonacci sequence formula

$$f_{n+1} = f_n + f_{n-1}.$$

The next number $f_{n+1} \in \mathbb{N}$ is the sum of the previous two.

Solution example: $f_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots]$. Check that the third number (1) is the sum of the previous two (0+1), and the fourth number (3) is the sum of the previous two (1+2), and 34=21+13.

6. **Eigenvalue analysis** of 2x2 matrices.

7. **History**

Question: Can the GCD, CFA, Pythagorean triplets and the Fibonacci equation be unified?

yes! Use *eigen analysis*.

The GCD, CFA, PT and Fibonacci equations were used to teach the application of 2x2 matrix analysis, as each of these equations has a general eigenvalue solution, and each of these is related to engineering problems. For example, the Fibonacci difference equation provides an introduction to digital signal processing. Each has a matrix formulation.

Based on the midterm exam grades, the students easily absorbed these topics, with enthusiasm. They easily made the connection to circuit theory (complex linear algebra) and digital signal processing (difference equations having complex roots). The key idea was the connection between differential equations, difference equations and the eigenvalue analysis.

By example, the eigen equation for the Fibonacci equation $f_{n+1} = f_n + f_{n-1}$ is

$$Te_\pm = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}_\pm = \lambda_\pm e_\pm$$

where $\pm$ label the two eigenvalues and eigenvectors. This equation may be solved for the two eigenvalues $\lambda_\pm$ and eigenvectors $e_\pm$. Once they are determined the system may be written in matrix form as

$$T\left[e_+, e_-\right] = \left[e_+, e_-\right] \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}. \quad \text{In matrix format:} \quad TE = E\Lambda.$$

which leads to the general form for powers of T^n in terms of powers of the eigenvalues

$$T^n = E^{-1}\Lambda^n E.$$

This relation gives a general simple series solution for any non-singular matrix T, in terms of powers of the eigenvalues. From this we can see that a solution is unstable (solution $\to \infty$) when $|\lambda| > 1$, and stable (solution $\to 0$) when $|\lambda| < 1$.

This very general and powerful result applies to all the cases (GCD, CFA, PT, and as discussed here, the Fibonacci's equation), once they are transformed to their matrix form. For example, for the case of differential equations, the eigen vectors, and thus the solutions, are $e^{\lambda_\pm t}$, where $\lambda_\pm$ are the complex eigenvalues.

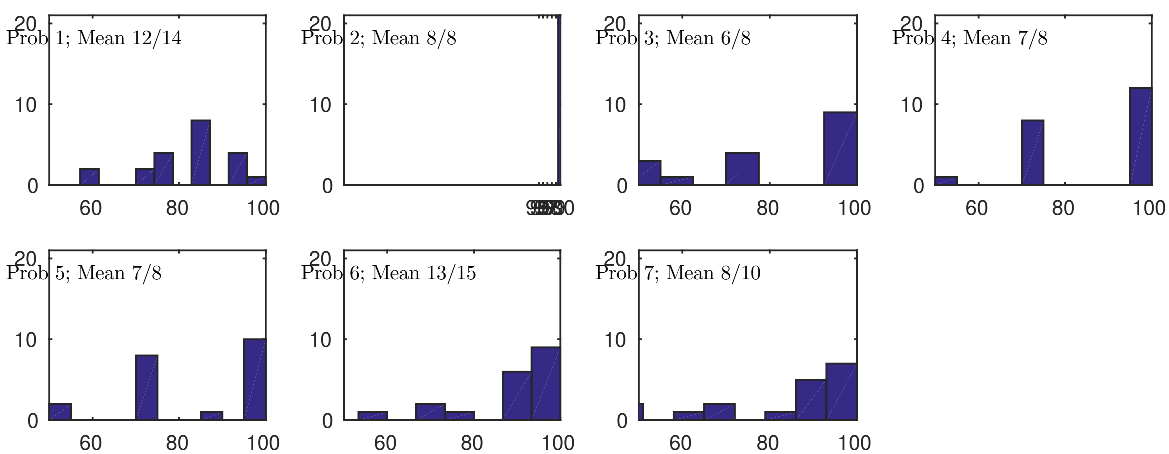

Figure 1: *Histograms of the seven problems for Exam I across the 24 students. Prob 1: Primes; P2: Pyth Trips; P3: CFA; P4: Pell; P5: Fib; P6: Eig-Diag; P7: History; With the exception of problem 2, the scores for each problem where widely distributed between 75-100%, but with a few students dropping between 50-60%. This is reflected in the final scores shown in Fig. 2 which were above 75% for 18 students. Seven had scores at or above 90%. For the histogram displays only, each score has each been normalized to 100%. For the final grade the points were simply added (not normalized).* EPS/ProbHist-Exm1X7-score

1.2 Midterm exam: 2x2 linear algebra questions

1.2.1 Summary of Student performance on Exam 1

ECE 298-CLA is being taught for the first time in Spring 2019. The hope is to give instructional relief to students taking ECE 210 and 310, in the area of complex analysis, especially Laplace transforms, and matrix computations of complex variables. The class is organized around functions of the complex variable $s = \sigma + \omega \jmath$, which is denoted the *Laplace frequency*. The class web page may be found at `https:\\auditorymodels.org`) (click on the link at the left `ECE298-CLA S2019`).

The analysis is shown in Fig. 1. There were eight problems on the midterm. More than half the class had scores above 80%, with 1-3 students scoring at and above 60%, giving bimodal distributions.

The student-t distributions do not apply to these data sets. The student-t is a unrealistic out of date idealization of actual student scores. If one masters the concept, a score near 100% will be achieved. The difficulty of the problems play a major role in the nature of the scores.

1. (14 pts) Primes and the *greatest common divisor* (GCD): The scores on this problem had a mode around 85% with a 10 point spread. The best score was 100%, and the lowest was 60%.

2. (8 pts) Pythagorean triplets: Perfect scores for everyone.

3. (8 pts) Continued fractions (CFA): 9 perfect scores and 3 75%.

4. (8 pts) 2x2 Matrix solution to Pell's equation: 11 perfect scores 7 75%.

5. (8 pts) 2x2 Matrix solution to Fibonacci equation: 10 perfect scores, 6 75%.

6. (15 pts) Eigenvalue decomposition (diagonalization) of a 2x2 matrix: 15 at or above 90%.

7. (10 pts) History (400 BCE–1720 CE): Generally high scores, skewed from 90% to 100%.

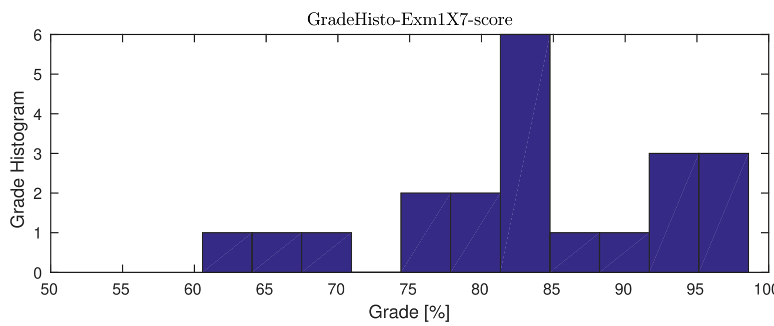

Figure 2: *Histograms of the grades for Exam I. The weights on each of the three components are HW (25%), Exam I (25%), Final Exam (50%). Seven bins divide the 24 students (3:1 overlap bin quantization).*

The first exam for 298-CLA was taken by 24 of the registered 27 students. The first student returned the exam almost exactly 1 hr into the exam. The last 3 hrs later. The students were given the solution to the exam once they handed in their exam. Since exact points were given for each of the eight problems, in theory it was possible for each student to compute their grade, and I requested that they do so.

The graded exams were returned one week later (April 12). On Monday, April 15, an initial version of this report was posted on the class website, with the distributions of scores on the exam. On April 19 the analysis was extended to include the distributions of each of the seven problems, shown in Fig. 1, and discussed below.

2 Final exam

Grade distribution across the 14 problems: The final exam had 14 problems:

1. Continued fraction algorithm (3/5 pts)

2. Fibonacci equation (6/6 pts)

3. Pell's equation (8/8 pts)

4. Matrix diagonalization (2x2 Eigen analysis) (12/14 pts)

5. Pythagorean triplets (3/3 pts)

6. Complex Power Series (5/15 pts)

7. Branch cuts and Riemann sheets (7/12 pts)

8. Transmission line analysis (10/20 pts)
9. Fundamental theorems of real and Complex Calculus (3/4 pts)
10. Cauchy-Riemann Equations (analytic functions) (3/4 pts)
11. Cauchy's theorems (6/7 pts)
12. Lowpass filter analysis (Laplace transforms) (3/5 pts)
13. Impedance (Laplace transforms) (8/12 pts)
14. History of mathematics (9/10 pts)

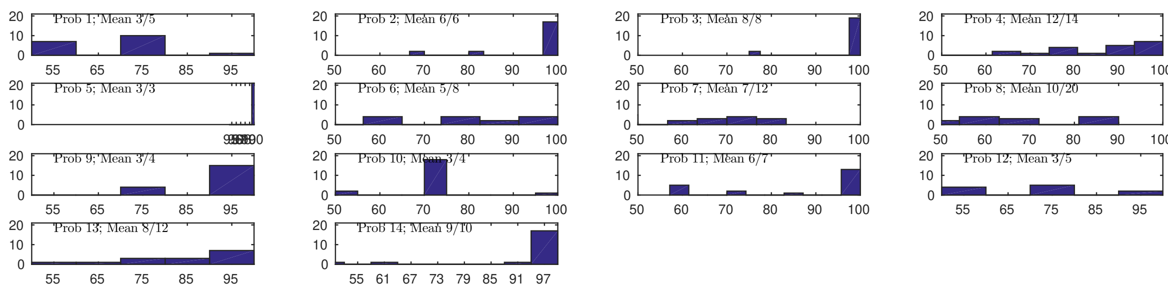

Figure 3: *Histograms for each of the the grade distributions of the 14 problems for the final. Many of the problems had almost perfect scores (problems: 2, 3, 5, 9, 11, 14) The most difficult problems were 1) (CFA) 6) Power series 7) Branch cuts 12) Lowpass filter with Laplace Transform.* EPS/ProbHist-Final-scores

2.1 Problem distributions

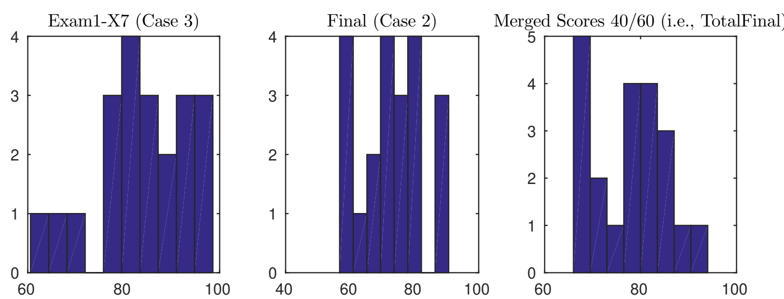

Figure 4: *Histograms of the Exam I scores with Problem 7 removed. Each grade is normalized to 100%. The final grad in the course will be based on three components: HW (25%), Exam I (25%), Final Exam (50%). These weights were changed because following HW-1, I modified the homework to include the solution to each problem, thus I increased the weight of the HW to 25% to reflect this change.* Fig(8): EPS/MergedTotalScore

The final letter grades: 2 A+; 11 A; 3 A-; 5 B+.