2.2 Problems AE-2

Topics of this homework:
Linear vs nonlinear systems of equations, Euclid’s formula, Gaussian elimination, matrix permutations, Ohm’s law, two-port networks,
Deliverables: Answers to problems

(2 pt) Gaussian elimination

Problem # 1 (: (2 pts) Gaussian elimination ).

– 1.1 (: (1 pts) Find the inverse of ).

\[ A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}. \]

Ans:

– 1.2 (: (1 pts) Verify that \[ A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \] ). Ans:

2.2.1 Two linear equations

Problem # 2 ( In this problem we transition from a general pair of equations ).

\[ f(x, y) = 0 \]
\[ g(x, y) = 0 \]

to the important case of two linear equations

\[ y = ax + b \]
\[ y = \alpha x + \beta. \]
Note that to help keep track of the variables, roman coefficients \((a, b)\) are used for the first equation and Greek \((\alpha, \beta)\) for the second.

− 2.1 ( : What does it mean, graphically, if these two linear equations have (1) a unique solution, (2) a nonunique solution, or (3) no solution? ). **Ans:**

− 2.2 ( : Assuming the two equations have a unique solution, find the solution for \(x\) and \(y\) ). **Ans:**

− 2.3 ( : When will this solution fail to exist (for what conditions on \(a, b, \alpha, \) and \(\beta\) )? ). **Ans:**

− 2.4 ( : Write the equations as a \(2 \times 2\) matrix equation of the form \(A\vec{x} = \vec{b}\), where \(\vec{x} = \{x, y\}^T\) ). **Ans:**

− 2.5 ( : State the inverse of the \(2 \times 2\) matrix, and solve the matrix equation for \(x\) and \(y\) ). **Ans:**
CHAPTER 2. ALGEBRAIC EQUATIONS

2.6 (Discuss (state) the properties of the determinant of the matrix (Δ) in terms of the slopes of the two equations (a and α).) \(\text{Ans:}\)

Problem # 3 (You are given the following pair of linear relationships between the input (source) variables \(V_1\) and \(I_1\) and the output (load) variables \(V_2\) and \(I_2\) of a transmission line:).

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
J & 1 \\
1 & -1
\end{bmatrix}\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}.
\]

3.1 (Let the output (the load) be \(V_2 = 1\) and \(I_2 = 2\) (i.e., \(V_2/I_2 = 1/2 \{\Omega\}\)). Find the input voltage and current, \(V_1\) and \(I_1\).) \(\text{Ans:}\)

3.2 (Let the input (source) be \(V_1 = 1\) and \(I_1 = 2\). Find the output voltage and current, \(V_2\) and \(I_2\).) \(\text{Ans:}\)

Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a Diophantine equation.

Problem # 4 (5 pts) A practical example of using a Diophantine equation:)

2.2.2 Vector algebra in \(\mathbb{R}^3\)

Problem # 5 (Scalar product \(A \cdot B\).)

5.1 (If \(A = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}\) and \(B = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}\), write out the definition of \(A \cdot B\).) \(\text{Ans:}\)
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--- 5.2 (: The dot product is often defined as $||A|| ||B|| \cos(\theta)$, where $||A|| = \sqrt{A \cdot A}$ and $\theta$ is the angle between $A$, $B$. If $||A|| = 1$, describe how the dot product relates to the vector $B$. ). Ans:

Problem # 6 (: Vector (cross) product $A \times B$ ).

--- 6.1 (: If $A = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$ and $B = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}$, write out the definition of $A \times B$. ). Ans:

--- 6.2 (: Show that the cross product is equal to the area of the parallelogram formed by $A$, $B$, namely $||A|| ||B|| \sin(\theta)$, where $||A|| = \sqrt{A \cdot A}$ and $\theta$ is the angle between $A$ and $B$. ). Ans:

Problem # 7 (: Triple product $A \cdot (B \times C)$ ). Let $A = [a_1, a_2, a_3]^T$, $B = [b_1, b_2, b_3]^T$, $C = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$.

--- 7.1 (: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $A \cdot (B \times C) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ ). Ans:

--- 7.2 (: Describe why $|A \cdot (B \times C)|$ is the volume of parallelepiped generated by $A$, $B$, and $C$. ). Ans:
7.3 ( Explain why three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are in one plane if and only if the triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$. ) Ans:

2.2.3 Ohm’s Law

As shown in the Table on page 28, the impedance concept also holds in mechanics, acoustics, and thermal circuits, and perhaps, even for non-relativistic gravity. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force is pressure and the flow is either the volume velocity or particle velocity of air molecules.

<table>
<thead>
<tr>
<th>Case</th>
<th>Force</th>
<th>Flow</th>
<th>Impedance</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>voltage (V)</td>
<td>current (I) / $\mathbf{D}$ / photon</td>
<td>$Z$</td>
<td>Ohms [Ω]</td>
</tr>
<tr>
<td>Mechanics</td>
<td>force (F)</td>
<td>velocity (V)</td>
<td>$Z$</td>
<td>Mechanical Ohms [Ω]</td>
</tr>
<tr>
<td>Acoustics</td>
<td>pressure (P)</td>
<td>particle velocity (U)</td>
<td>$Z_{A, \rho c}$</td>
<td>Acoustic Ohms [Ω]</td>
</tr>
<tr>
<td>Thermal</td>
<td>temperature (T)</td>
<td>entropy-rate ($\dot{S}$)</td>
<td>$Z_T$</td>
<td>Thermal Ohms [Ω]</td>
</tr>
<tr>
<td>Gravity</td>
<td>potential</td>
<td>momentum = $2\pi/\lambda_G$ / graviton</td>
<td>$Z_G$</td>
<td>gravitational Ohms [?]</td>
</tr>
</tbody>
</table>

Problem #8 ( The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As the bulb lights up, the resistance of the metal filament increases as the temperature $T$ rises.). Ohm’s law says that the current

$$\frac{V}{T} = R(T),$$

where $T$ is the temperature.

8.1 ( In the United States, the voltage is 120 volts (RMS) at 60 (Hz). Find the current when the light is first switched on. ) Ans:

Problem #9 ( 1 pts) The power in watts is the product of the force and the flow. What is the power of the lightbulb of Problem 8? ) Ans:
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Problem # 10 (: 1 pts) State the impedance $Z(s)$ of each of the following circuit elements: (1) a resistor with resistance $R$, (2) an inductor with inductance $L$, and (3) a capacitor with capacitance $C$.

Ans:

2.2.4 Nonlinear (quadratic) to linear equations

Problem # 11 (: Solve the equations:).

– 11.1 (: Solve for $x(p, q)$ (remove $y$):).

\[
\begin{align*}
    x + y &= p \\
    xy &= q.
\end{align*}
\]

Ans: