2.2 Problems AE-2

Topics of this homework:

Linear vs nonlinear systems of equations, Euclid's formula, Gaussian elimination, matrix permutations, Ohm's law, two-port networks,

Deliverables: Answers to problems

(2 pt) Gaussian elimination

Problem # *1: (2 pts) Gaussian elimination*

– 1.1: (1 pts) Find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Sol:

$$A^{-1} = \frac{1}{3-8}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}.$$

■

– 1.2: (1 pts) Verify that $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Sol: Multiply them to show this. ■

2.2.1 Two linear equations

Problem # *2 In this problem we transition from a general pair of equations*

$$f(x,y) = 0$$
$$g(x,y) = 0$$

to the important case of two linear equations

$$y = ax + b$$
$$y = \alpha x + \beta.$$

Note that to help keep track of the variables, roman coefficients (a, b) *are used for the first equation and Greek* (α, β) *for the second.*

– 2.1: What does it mean, graphically, if these two linear equations have (1) a unique solution, (2) a nonunique solution, or (3) no solution?

Sol: There are three possibilities:

1. When they have different slopes, they meet at one (x,y) point, which is the solution.
2. If the two lines are identical, any point on the line is a solution.
3. If they have the same slope but different intercepts (are parallel to each other) there is no solution.

■

– 2.2: Assuming the two equations have a unique solution, find the solution for x and y.
Sol: Since there must be one point where the two are equal, we may solve for that by setting the y values equal to each other:

$$ax + b = \alpha x + \beta$$

Thus

$$x = \frac{\beta - b}{a - \alpha}$$
$$y = a\frac{\beta - b}{a - \alpha} + b$$

■

– 2.3: When will this solution fail to exist (for what conditions on a, b, α, and β)?
Sol: As stated above, if they have the same slope $\alpha = a$ but different intercepts $\beta \neq b$, there is no solution. When $\beta = b$ and $\alpha = a$ every point on the line is a solution. ■

– 2.4: Write the equations as a $2{\times}2$ matrix equation of the form $A\vec{x} = \vec{b}$, where $\vec{x} = \{x, y\}^T$.

Sol:

$$\begin{bmatrix} 1 & -a \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ \beta \end{bmatrix}$$

■

– 2.5: State the inverse of the 2×2 matrix, and solve the matrix equation for x and y.
Sol:

$$\begin{bmatrix} y \\ x \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -\alpha & a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b \\ \beta \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -\alpha b + a\beta \\ -b + \beta \end{bmatrix}$$

where the determinant is $\Delta \equiv a - \alpha$. ■

– 2.6: Discuss (state) the properties of the determinant of the matrix (Δ) in terms of the slopes of the two equations (a and α).
Sol: When the slopes are the same there is no solution and $\Delta = 0$. Thus the matrix solution is consistent with the geometry. This is our first result in analytic geometry. ■

Problem # *3: You are given the following pair of linear relationships between the input (source) variables V_1 and I_1 and the output (load) variables V_2 and I_2 of a transmission line:*

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

– 3.1: Let the output (the load) be $V_2 = 1$ and $I_2 = 2$ (i.e., V_2/I_2 =1/2 $\{\Omega\}$). Find the input voltage and current, V_1 and I_1.

Sol: This case corresponds to

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\jmath + 2 \\ 1 - 2 \end{bmatrix}$$

Thus $V_1 = 2 + 1\jmath$ and $I_1 = -1$. ■

– 3.2: Let the input (source) be $V_1 = 1$ and $I_1 = 2$. Find the output voltage and current, V_2 and I_2.
Sol: With the input specified the two equations are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

Euclid's formula derivation

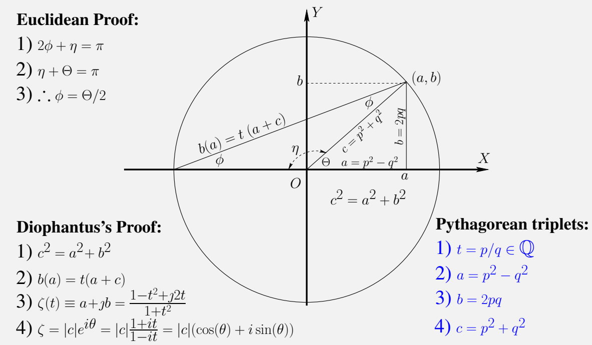
Euclid's formula for Pythagorean triplets (Eq. NS-2 2.6, p. 42) can be derived by intersecting a circle and a secant line. Consider the nonlinear equation of a unit circle having radius 1, centered at $(x, y) = (0, 0)$,

$$x^2 + y^2 = 1,$$

and the secant line through $(-1, 0)$,

$$y = t(x + 1),$$

a linear equation having slope t and intercept $x = -1$. If the slope $0 < t < 1$, the line intersects the circle at a second point (a, b) in the positive x, y quadrant. The goal is to find $a, b \in \mathbb{N}$ and then show that $c^2 = a^2 + b^2$. Since the construction gives a right triangle with short sides $a, b \in \mathbb{N}$, then it follows that $c \in \mathbb{N}$.



Derivation of Euclid's formula for the Pythagorean triplets (PT) $[a, b, c]$, based on a composition of a line, having a rational slope $t = p/q \in \mathbb{F}$, and a circle $c^2 = a^2 + b^2$, $[a, b, c] \in \mathbb{N}$. This analysis is attributed to Diophantus (Di·o·phan′·tus) (250 CE), and today such equations are called Diophantine (Di·o·phan′·tine) equations. *PTs have applications in architecture and scheduling, and many other practical problems. Most interesting is their relation to Rydberg's formula for the eigenstates of the hydrogen atom (Appendix H).*

Problem # *12: Derive Euclid's formula*

– 12.1: Draw the circle and the line, given a positive slope $0 < t < 1$.
Sol: Sol in given in Fig. 2.2.4 ■

Problem # *13: Substitute $y = t(x + 1)$ (the line equation) into the equation for the circle, and solve for $x(t)$.*
Hint: Because the line intersects the circle at two points, you will get two solutions for x. One of these solutions is the trivial solution $x = -1$. **Sol:** $x(t) = (1 - t^2)/(1 + t^2)$ ■

– 13.1: Substitute the $x(t)$ you found back into the line equation, and solve for $y(t)$.
Sol: $y(t) = 2t/(1 + t^2)$ ■

– 13.2: Let $t = q/p$ be a rational number, where p and q are integers. Find $x(p, q)$ and $y(p, q)$.
Sol: $x(p, q) = 2pq/(p^2 + q^2)$ and $y(p, q) = (p^2 - q^2)/(p^2 + q^2)$ ■

– 13.3: Substitute $x(p, q)$ and $y(p, q)$ into the equation for the circle, and show how Euclid's formula for the Pythagorean triples is generated.
Sol: Multiplying out gives $(p^2 + q^2) = (p^2 - q^2) + 2pq$ ■

For full points you must show that you understand the argument. Explain the meaning of the comment "magic happens" when t^4 cancels.

Problem # *8: The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As the bulb lights up, the resistance of the metal filament increases as the temperature T rises.*

Ohm's law says that the current

$$\frac{V}{I} = R(T),$$

where T is the temperature.

– 8.1: In the United States, the voltage is 120 volts (RMS) at 60 (Hz). Find the current when the light is first switched on.

Sol: Thus the current is

$$I = 120/R = 120/100 = 1.2. \qquad \text{[Amps]}$$

As the bulb heats up, the current rapidly drops, and the resistance increases. This typically takes less than a milliseconds [ms], which depends on the wattage of the light bulb. Such light-bulbs are *nonlinear.* These rules don't apply to LED bulbs. ■

Problem # *9: (1 pts) The power in watts is the product of the force and the flow. What is the power of the lightbulb of Problem 8?*

Sol: $P = V \cdot I = 120 \times 1.2 = 120 + 24 = 144$ [W]. ■

Problem # *10: (1 pts) State the impedance $Z(s)$ of each of the following circuit elements: (1) a resistor with resistance R, (2) an inductor with inductance L, and (3) a capacitor with capacitance C.*

Sol: (1) For the resistor, $Z = R$.

(2) For the inductor, $Z = sL$ with $s = \sigma + \omega \jmath$. Note the flux $\psi(t) = Li(t)$. The voltage $v(t)$ is the time derivative of the flux

$$v(t) = \frac{d\psi(t)}{dt} = L\frac{di(t)}{dt}.$$

(3) For the capacitor, $Z = 1/sC$. Note the charge $q(t) = Cv(t)$, thus the current $i(t)$ is the time derivative of the charge

$$i(t) = \frac{d}{dt}q(t) = C\frac{dv(t)}{dt}.$$

■

2.2.4 Nonlinear (quadratic) to linear equations

Problem # *11: Solve the equations:*

– 11.1: Solve for $x(p,q)$ (remove y):

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Sol: Solve the first equation for y as $y = p - x$, and then substitute it into the second equation, resulting in

$$x(p - x) = -x^2 + px = q.$$

Collecting terms gives the quadratic equation in x

$$x^2 - px + q = 0.$$

Completing the square

$$(x - p/2)^2 = (p/2)^2 - q$$

providing the roots $x_\pm = p/2 \pm \sqrt{(p/2)^2 - q}$, and the corresponding values for $y_\pm(x_\pm)$ $y_\pm = p - x\pm$.

Summary: Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots. For certain values of $\{p, q\}$ (a negative discriminant) the roots are complex. ■

To find the input we must invert the matrix ($\Delta = -j - 1$)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \frac{1}{1+j}\begin{bmatrix} 1 & 1 \\ 1 & -\jmath \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Thus $V_2 = 3/(1+\jmath) = 3(1-\jmath)/2$, $I_2 = (1-2\jmath)/(1+1\jmath) = -(1+3\jmath)/2$. The point of this exercise is that the two lines have a complex intersection point, not easily visualized. ■

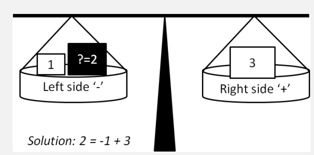
Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

Problem # *4: (5 pts) A practical example of using a Diophantine equation:*

"A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?" - *Bachet de Bèziriac (1623)*[a].

Here, weighing is performed using a balance scale that has two pans, with weights on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then the scale will be balanced. A solution to the four weights for Bachet's problem is $1 + 3 + 9 + 27 = 40$ pounds.



[a] Taken from **?**, p. 50

– 4.1: Show how the combination of 1-, 3-, 9-, and 27-pound weights can be used to weigh $1, 2, 3, \dots, 8, 28$, and 40 pounds of milk (or something else, such as flour). Assuming that the milk is in the left *pan, provide the position of the weights using a negative sign $-$ to indicate the left pan and a positive sign $+$ to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan, $+1$, will balance the scales.*

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should contain only the values -1 (left pan), $+1$ (right pan), and 0 (leave out). You can indicate these using $-$, $+$, and blanks.

Sol: Any integer between 1 and 40 may be expanded using the weights 1, 3, 9, 27. Here is the problem stated in matrix form:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \dots \\ 28 \\ \dots \\ 40 \end{bmatrix} = \begin{bmatrix} + & & & \\ - & + & & \\ & + & & \\ + & + & & \\ - & - & + & \\ & - & + & \\ + & - & + & \\ - & & + & \\ \dots & & & \\ + & & & + \\ \dots & & & \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix}$$

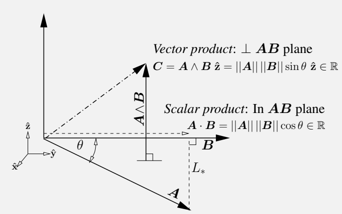
The left column is the weight of the milk. The right-most column are the four weights. It should be clear that these four weights span the integers from 1-40 with binary weights. Each weight may be computed recursively from twice the sum of the previous weights +1, that is

$$W_{n+1} = 2W_n + 1 = 2^{n+1} \quad \text{since} \quad W_n = 2^n.$$

For example to get 26 we place weights 9+3+1 in the pan with 26, and get 27-1. For example 27 = 2*(9+3+1)+1 is the next weight. Recursively, the weights are 3=2*1+1, 9=2*(3+1)+1, 27=2*(9+3+1)+1. The next weight (not shown) would be: 81=2*(27+9+3+1)+1 = 2*40+1. ■

2.2.2 Vector algebra in $\mathbb{R}^3$

Definitions of the vector scalar $\boldsymbol{A} \cdot \boldsymbol{B}$ and wedge $\boldsymbol{A} \wedge \boldsymbol{B}$ products.



Problem # 5: Scalar product A · B

– 5.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$, write out the definition of $\mathrm{A} \cdot \mathrm{B}$.

Sol: See the definition in the above figure. $\boldsymbol{A} \cdot \boldsymbol{B} = a_x b_x + a_y b_y + a_z b_z$. In general: $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_k A_k B_k$. ■

– 5.2: The dot product is often defined as $||\boldsymbol{A}||\,||\boldsymbol{B}||\cos(\theta)$, where $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A}, \boldsymbol{B}$. If $||\boldsymbol{A}|| = 1$, describe how the dot product relates to the vector B.

Sol: See the definition in the above figure. The vector product is the portion of $\mathbf{B}$ in the direction of $\mathbf{A}$. ■

Problem # 6: Vector (cross) product A × B

– 6.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$, write out the definition of $\mathrm{A} \times \mathrm{B}$.

Sol:

$$\boldsymbol{A} \times \boldsymbol{B} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}.$$

■

– 6.2: Show that the cross product is equal to the area of the parallelogram formed by $\boldsymbol{A}, \boldsymbol{B}$, namely $||\boldsymbol{A}||\,||\boldsymbol{B}||\sin(\theta)$, where $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

Sol: A parallelogram's area is equal to its base times its height. Therefore, let's say the base is length $||A||$, and the height $||B||\sin(\theta)$, which is the portion of $\mathbf{B}$ that is perpendicular to $\mathbf{A}$. ■

Problem # 7: *Triple product $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$*

Let $\boldsymbol{A} = [a_1, a_2, a_3]^T$, $\boldsymbol{B} = [b_1, b_2, b_3]^T$, $\boldsymbol{C} = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$.

– 7.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Sol: Using the determinate-definition of the cross product,

$$\boldsymbol{B} \times \boldsymbol{C} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}.$$

Let $\boldsymbol{D} = \boldsymbol{B} \times \boldsymbol{C}$ and compute $\boldsymbol{A} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. Finally compute the requested right-hand side, and compare the two. It should be clear that they are the same, because the dot product transfers the elements of vector $\boldsymbol{A}$ to cross product and reduces the product to the scalar. ■

– 7.2: Describe why $|\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})|$ is the volume of parallelepiped generated by $\boldsymbol{A}$,$\boldsymbol{B}$, and $\boldsymbol{C}$.

Sol: Note that the norm of $\boldsymbol{B} \times \boldsymbol{C}$ is the area of the parallelogram generated by $\boldsymbol{C}$ and $\boldsymbol{B}$. Taking the dot product with $\mathbf{A}$ results in the volume of the corresponding parallelepiped (prism). So the absolute value of triple product is volume of parallelepiped. ■

– 7.3: Explain why three vectors A, B, C are in one plane if and only if the triple product $\mathrm{A} \cdot (\mathrm{B} \times \mathrm{C}) = 0$.

Sol: (triple product is zero) if and only if: (volume is zero), if and only if: (they are in the same plane) ■

2.2.3 Ohm's Law

As shown in the Table on page 94, the impedance concept also holds in mechanics, acoustics, and thermal circuits, and perhaps, even for non-relativistic gravity. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force is pressure and the flow is either the volume velocity or particle velocity of air molecules.

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I) / $\dot{\boldsymbol{D}}$ / photon	Z	Ohms [Ω]
Mechanics	force (F)	velocity (V)	Z	Mechanical Ohms [Ω]
Acoustics	pressure (P)	particle velocity (U)	Z_A, ρc	Acoustic Ohms [Ω]
Thermal	temperature (T)	entropy-rate $(\dot{S})$	Z_T	Thermal Ohms [Ω]
Gravity	potential	momentum = $2\pi/\lambda_G$ / graviton	Z_G	gravitational Ohms [?]