Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem #1: In each case derive (e.g., using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and give the RoC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).

- 1.1: \( 1/(1 - s) \)

**Sol:** \( 1/(1 - s) = \sum_{n=0}^{\infty} s^n \), which converges for \( |s| < 1 \) (e.g., the RoC is \( |s| < 1 \)). ■

- 1.2: \( 1/(1 - s^2) \)

**Sol:** \( 1/(1 - s^2) = \sum_{n=0}^{\infty} s^{2n} \), which converges for \( |s^2| < 1 \) (e.g., the RoC is \( |s| < 1 \)). One can also factor the polynomial, thus write it as: \( 1/(1-s)(1+s) \). There are two poles, at \( s = \pm 1 \), and each has an RoC of 1. ■

- 1.3: \( 1/(1 + s^2) \).

**Sol:** The resulting series is \( 1/(1 + s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n) \). The RoC is \( |s| < 1 \). We can see this by considering the poles of the function at \( s = \pm i \); both poles are 1 from \( s = 0 \), the point of expansion. An alternative is to write the function as \( 1/(1 - (is)^2) = \sum(is)^n \). ■

- 1.4: \( 1/s \)

**Sol:** If you try to do a Taylor expansion at \( s = 0 \), the first term, \( w(0) \rightarrow \infty \). Thus, the Taylor series expansion in \( s \) does not exist. ■

- 1.5: \( 1/(1 - |s|^2) \)

**Sol:** The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed. ■

Problem #2: Consider the function \( w(s) = 1/s \)

- 2.1: Expand this function as a power series about \( s = 1 \). Hint: Let \( 1/s = 1/(1 - 1 + s) = 1/(1 - (1 - s)) \).
The power series is

\[ w(s) = \sum_{n=0}^{\infty} (-1)^n (s-1)^n, \]

which converges for \(|s-1| < 1\).

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,’ExpansionPoint’,1)`, which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to \(z = -1\) via the Möbius “translation” \(s = z + 1\), and expand using the Taylor series

\[ \frac{1}{s} = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n. \]

Next back-substitute \(z = s - 1\) giving

\[ \frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n (s-1)^n. \]

It follows that the RoC is \(|z| = |s-1| < 1\), as provided by Matlab/Octave.

- **2.2:** What is the RoC?
  - **Sol:** As stated in the solution of 2.1, \(|s-1| < 1\).

- **2.3:** Expand \(w(s) = 1/s\) as a power series in \(s^{-1} = 1/s\) about \(s^{-1} = 1\).
  - **Sol:** Let \(z = s^{-1}\) and expand about 1: The solution is \(w(z) = z\), which has a zero at 0 thus a pole at \(\infty\).

- **2.4:** What is the RoC?
  - **Sol:** \(|s| > 0\) or \(|z| < \infty\).

- **2.5:** What is the residue of the pole?
  - **Sol:** The pole is at \(\infty\). Since \(w(s) = 1/s\) and applying the definition for the residue \(c_{-1} = \lim_{s \to \infty} s(1/s) = 1\). Thus residue is 1. Note that it is the amplitude of the pole, which is 1.

**Problem # 3: Consider the function \(w(s) = 1/(2 - s)\)**

- **3.1:** Expand \(w(s)\) as a power series in \(s^{-1} = 1/s\). State the RoC as a condition on \(|s^{-1}|\).
  - **Hint:** Multiply top and bottom by \(s^{-1}\).
  - **Sol:** \(1/(2-s) = (-s^{-1}/(1-2s^{-1})) = -s^{-1} \sum_{n=0}^{\infty} 2^n s^{-n}\). The RoC is \(|2/s| < 1\), or \(|s| > 2\).

- **3.2:** Find the inverse function \(s(w)\). Where are the poles and zeros of \(s(w)\), and where is it analytic?
  - **Sol:** Solving for \(s(w)\) we find \(2 - s = 1/w\) and \(s = 2 - 1/w = (2w - 1)/w\). This has a pole at 0 and a zero at \(w = 1/2\). The RoC is therefore from the expansion point out to, but not including \(w = 0\).

**Problem # 4: Summing the series**

The Taylor series of functions have more than one region of convergence.

- **4.1:** Given some function \(f(x)\), if \(a = 0.1\), what is the value of \(f(a) = 1 + a + a^2 + a^3 + \cdots\)?
  - **Show your work.** **Sol:** To sum this series, we may use the fact that
    \[ f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1 - 1) + a^2(1 - 1) + \cdots \]
  This gives \((1 - a)f(a) = 1\), or \(f(a) = 1/(1 - a)\). Now since \(a = .1\), the sum is \(1/(1 - 0.1) = 1.11\).

- **4.2:** Let \(a = 10\). What is the value of \(f(a) = 1 + a + a^2 + a^3 + \cdots\)?
  - **Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for \(y = 1/x\) rather than for \(x\).
    \[ f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + a1/2) = 1 + (1 - 1)/a + (1 - 1)/a^2 + \cdots \]
  This gives \(f(1/a) = -a^{-1}/(1 - a^{-1})\). Now since \(a = 10\), the series sums to \(f(10) = -0.1/(1 - 0.1) = -1/9\).
3.1.3 Cauchy-Riemann Equations

**Problem # 5:** For this problem \( j = \sqrt{-1} \), \( s = \sigma + j \omega \), and \( F(s) = u(\sigma, \omega) + jv(\sigma, \omega) \). According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of \( F(s) \) is defined as

\[
\frac{dF}{ds} = \frac{d}{ds}[u(\sigma, \omega) + jv(\sigma, \omega)].
\] (DE-1.1)

If the integral is independent of the path, then the derivative must also be independent of the direction:

\[
\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \omega}.
\] (DE-1.2)

The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega}
\] and

\[
\frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation DE-1.2 holds.

- 5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

**Sol:** First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations.

- 5.2: Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equations.

\[
\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.
\]

**Sol:** Take partial derivatives with respect to \( \sigma \) and \( \omega \) and solve for one equation in each of \( u \) and \( v \). ■

- 5.3: What can you conclude?

**Sol:** We can conclude that the real and imaginary parts of complex analytic functions must obey these conditions. ■

**Problem # 6:** Apply the CR equations to the following functions. State for which values of \( s = \sigma + j \omega \) the CR conditions do or do not hold (e.g., where the function \( F(s) \) is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

- 6.1: \( F(s) = e^s \)

**Sol:** CR conditions hold everywhere. ■

- 6.2: \( F(s) = 1/s \)

**Sol:** CR conditions are violated at \( s = 0 \). The function is analytic everywhere except \( s = 0 \). ■

3.1.4 Branch cuts and Riemann sheets

**Problem # 7:** Consider the function \( w^2(z) = z \). This function can also be written as \( w_{\pm}(z) = \sqrt{z_{\pm}} \). Assume \( z = re^{j\theta} \) and \( w(z) = \rho e^{j\theta/2} \).

- 7.1: How many Riemann sheets do you need in the domain \( z \) and the range \( w \) to fully represent this function as single-valued?

**Sol:** There is one sheet for \( z \) and two sheet for \( w = \pm \sqrt{z} \). When any point in the domain \( z \) (being mapped to \( w(z) \)) crosses the \( z \) branch cut, the codomain (range) \( w_{\pm}(z) \) switches from the \( w_+ \) sheet to the \( w_- \) sheet. \( w(z) \) remains analytic on the cut. Look at Fig. 4.4 in Chap. 4 (p. 130) to see how this works. ■