Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem # 1: *In each case derive (e.g., using Taylor's formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn't exist, state why!* Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.

– *1.1:* $1/(1-s)$
Sol: $1/(1-s) = \sum_{n=0}^{\infty} s^n$, which converges for $|s| < 1$ (e.g., the RoC is $|s| < 1$) ■

– *1.2:* $1/(1-s^2)$
Sol: $1/(1-s^2) = \sum_{n=0}^{\infty} s^{2n}$, which converges for $|s^2| < 1$. (e.g., the RoC is $|s| < 1$). One can also factor the polynomial, thus write it as: $\frac{1}{(1-s)(1+s)}$. There are two poles, at $s = \pm 1$, and each has an RoC of 1. ■

– *1.3:* $1/(1+s^2)$.
Sol: The resulting series is $1/(1+s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n)$. The RoC is $|s| < 1$. We can see this by considering the poles of the function at $s = \pm i$; both poles are 1 from $s = 0$, the point of expansion. An alternative is to write the function as $1/(1-(is)^2) = \sum (is)^n$. ■

– *1.4:* $1/s$
Sol: If you try to do a Taylor expansion at $s = 0$, the first term, $w(0) \to \infty$. Thus, the Taylor series expansion in s does not exist. ■

– *1.5:* $1/(1-|s|^2)$
Sol: The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed. ■

Problem # 2: *Consider the function $w(s) = 1/s$*

– *2.1:* *Expand this function as a power series about $s = 1$.* Hint: Let $1/s = 1/(1-1+s) = 1/(1-(1-s))$.

Sol: The power series is

$$w(s) = \sum_{n=0}^{\infty} (-1)^n (s-1)^n,$$

which converges for $|s-1| < 1$.

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,'ExpansionPoint',1)`, which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to $z = -1$ via the Möbius "translation" $s = z + 1$, and expand using the Taylor series

$$\frac{1}{s} = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Next back-substitute $z = s - 1$ giving

$$\frac{1}{s} = \sum (-1)^n (s-1)^n.$$

It follows that the RoC is $|z| = |s-1| < 1$, as provided by Matlab/Octave. ■

– *2.2: What is the RoC?*
Sol: As stated in the solution of 2.1, $|s-1| < 1$. ■

– *2.3: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.*
Sol: Let $z = s^{-1}$ and expand about 1: The solution is $w(z) = z$, which has a zero at 0 thus a pole at ∞. ■

– *2.4: What is the RoC?*
Sol: $|s| > 0$ or $|z| < \infty$. ■

– *2.5: What is the residue of the pole?*
Sol: The pole is at ∞. Since $w(s) = 1/s$ and applying the definition for the residue $c_{-1} = \lim_{s\to\infty} s(1/s) = 1$. Thus residue is 1. Note that it is the amplitude of the pole, which is 1. ■

Problem # *3: Consider the function $w(s) = 1/(2-s)$*

– *3.1: Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the RoC as a condition on $|s^{-1}|$.*
Hint: Multiply top and bottom by s^{-1}.
Sol: $1/(2-s) = -s^{-1}/(1-2s^{-1}) = -s^{-1} \sum 2^n s^{-n}$. The RoC is $|2/s| < 1$, or $|s| > 2$. ■

– *3.2: Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?*
Sol: Solving for $s(w)$ we find $2 - s = 1/w$ and $s = 2 - 1/w = (2w-1)/w$. This has a pole at 0 and a zero at $w = 1/2$. The RoC is therefore from the expansion point out to, but not including $w = 0$. ■

Problem # *4:Summing the series*

The Taylor series of functions have more than one region of convergence.

– *4.1: Given some function $f(x)$, if $a = 0.1$, what is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

Show your work. **Sol:** To sum this series, we may use the fact that

$$f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1-1) + a^2(1-1) + \cdots$$

This gives $(1-a)f(a) = 1$, or $f(a) = 1/(1-a)$. Now since $a = .1$, the sum is $1/(1-0.1) = 1.11$. ■

– *4.2: Let $a = 10$. What is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

Sol: In this case the series clearly does not converge. To make it converge we need to write a formula for $y = 1/x$ rather than for x.

$$f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + a1/^2) = 1 + (1-1)/a + (1-1)/a^2 + \cdots$$

This gives $f(1/a) = -a^{-1}/(1-a^{-1})$. Now since $a = 10$, the series sums to $f(10) = -0.1/(1-0.1) = -1/9$. ■

3.1.3 Cauchy-Riemann Equations

Problem # 5: *For this problem $\jmath = \sqrt{-1}$, $s = \sigma + \omega\jmath$, and $F(s) = u(\sigma,\omega) + \jmath v(\sigma,\omega)$. According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of $F(s)$ is defined as*

$$\frac{dF}{ds} = \frac{d}{ds}\left[u(\sigma,\omega) + \jmath v(\sigma,\omega)\right]. \tag{DE-1.1}$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \jmath\omega}. \tag{DE-1.2}$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma,\omega)}{\partial \sigma} = \frac{\partial v(\sigma,\omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma,\omega)}{\partial \omega} = -\frac{\partial v(\sigma,\omega)}{\partial \sigma}$$

may be used to show where Equation DE-1.2 holds.

– *5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.*

Sol: First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations. ■

– *5.2: Merge the CR equations to show that u and v obey Laplace's equations.*

$$\nabla^2 u(\sigma,\omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma,\omega) = 0.$$

Sol: Take partial derivatives with respect to σ and ω and solve for one equation in each of u and v. ■

– *5.3: What can you conclude?*

Sol: We can conclude that the real and imaginary parts of complex analytic functions must obey these conditions. ■

Problem # 6: *Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g., where the function $F(s)$ is or is not analytic).* Hint: Review where CR-1 and CR-2 hold.

– *6.1: $F(s) = e^s$*

Sol: CR conditions hold everywhere. ■

– *6.2: $F(s) = 1/s$*

Sol: CR conditions are violated at $s = 0$. The function is analytic everywhere except $s = 0$. ■

3.1.4 Branch cuts and Riemann sheets

Problem # 7: *Consider the function $w^2(z) = z$. This function can also be written as $w_\pm(z) = \sqrt{z_\pm}$. Assume $z = re^{\phi\jmath}$ and $w(z) = \rho e^{\theta\jmath} = \sqrt{r}e^{\phi\jmath/2}$.*

– *7.1: How many Riemann sheets do you need in the* domain *(z) and the* range *(w) to fully represent this function as single-valued?*

Sol: There is one sheet for z and two sheet for $w = \pm\sqrt{z}$. When any point in the domain z (being mapped to $w(z)$) crosses the z branch cut, the codomain (range) $w_\pm(z)$ switches from the w_+ sheet to the w_- sheet. $w(z)$ remains analytic on the cut. Look at Fig. 4.4 in Chap. 4 (p. 130) to see how this works. ■