Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:
Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem # 1: In each case derive (e.g., using Taylor’s formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.

- 1.1: $1/(1 - s)$

**Sol:** $1/(1 - s) = \sum_{n=0}^{\infty} s^n$, which converges for $|s| < 1$ (e.g., the RoC is $|s| < 1$).

- 1.2: $1/(1 - s^2)$

**Sol:** $1/(1 - s^2) = \sum_{n=0}^{\infty} s^{2n}$, which converges for $|s^2| < 1$. (e.g., the RoC is $|s| < 1$). One can also factor the polynomial, thus write it as: $1/(1 - (1 - s)(1 + s))$. There are two poles, at $s = \pm 1$, and each has an RoC of 1.

- 1.3: $1/(1 + s^2)$.

**Sol:** The resulting series is $1/(1 + s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n)$. The RoC is $|s| < 1$. We can see this by considering the poles of the function at $s = \pm i$; both poles are 1 from $s = 0$, the point of expansion. An alternative is to write the function as $1/(1 - (is)^2) = \sum (is)^n$.

- 1.4: $1/s$

**Sol:** If you try to do a Taylor expansion at $s = 0$, the first term, $w(0) \to \infty$. Thus, the Taylor series expansion in $s$ does not exist.

- 1.5: $1/(1 - |s|^2)$

**Sol:** The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed.

Problem # 2: Consider the function $w(s) = 1/s$

- 2.1: Expand this function as a power series about $s = 1$. Hint: Let $1/s = 1/(1 - 1 + s) = 1/(1 - (1 - s))$. 
The power series is

\[ w(s) = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n, \]

which converges for \(|s - 1| < 1\).

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s, s, 'ExpansionPoint', 1)`, which is equivalent to the shorthand `syms s; taylor(1/s, s, 1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to \(z = -1\) via the Möbius “translation” \(s = z + 1\), and expand using the Taylor series

\[ \frac{1}{s} = \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-z)^n. \]

Next back-substitute \(z = s - 1\) giving

\[ \frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n. \]

It follows that the RoC is \(|z| = |s - 1| < 1\), as provided by Matlab/Octave.

**Problem # 3: Consider the function \(w(s) = 1/(2 - s)\)**

Hint: Multiply top and bottom by \(s^{-1}\).

**Sol:** \(1/(2 - s) = -s^{-1}/(1 - 2s^{-1}) = -s^{-1} \sum_{n=0}^{\infty} 2^n s^{-n}\). The RoC is \(|2/s| < 1\), or \(|s| > 2\).

**Problem # 4: Summing the series**

The Taylor series of functions have more than one region of convergence.

- **4.1:** Given some function \(f(x)\), if \(a = 0.1\), what is the value of

\[ f(a) = 1 + a + a^2 + a^3 + \cdots? \]

Show your work. **Sol:** To sum this series, we may use the fact that

\[ f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1 - 1) + a^2(1 - 1) + \cdots \]

This gives \((1 - a)f(a) = 1\), or \(f(a) = 1/(1 - a)\). Now since \(a = .1\), the sum is \(1/(1 - 0.1) = 1.11\).

**4.2:** Let \(a = 10\). What is the value of

\[ f(a) = 1 + a + a^2 + a^3 + \cdots? \]

**Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for \(y = 1/x\) rather than for \(x\).

\[ f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + a^2 + \cdots) = 1 + (1 - 1)/a + (1 - 1)/a^2 + \cdots \]

This gives \(f(1/a) = -a^{-1}/(1 - a^{-1})\). Now since \(a = 10\), the series sums to \(f(10) = -0.1/(1 - 0.1) = -1/9\).
3.1.3 Cauchy-Riemann Equations

**Problem # 5:** For this problem \( \gamma = \sqrt{-1} \), \( s = \sigma + \omega \gamma \), and \( F(s) = u(\sigma, \omega) + \gamma v(\sigma, \omega) \). According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of \( F(s) \) is defined as

\[
\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + \gamma v(\sigma, \omega)]. \tag{DE-1.1}
\]

If the integral is independent of the path, then the derivative must also be independent of the direction:

\[
\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \omega}. \tag{DE-1.2}
\]

The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation DE-1.2 holds.

- **5.1:** Assuming Equation DE-1.2 is true, use it to derive the CR equations.

**Sol:** First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations.

- **5.2:** Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equations.

\[
\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.
\]

**Sol:** Take partial derivatives with respect to \( \sigma \) and \( \omega \) and solve for one equation in each of \( u \) and \( v \).

- **5.3:** What can you conclude?

**Sol:** We can conclude that the real and imaginary parts of complex analytic functions must obey these conditions.

**Problem # 6:** Apply the CR equations to the following functions. State for which values of \( s = \sigma + i\omega \) the CR conditions do or do not hold (e.g., where the function \( F(s) \) is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

- **6.1:** \( F(s) = e^s \)

**Sol:** CR conditions hold everywhere.

- **6.2:** \( F(s) = 1/s \)

**Sol:** CR conditions are violated at \( s = 0 \). The function is analytic everywhere except \( s = 0 \).

3.1.4 Branch cuts and Riemann sheets

**Problem # 7:** Consider the function \( w^2(z) = z \). This function can also be written as \( w_{\pm}(z) = \sqrt{z_{\pm}} \). Assume \( z = r e^{i\gamma} \) and \( w(z) = \rho e^{i\delta/2} \).

- **7.1:** How many Riemann sheets do you need in the domain \( z \) and the range \( w \) to fully represent this function as single-valued?

**Sol:** There is one sheet for \( z \) and two sheet for \( w = \pm \sqrt{z} \). When any point in the domain \( z \) (being mapped to \( w(z) \)) crosses the \( z \) branch cut, the codomain (range) \( w_{\pm}(z) \) switches from the \( w_+ \) sheet to the \( w_- \) sheet. \( w(z) \) remains analytic on the cut. Look at Fig. 4.4 in Chap. 4 (p. 132) to see how this works.