3.2 Problems DE-2

3.2.1 Topics of this homework:

Integration of complex functions, Cauchy's theorem, integral formula, residue theorem, power series, Riemann sheets and branch cuts, inverse Laplace transforms, Quadratic forms.

Two fundamental theorems of calculus

Fundamental Theorem of Calculus (Leibniz):

According to the fundamental theorem of (real) calculus (FTC),

$$f(x) = f(a) + \int_a^x F(\xi)d\xi, \tag{DE-2.1}$$

where $x, a, \xi, F, f \in \mathbb{R}$. This is an indefinite integral (since the upper limit is unspecified). It follows that

$$\frac{df(x)}{dx} = \frac{d}{dx}\int_a^x F(x)dx = F(x).$$

This justifies also calling the indefinite integral the *antiderivative*.

For a closed interval $[a, b]$, the FTC is

$$\int_a^b F(x)dx = f(b) - f(a), \tag{DE-2.2}$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Fundamental Theorem of Complex Calculus:

According to the fundamental theorem of complex calculus (FTCC),

$$f(z) = f(z_0) + \int_{z_0}^z F(\zeta)d\zeta, \tag{DE-2.3}$$

where $z_0, z, \zeta, f, F \in \mathbb{C}$. It follows that

$$\frac{df(z)}{dz} = \frac{d}{dz}\int_{z_0}^z F(\zeta)d\zeta = F(z). \tag{DE-2.4}$$

For a closed interval $[s, s_o]$, the FTCC is

$$\int_{s_o}^s F(\zeta)d\zeta = f(s) - f(s_o), \tag{DE-2.5}$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Problem # *1*

– 1.1: (2 pts) Consider Equation DE-2.1. What is the condition on $F(x)$ for which this formula is true?

Sol: The sufficient condition is that the integrand $F(x)$ is be analytic, namely $F(x) = \sum_{x=0}^{\infty} a_n x^n$. This assures that $F(x)$ is single valued and may be integrated, since it may integrated term by term. It follows that as long as $x <$ ROC, this integral exists. Thus the integral equals $F(x) - F(a)$. Note that if the integrand has a Taylor series, all of its derivatives exist within the ROC, because the coefficients depend on derivatives of $F(x)$. ■

– 1.2: (2 pts) Consider Equation DE-2.3. What is the condition on $F(z)$ for which this formula is true?

Sol: The sufficient condition is that the integrand $F(z)$ must be complex analytic, namely $F(z) = \sum_{z=0}^{\infty} c_n z^n$, with $c \in \mathbb{C}$. ■

– 1.3: Let $F(z) = \sum_{k=0}^{\infty} c_k z^k$.

Sol: Applying term by term integration gives

$$I = \int_C \sum_{k=0}^{\infty} c_k z^k dz = \sum_{k=0}^{\infty} c_k \int_C z^k dz$$
$$= \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}$$

■

– 1.4: Let

$$F(z) = \frac{\sum_{k=0}^{\infty} c_k z^k}{z - \jmath}.$$

Sol: Applying term by term integration, and using CT-3, gives

$$I = \int_C \frac{\sum_{k=0}^{\infty} c_k z^k}{z-\jmath} dz = \sum_{k=0}^{\infty} c_k \int_C \frac{z^k}{z-\jmath} dz$$
$$= \begin{cases} 0 & z = \jmath \notin \mathcal{C} \\ \frac{1}{2\pi\jmath} \sum_{k=0}^{\infty} c_k \jmath^k & z = \jmath \in \mathcal{C} \end{cases}$$

Note:

$$\int_C \frac{z^k}{z-\jmath} dz = \sum_{m=1}^{k} \frac{(\jmath z)^m}{m} + \ln(z - \jmath),$$

but the residue is $\jmath^k$ which saves the day. ■

Problem # *2: In the following problems, solve the integral*

$$I = \int_C F(z) dz$$

for a given path $\mathcal{C} \in \mathbb{C}$.

– 2.1: Perform the following integrals ($z = x + iy \in \mathbb{C}$):
$I = \int_0^{1+\jmath} zdz$

Sol: $I = \frac{1}{2}z^2\big|_0^{1+\jmath} = \frac{1}{2}(1+\jmath)^2 = \frac{1}{2}(1-1+2\jmath) = \jmath$ ■

– 2.2: $I = \int_0^{1+j} zdz$, but this time make the path explicit: from 0 to 1, with $y = 0$, and then to $y = 1$, with $x = 1$.

Sol: Some text to start the ball rolling

$$
\begin{aligned}
I &= \int_{x=0}^{1} (x + 0\jmath)\, dx + \int_{y=0}^{1} (1 + y\jmath)\, dy\, \jmath \\
I &= \frac{1}{2}\, x^2\big|_0^1 + \int_{y=0}^{1} (\jmath - y)\, dy \\
&= \frac{1}{2} + \left(y\jmath - \frac{1}{2}y^2\right)\Big|_0^1 \\
&= \frac{1}{2} + j - \frac{1}{2} \\
&= j
\end{aligned}
$$

We conclude that the integration of z is independent of the path. This is true for any integrand z^n with $n \in \mathbb{Z}$. ■

– 2.3: Discuss whether your results agree with Eq. DE-2.4?

Sol: Yes the two integrals must agree, because the function is analytic, and the integral must be the same, independent of the path. ■

Problem # *3: Perform the following integrals on the closed path $\mathcal{C}$, which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$, and integrate from $\{-\pi, \pi\}$ to go once around the unit circle.*

Discuss whether your results agree with Eq. DE-2.4?

– 3.1: (2 pts) $\int_C zdz$

Sol: $\int_C zdz = \int_{-\pi}^{\pi} e^{i\theta}de^{i\theta} = \int_{-\pi}^{\pi} e^{i2\theta}id\theta = e^{i2\theta}|_{-\pi}^{\pi} = 0$.
This example obeys the FTCC because $f(z) = z$ is analytic everywhere; ■

– 3.2: (2 pts) $\int_C \frac{1}{z}dz$

Sol: $\int_{-\pi}^{\pi} id\theta = 2\pi i$.
This example does not obey FTCC because $f(z) = 1/z$ is not analytic at $z = 0$ (inside C), instead it satisfies CT-2; ■

– 3.3: (2 pts) $\int_C \frac{1}{z^2}dz$

Sol: $\int_C \frac{dz}{z^2} = \int_{-\pi}^{\pi} e^{-i2\theta}de^{i\theta} = \int_{-\pi}^{\pi} e^{-i\theta}id\theta = -\frac{e^{-i\theta}}{\theta}|_{-\pi}^{\pi} = -\frac{e^{-i\pi}}{\pi} + \frac{e^{i\pi}}{\pi} = \frac{1-1}{\pi} = 0$.
This example obeys the FTCC because the residue is $-e^{-i\theta}/\theta$, and the loop is closed (starting and ending points are the same). ■

– *3.4:* $I = \int_C \frac{1}{(z+2\jmath)^2} dz$.

Recall that the path of integration is the unit circle, starting and ending at -1.

Sol: Let $\zeta = z + 2\jmath$, then the limits become $[-1+2\jmath, 1+2\jmath] = [2\jmath + e^{-\jmath\pi}, 2\jmath + e^{\jmath\pi+2\jmath}]$.

$$I = \int_C \frac{d\zeta}{\zeta^2} = \int_{-\pi}^{\pi} e^{-i2\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{-i\theta} i d\theta = -\frac{e^{-i\theta}}{\theta}|_{-\pi}^{\pi} = -\frac{e^{-i\pi}}{\pi} + \frac{e^{i\pi}}{\pi} = \frac{1-1}{\pi} = 0.$$

This example reduces to the case of (3), and therefore must have the same conclusion as (3). But in this case the reasoning is different because the second order pole (singular point) is outside the unit circle, thus the function is analytic inside $\mathcal{C}$, so CT-1 applies. ■

Problem # *4: FTCC and integration in the complex plane*

Let the function $F(z) = c^z$, where $c \in \mathbb{C}$ is given for each question. *Hint: Can you apply the FTCC?*

– *4.1: For the function $f(z) = c^z$, where $c \in \mathbb{C}$ is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that $f(z)$ is analytic for all $z \in \mathbb{C}$.*

Sol: We may rewrite this function as $f(z) = e^{\ln(c)z}$, where $z = x + iy$ and $f = u + iv$. Thus

$$u(x, y) = e^{\ln(c)x} \cos(\ln(c)y),$$
$$v(x, y) = e^{\ln(c)x} \sin(\ln(c)y)$$
$$\frac{\partial u}{\partial x} = \ln(c)\, e^{\ln(c)x} \cos(\ln(c)y) = \frac{\partial v}{\partial y} = \ln(c)\, e^{\ln(c)x} \cos(\ln(c)y)$$

$$\frac{\partial u}{\partial y} = -\ln(c)\, e^{\ln(c)x} \sin(\ln(c)y) = -\frac{\partial v}{\partial x} = -\ln(c)\, e^{\ln(c)x} \sin(\ln(c)y)$$

Thus the CR conditions are satisfied everywhere and the function is analytic for all $z \in \mathbb{C}$. ■

– *4.2: Find the antiderivative of $F(z)$.*

Sol: Since $c^z = e^{\ln(c)z}$, the indefinite integral (anti-derivative) is

$$I(z) = \frac{1}{\ln c} e^{\ln(c)z} \quad \text{since} \quad \frac{d}{dz} I(z) = \frac{d}{dz} \frac{1}{\ln c} e^{\ln(c)z} = e^{\ln(c)z} = F(z).$$

■

– *4.3: $c = 1/e = 1/2.7183, \ldots$ where $\mathcal{C}$ is $\zeta = 0 \rightarrow i \rightarrow z$*

Sol: The integrand is $F(z) = e^{-z}$, which is *entire*. Thus the the integral is independent of the path (i.e., $\mathcal{C}$ is not relevant to the final answer).

$$I(z) = \int_0^i e^{-\zeta} d\zeta + \int_i^z e^{-\zeta} d\zeta = F(z) - F(i) + F(i) - F(0)$$
$$= \int_0^z e^{-\zeta} d\zeta = -e^z \Big|_0^z = -(e^{-z} - 1)$$

■

– *4.4: $c = 2$, where $\mathcal{C}$ is $\zeta = 0 \rightarrow (1+i) \rightarrow z$*

Sol: The integrand is $F(z) = 2^z$, where $2 = e^{\ln 2}$. The path $\mathcal{C}$ is not relevant to the final answer.

$$I(z) = \int_0^z 2^\zeta d\zeta = \int_0^z e^{\zeta \ln 2} d\zeta = \frac{e^{\zeta \ln 2}}{\ln 2}\Big|_0^z = (e^{z \ln 2} - 1)/\ln 2 \approx 1.443(e^{0.693z} - 1)$$

■

– *4.5: $c = i$, where the path $\mathcal{C}$ is an inward spiral described by $z(t) = 0.99^t e^{i2\pi t}$ for $t = 0 \to t_0 \to \infty$*

Sol: $i = e^{i\pi/2}e^{i2\pi n}$. We have already proved that the path doesn't matter for any $F(z) = c^z$, so we just need to evaluated $z(t)$ for $t = 0$ and $t \to \infty$. This gives $z(0) = 1$ and $z(t \to \infty) = 0$.

$$I = \int_{z(0)}^{z(t\to\infty)} i^z dz = \int_{z(0)}^{z(t\to\infty)} e^{i\pi z/2} dz = \frac{2e^{i\pi z/2}}{i\pi}\Bigg|_1^0 = \frac{2}{i\pi}(1 - e^{i\pi/2}) = \frac{-2(i+1)}{\pi}$$

■

– *4.6: $c = e^{t-\tau_0}$, where $\tau_0 > 0$ is a real number and $\mathcal{C}$ is $z = (1 - i\infty) \to (1 + i\infty)$.* Hint: Do you recognize this integral? If you do not, please do not spend a lot of time trying to solve it via the "brute force" method.

Sol: This is the basically the inverse Laplace transform of $e^{-\tau_0 z}$, we are just missing the scale factor $\frac{1}{2\pi i}$.

$$I(t) = \int_{1-i\infty}^{1+i\infty} e^{(t-\tau_0)z} dz = \int_{1-i\infty}^{1+i\infty} e^{-\tau_0 z} e^{zt} dz = 2\pi i \delta(t - \tau_0)$$

■

3.2.2 Cauchy's theorems CT-1, CT-2, CT-3

There are three basic definitions related to Cauchy's integral formula. They are all related and can greatly simplify integration in the complex plane. When a function depends on a complex variable, we use uppercase notation, consistent with the engineering literature for the Laplace transform.

Problem # 5: *Describe the relationships between the theorems:*

– *5.1: CT-1 and CT-2*

Sol: When z_0 falls outside of $\mathcal{C}$, CT-2 reduces to CT-1. ■

– *5.2: CT-1 and CT-3*

Sol: When there are no poles inside $\mathcal{C}$, all the residues are zero, and CT-3 reduces to CT-1. ■

– *5.3: CT-2 and CT-3*

Sol: Case CT-2 has only one induced pole at $z = z_0$, having residue $F(z_0)$. Thus CT-3 is the same as CT-2 when $K = 1$, the pole at z_0 is within contour $\mathcal{C}$, and the single residue is $F(z_0)$. ■

– *5.4: Consider the function with poles at $z = \pm j$,*

$$F(z) = \frac{1}{1+z^2} = \frac{1}{(z-j)(z+j)}.$$

Find the residue expansion.

Sol:

$$F(z) = \frac{j}{2}\Big(\frac{1}{z+j} - \frac{1}{z-j}\Big).$$

■

Problem # 6: *Apply Cauchy's theorems to solve the following integrals. State which theorem(s) you used and show your work.*

– *6.1: $\oint_{\mathcal{C}} F(z)dz$, where $\mathcal{C}$ is a circle centered at $z = 0$ with a radius of $\frac{1}{2}$*

Sol: Because the contour $\mathcal{C}$ does not include the poles, $F(z)$ is analytic everywhere inside $\mathcal{C}$). Using *Cauchy's integral theorem*, the integral is 0. ■

– *6.2: $\oint_{\mathcal{C}} F(z)dz$, where $\mathcal{C}$ is a circle centered at $z = j$ with a radius of 1*

Sol: Since we only enclose the pole at $z = j$, use the *integral formula* with $F(z) = 1/(z + j)$:

$$\oint_{\mathcal{C}} \frac{F(z)}{z-j} dz = 2\pi j \mathrm{Res}_j = 2\pi j \Big[\frac{1}{z+j}\Big]_{z=j} = 2\pi j \frac{1}{2j} = \pi$$

■

– *6.3:* $\oint_C F(z)dz$*, where* $\mathcal{C}$ *is a circle centered at* $z = 0$ *with a radius of* 2

Sol: Since we enclose both poles, using the *residue theorem*:

$$\oint_C F(z)dz = 2\pi j(\text{Res}_j + \text{Res}_{-j}) = 2\pi j\left(\frac{1}{2j} - \frac{1}{2j}\right) = 0$$

As a side note, the inverse Laplace transform for $F(z)$ is $\sin(t)$, which is zero for $t = 0$, consistent with this result. ■

Integration of analytic functions

Problem # 7: *In the following questions, you'll be asked to integrate* $F(s) = u(\sigma,\omega)+iv(\sigma,\omega)$ *around the contour* $\mathcal{C}$ *for complex* $s = \sigma + i\omega$*,*

$$\oint_C F(s)ds. \tag{DE-2.6}$$

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations

– *7.1:* $F(s) = \sin(s)$

Sol: Analytic everywhere. $-\cos(s) = \int_{\theta=0}^{2\pi} \sin(s)ds = 0$. This function is *entire* (i.e., has no poles) so the integral must be zero. ■

– *7.2: Given function* $F(s) = \frac{1}{s}$ *State where the function is and is not analytic.*

Sol: Analytic everywhere except at $s = 0$, where it has a pole. ■

– *7.3: Explicitly evaluate the integral when* $\mathcal{C}$ *is the unit circle, defined as* $s = e^{i\theta}$*,* $0 \le \theta \le 2\pi$*.*

Sol:

$$\oint_C F(s)ds = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} id\theta = 2\pi i$$

■

– *7.4: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.*

Sol: The residue is 1 so the integral is $2\pi i$. ■

– *7.5:* $F(s) = \frac{1}{s^2}$ *State where the function is and is not analytic.*

Sol: Analytic everywhere except at $s = 0$, where it has a 2^{nd} order pole. ■

– *7.6: Explicitly evaluate the integral when* $\mathcal{C}$ *is the unit circle, defined as* $s = e^{i\theta}$*,* $0 \le \theta \le 2\pi$*.*

Sol:

$$\oint_C F(s)ds = \int_0^{2\pi} \frac{1}{e^{i2\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} ie^{-i\theta} d\theta = i\frac{1}{-i}e^{-i\theta}\Big|_0^{2\pi} = 1(e^{-i2\pi} - e^0) = 0$$

■

– *7.7: What does your result imply about the residue of the second-order pole at* $s = 0$*?*

Sol: The residue is 0. ■

– *7.8:* $F(s) = e^{st}$*: State where the function is and is not analytic.*

Sol: Analytic everywhere. ■

– *7.9: Explicitly evaluate the integral when* $\mathcal{C}$ *is the square* $(\sigma,\omega) = (1,1) \to (-1,1) \to (-1,-1) \to (1,-1) \to (1,1)$.

Sol: When you perform this integral piece-wise, you will find that all terms cancel out and the result is 0. ■

– *7.10: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.*

Sol: The function is analytic everywhere, so the integral is 0 by Cauchy's theorem. ■

– 7.11: $F(s) = \frac{1}{s+2}$: State where the function is and is not analytic.
Sol: Analytic everywhere except at $s = -2$, where it has a pole. ■

– 7.12: Let $\mathcal{C}$ be the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Sol: The function is analytic everywhere inside $\mathcal{C}$, so the integral is 0 by Cauchy's theorem. ■

– 7.13: Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Sol: The contour contains the pole and the residue 1, therefore the integral is $2\pi\imath$. ■

– 7.14: $F(s) = \frac{1}{2\pi i}\frac{e^{st}}{(s+4)}$ State where the function is and is not analytic.
Sol: Analytic everywhere except at $s = -4$, where it has a pole. ■

– 7.15: Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Sol: The contour does not contain the pole. Thus the integral is zero. ■

– 7.16: Let $\mathcal{C}$ contain the entire left half s plane. Evaluate the integral using Cauchy's theorem and/or the residue theorem. Do you recognize this integral?
Sol: This contour contains the pole. The residue is $\frac{1}{2\pi i}e^{-2t}$, therefore the integral is equal to e^{-2t}. This contour is the inverse Laplace transform. ■

– 7.17: $F(s) = \pm\frac{1}{\sqrt{s}}$ (e.g., $F^2 = \frac{1}{s}$) State where the function is and is not analytic.
Sol: Analytic everywhere except $s = 0$, where there is a pole. ■

– 7.18: This function is multivalued. How many Riemann sheets do you need in the domain (s) and the range (f) to fully represent this function? Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

Sol: There are 2 sheets in the domain (for the $\pm$ square root) which map to 1 sheet in the range. ■

– 7.19: Explicitly evaluate the integral $\int_{\mathcal{C}} \frac{1}{\sqrt{z}} dz$ when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$. Is this contour closed? State why or why not.

Sol: The solution is

$$2\sqrt{z}\Big|_{\theta=0}^{2\pi} = 2e^{j\theta/2}\Big|_0^{2\pi} = 2(e^{j\pi} - e^0) = -4.$$

In polar coordinates

$$\begin{aligned}\int_0^{2\pi} \frac{ds}{\sqrt{s}} &= \int_0^{2\pi} \frac{de^{i\theta}}{e^{i\theta/2}} \\ &= i\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta \\ &= i\int_0^{2\pi} e^{i\theta/2} d\theta \\ &= 2\, e^{i\theta/2}\Big|_0^{2\pi} \\ &= 2[e^{i\pi} - 1] = 2(-2) = -4.\end{aligned}$$

This contour is *not* closed. One way to determine this is to see if going once around the unit circle returns $F(s)$ to its original value.

$$F(e^{i0}) = 1 \ne F(e^{i2\pi}) = e^{-i\pi} = -1.$$

■

– 7.20: Explicitly evaluate the integral $\int_C \frac{1}{\sqrt{z}} dz$ *when* C *is* twice *around the unit circle, defined as* $s = e^{i\theta}$, $0 \le \theta \le 4\pi$. *Is this contour closed? State why or why not. Hint: Note that* $\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi}\sqrt{e^{i\theta}} = -1\sqrt{e^{i\theta}}$.

Sol:

$$\begin{aligned}
\int_0^{4\pi} \frac{ds}{\sqrt{s}} &= \int_0^{4\pi} \frac{de^{i\theta}}{e^{i\theta/2}} \\
&= i\int_0^{4\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta \\
&= i\int_0^{4\pi} e^{i\theta/2} d\theta \\
&= 2\, e^{i\theta/2}\Big|_0^{4\pi} \\
&= 2[e^{i2\pi} - 1] = 2(0) = 0.
\end{aligned}$$

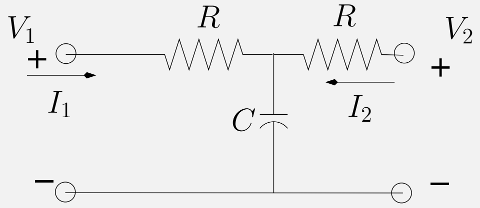
This contour *is* closed. One way to determine this is to see if going twice around the unit circle returns $F(s)$ to its original value. $F(e^{i0}) = 1 = F(e^{i4\pi}) = e^{-i2\pi} = 1$. ■

– 7.21: What does your result imply about the residue of the (twice-around $\frac{1}{2}$ *order) pole at* $s = 0$*?*
Sol: The residue is 0. ■

– 7.22: Show that the residue is zero. Hint: Apply the definition of the residue.
Sol: $c_{-1} = \lim_{z\to z_k} z/\sqrt{z} = \lim_{z\to z_k} \sqrt{z} = 0$. ■

3.2.3 Laplace transform applications

Problem # *8: A two-port network application for the Laplace transform*



This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$. It is convenient to define the dimensionless ratio $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.

3.2.4 Computer exercises with Matlab/Octave

Problem # *9: With the help of a computer*

Now we look at a few important concepts using Matlab/Octave's `syms` commands or Wolfram Alpha's symbolic math toolbox.[1]

For example, to find the Taylor series expansion about $s = 0$ of

$$F(s) = -\log(1 - s),$$

we first consider the derivative and its Taylor series (about $s = 0$)

$$F'(s) = \frac{1}{1-s} = \sum_{n=0}^{\infty} s^n.$$

[1] `https://www.wolframalpha.com/`

Then, we integrate this series term by term:

$$F(s) = -\log(1-s) = \int^s F'(s)ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.$$

Alternatively we can use Matlab/Octave commands:

```
syms s
taylor(-log(1-s),'order',7)
```

– 9.1: Use Octave's `taylor(-log(1-s))` *to the seventh order, as in the example above. Try the above Matlab/Octave commands. Give the first seven terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).*

Sol:

$$F(s) = \cdots + \frac{s^7}{7} + \frac{s^6}{6} + \frac{s^5}{5} + \frac{s^4}{4} + \frac{s^3}{3} + \frac{s^2}{2} + s$$

■

– 9.2: What is the inverse Laplace transform of this series? Consider the series term by term.

Sol: $f(t) = \sum \delta^{(n)}/n$ ■

– 9.3: The function $1/\sqrt{z}$ has a branch point at $z = 0$; thus it is singular there. Can you apply Cauchy's integral theorem when integrating around the unit circle?

Sol: No, one cannot apply the Cauchy Theorem since it is not analytic at $z = 0$. But the integral may be evaluated. ■

– 9.4: This Matlab/Octave code computes $\int_0^{4\pi} \frac{dz}{\sqrt{z}}$ using Matlab's/Octave's symbolic analysis package. Run the following script:

```
syms  z
I=int(1/sqrt(z))
J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

What answers do you get for I and J?

Sol: This script returns the answers $I = 2 * \sqrt{z}$ and $J = 2.4493e - 16$, which is numerically the same as zero. ■

– 9.5: Modify this code to integrate $f(z) = 1/z^2$ once *around the unit circle. What answers do you get for I and J?*

Sol: This function has a 2d order pole at $s = 0$. Thus from the CIT, the integral evaluates to zero.

Proof:

$$I = \oint \frac{ds}{s^2} = -\left.\frac{1}{s}\right|_0^{2\pi} = -\left. e^{-i\theta}\right|_0^{2\pi} = -(1-1) = 0$$

More generally $I = \oint \frac{ds}{s^n} = 0$ for $n \neq 1$. As best I know, this holds for any $n \in \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{R}, \mathbb{C}$. For $n = 1$ it has a value of $2\pi j$. ■

– 9.6: Bessel functions can describe waves in a cylindrical geometry.

The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$

Draw a hand sketch showing the nature of the branch cut. Hint: Use `zviz`. **Sol:** The roots are given by $s_\pm = \pm \jmath$. The branch cut connects the two roots, or can go from each root to ∞. Either choice is valid. ■

Problem # *10: Matlab/Octave exercises*

– *10.1: Try the following Matlab/Octave commands, and then comment on your findings.*

```
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)
```

Sol: $I = J_o(t)u(t)$. ■

```
%Find the Taylor series of the LT
T = taylor(1/sqrt(1+s^2),10); disp(T);
```

Sol:

$$T = \cdots + \frac{3s^4}{8} - \frac{s^2}{2} + 1$$

■

```
%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);
```

Sol: $I = \frac{1}{\sqrt{1+s^2}}$. ■

```
%plot the Bessel function
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

Sol: Plot of $J_o(t)u(t)$. ■

– *10.2: When did Friedrich Bessel live?*
Sol: 1784–1846, in Königsberg, Germany. ■

– *10.3: What did he use Bessel functions for?*
Sol: Solving the Bessel equation, which is the wave equation in 2D. Bessel functions were first introduced by the Daniel Bernoulli. ■

Problem # *11: Colorized plots of analytic functions. Use `zviz` for each of the following.*

– *11.1: Describe the plot generated by `zviz S=Z`.*
Sol: It is a polar plot of the function, with intensity coding the magnitude and color coding the phase. Red is a positive real number while and blue is a negative real number. ■

– *11.2: Describe the plot generated by* `zviz` $1./sqrt(1 + S.^2)$.
Sol: No. The RHP has blue near the branch cut, in the RHP. ■

– *11.3: Describe the plot generated by* `zviz` $1./sqrt(1 - S.^2)$. *Is this function a Brune impedance (i.e., does this function obey*
Sol: NO, there is a branch cut in the RHP. ■

– *11.4:* `zviz` $1./(1 + sqrt(S))$
Sol: Yes, it's red almost everywhere even though it has a branch cut from $[-\infty < \sigma \le -10]$. Since $1/\sqrt{s}$ has an $\mathcal{LT}^{-1}$, this function must as well. Matlab found

$$\frac{1}{\sqrt{1+s}} \leftrightarrow \frac{e^{-t}}{\sqrt{\pi}\sqrt{t}}u(t),$$

however Octave failed to find the inverse transform, (but was able to find the forward transform). ■

3.2.5 Inverse of Riemann $\zeta(s)$ function

Problem # *12: Inverse zeta function (This problem is for extra credit).*

– *12.1: Find the* $\mathcal{LT}^{-1}$ *of one factor of the Riemann zeta function* $\zeta_p(s)$, *where* $\zeta_p(s) \leftrightarrow z_p(t)$. *Describe your results in words. Hint: Consider the geometric series representation*

$$\zeta_p(s) = \frac{1}{1 - e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \tag{DE-2.7}$$

for which you can look up the $\mathcal{LT}^{-1}$ of each term.
Sol: Since each term in the series is a pure delay[2]

[2]Here we use a shorthand double-parentheses notation to define the infinite (one-sided) sum $f(t))_T \equiv \sum_{k=0}^{\infty} f(t - kT)$.

$$z_p(t) = \delta(t))_{T_p} \equiv \sum_{k=0}^{\infty} \delta(t - kT_p) \leftrightarrow \frac{1}{1 - e^{-sT_p}}. \tag{DE-2.8}$$

■

Problem # *13: Inverse transform of products:*
The time-domain version of Eq. DE-2.7 may be written as the convolution of all the $z_k(t)$ factors:

$$z(t) \equiv z_2(t) \star z_3(t) \star z_5(t) \star z_7(t) \star \cdots \star z_p(t) \star \cdots, \tag{DE-2.9}$$

where $\star$ represents time convolution.

Explain what this means in physical terms. Start with two terms (e.g., $z_1(t) \star z_2$). Hint: The input admittance of this cascade may be interpreted as the analytic continuation of $\zeta(s)$ by defining a cascade of eigenfunctions with eigenvalues derived from the primes. For a discussion of this idea see Sec. 3.2.3 and C.1.1.

Sol: In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines, each having a delay given by $T_p = \ln \pi_p$. ■

Physical interpretation: Such functions may be generated in the time domain, as shown in Fig. 3.3 (p. 61), using a feedback delay of T_p seconds, described by the two equations in the Fig. 3.3 with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation, we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is an all-pole function, since

$$Q(s) = e^{-sT_n} Q(s) + V(s), \text{ or } \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \tag{DE-2.10}$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$—namely,

$$Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \tag{DE-2.11}$$

If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification, $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j\tan(sT_p/2). \tag{DE-2.12}$$

Every impedance $Z(s)$ has a corresponding *reflectance* function given by a Möbius transformation, which may be read off of Eq. DE-2.11 as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}, \tag{DE-2.13}$$

since impedance is also related to the round-trip delay T_p on the line. The inverse Laplace transform of $\Gamma(s)$ is the round-trip delay T_p on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \tag{DE-2.14}$$

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations that are not obvious in the frequency domain (See Sec. 3.2.3). Transforming to the time domain is a form of analytic continuation of $\zeta(s)$ that depends on the assumption that $Z^{eta}(t) \leftrightarrow \zeta(s)$ is one-sided in time (causal).

3.2.6 Quadratic forms

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy, since the power is the voltage times the current. Given an impedance matrix

$$\mathbf{V} = \mathcal{Z}\mathbf{I},$$

the power $\mathcal{P}$ is

$$\mathcal{P} = \mathbf{I} \cdot \mathbf{V} = \mathbf{I} \cdot \mathcal{Z}\mathbf{I},$$

which must be positive-definite for the system to obey conservation of energy.

Problem # *14: In this problem, consider the 2×2 impedance matrix*

$$\mathcal{Z} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

– *14.1: Solve for the power $\mathcal{P}(i_1, i_2)$ by multiplying out this matrix equation (which is a* quadratic form*):*

$$\mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}.$$

Sol:

$$\mathcal{P}(i_1, i_2) = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2i_1 + i_2 \\ i_1 + 4i_2 \end{bmatrix} = 2i_1^2 + 2i_1 i_2 + 4i_2^2.$$

■

– *14.2: Is the impedance matrix positive-definite? Show your work by finding the eigenvalues of the matrix* **Z**.

Sol: Yes, as it is positive-definite if the eigenvalues are both positive. You need to show that the eigenvalues are positive (not zero or negative). They are, so it is.

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3 \pm \sqrt{2} > 0$$

■

– *14.3: Should an impedance matrix always be positive-definite? Explain.*

Sol: Yes. ■