3.3 Problems DE-3

3.3.1 Topics of this homework: Brune impedance

lattice transmission line analysis

3.3.2 Brune Impedance

Problem # *1: Residue form*

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \tag{DE-3.1}$$

with

$$D(s) = \prod_{k=1}^{K} (s - s_k) \qquad \text{and} \qquad c_k = \lim_{s \to s_k} (s - s_k) D(s) = \prod_{n'=1}^{K-1} (s - s_n).$$

The prime on the index n' means that $n = k$ is not included in the product.

– 1.1: Find the Laplace transform ($\mathcal{LT}$) of a (1) spring, (2) dashpot, and (3) mass.

Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke's law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton's law for mass $f(t) = M dv(t)/dt$. **Sol:**

1. Hooke's Law $f(t) = Kx(t)$. Taking the $\mathcal{LT}$ gives

$$F(s) = KX(s) = KV(s)/s \leftrightarrow f(t) = Ku(t) \star v(t) = K \int^{t} v(t),$$

since

$$v(t) = \frac{d}{dt} x(t) \leftrightarrow V(s) = sX(s).$$

Thus the impedance of the spring is

$$Z_s(s) = \frac{K}{s} \leftrightarrow z(t) = Ku(t),$$

which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as $C = 1/K$.

2. Dashpot resistance $f(t) = Rv(t)$. From the $\mathcal{LT}$ this becomes

$$F(s) = RV(s)$$

and the impedance of the dashpot is then

$$Z_r = R \leftrightarrow R\delta(t),$$

analogous to that of an electrical resistor.

3. Newton's law for mass $f(t) = M dv(t)/dt$. Taking the $\mathcal{LT}$ gives

$$f(t) = M \frac{d}{dt} v(t) \leftrightarrow F(s) = M\, sV(s),$$

thus

$$Z_m(s) = sM \leftrightarrow M \frac{d}{dt},$$

analogous to an electrical inductor.

■

– 1.2: Take the Laplace transform ($\mathcal{LT}$) of Eq. DE-3.2 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \tag{DE-3.2}$$

Sol: From the properties of the $\mathcal{LT}$ that $dx/dt \leftrightarrow sX(s)$, we find

$$f(t) \leftrightarrow F(s) = Ms^2X(s) + RsX(s) + KX(s).$$

In terms of velocity this is $(Ms + R + K/s)V(s) = F(s)$. Thus the circuit impedance is

$$z(t) \leftrightarrow Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s}.$$

■

– 1.3: What are $N(s)$ and $D(s)$ (see Eq. DE-3.1)?
Sol: $D(s) = s$ and $N(s) = K + Rs + Ms^2$. ■

– 1.4: Assume that $M = R = K = 1$ and find the residue form of the admittance $Y(s) = 1/Z(s)$ (see Eq. DE-3.1) in terms of the roots $s_\pm$. Hint: Check your answer with Octave's/Matlab's `residue` command.
Sol: First find the roots of the numerator of $Z(s)$ (the denominator of $Y(s)$):

$$s_\pm^2 + s_\pm + 1 = (s_\pm + 1/2)^2 + 3/4 = 0,$$

which is

$$s_\pm = \frac{-1 \pm \jmath\sqrt{3}}{2}.$$

Second form a partial fraction expansion

$$\frac{s}{1+s+s^2} = c_0 + \frac{c_+}{s - s_+} + \frac{c_-}{s - s_-} = \frac{s(c_+ + c_-) - (c_+s_- + c_-s_+)}{1 + s + s^2}.$$

Comparing the two sides shows that $c_0 = 0$. We also have two equations for the residues $c_+ + c_- = 1$ and $c_+s_- + c_-s_+ = 0$. The best way to solve this is to set up a matrix relation and take the inverse

$$\begin{bmatrix} 1 & 1 \\ s_- & s_+ \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{thus:} \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{s_+ - s_-} \begin{bmatrix} s_+ & -1 \\ -s_- & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which gives $c_\pm = \pm\frac{s_\pm}{s_+ - s_-}$ The denominator is $s_+ - s_- = \jmath\sqrt{3}$ and the numerator is $\pm 1 + \jmath\sqrt{3}$. Thus

$$c_\pm = \pm\frac{s_\pm}{s_+ - s_-} = \frac{1}{2}\Big(1 \pm \frac{\jmath}{\sqrt{3}}\Big).$$

As always, finding the coefficients is always the most difficult part. Using 2x2 matrix algebra automates the process. Always check your final result as correct. ■

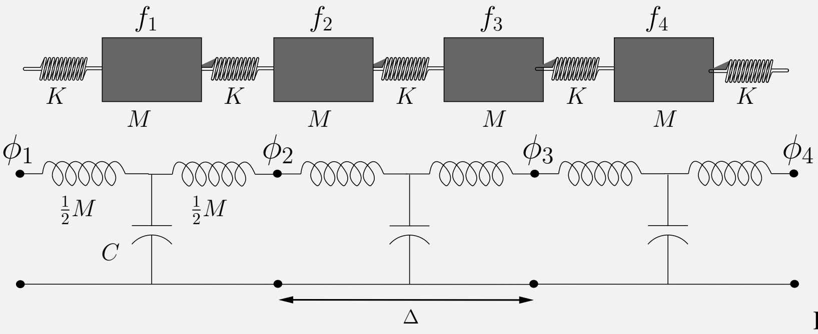
– 1.5: By applying Eq. NS-3.3 (page 135), find the inverse Laplace transform ($\mathcal{LT}^{-1}$). Use the residue form of the expression that you derived in question 1.4.
Sol:

$$z(t) = \frac{1}{2\pi j}\oint_{\mathcal{C}} Z(s)e^{st}ds.$$

were $\mathcal{C}$ is the Laplace contour which encloses the entire left-half s plane. Applying the CRT

$$z(t) = c_+e^{s_+t} + c_-e^{s_-t}.$$

where $s_\pm = -1/2 \pm \jmath\sqrt{3}/2$ and $c_\pm = 1/2 \pm \jmath/(2\sqrt{3})$. ■



Depiction of a train consisting of cars treated as masses M and linkages treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit for comparison. The masses are modeled as inductors and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a *Cauer synthesis*, is based on the ABCD transmission line method of Sec. 3.6 (p. 92).

3.3.3 Transmission-line analysis

Problem # *2:(14 pts)* **Train-mission-line** *We wish to model the dynamics of a freight train that has N such cars and study the* velocity transfer function *under various load conditions.*

As shown in Fig. 4.8.2, the train model consists of masses connected by springs.

Problem # *3: Transfer functions*

Use the ABCD method (see the discussion in Appendix B.3, p. 212) to find the matrix representation of the system of Fig. 4.8.2. Define the force on the nth train car $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$. Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrices $\mathcal{T}(s)$.

– 3.1: Find the elements of the ABCD matrix $\mathcal{T}$ for the single cell that relate the input node 1 to output node 2

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2 . \tag{DE-3.3}$$

Sol:

$$\begin{aligned} \mathcal{T} &= \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + s^2MC/2 & (sM)(1 + s^2MC/4) \\ sC & 1 + s^2MC/2 \end{bmatrix} \end{aligned} \tag{DE-3.4a}$$

■

– 3.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi jf$, $s_c = 2\pi jf_c$). The Nyquist wavelength sampling condition is $\lambda_c > 2\Delta$. It says the critical wavelength $\lambda_c > 2\Delta$. condition is $\lambda_c > 2\Delta$.[a] It says the critical wavelength $\lambda_c > 2\Delta$. Namely it is defined in terms the minimum number of cells 2Δ, per minimum wavelength λ_c.

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.
Proof: From the figure, the distance between cars $\Delta = c_o T_o$ [m], where

$$c_o = \frac{1}{\sqrt{MC}} \qquad \text{[m/s]}.$$

The cutoff frequency obeys $f_c\lambda_c = c_o$. The Nyquist critical wavelength is $\lambda_c = c_o/f_c > 2\Delta$. Therefore the Nyquist sampling condition is

$$f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}} \qquad \text{[rad/sec]}. \tag{DE-3.5}$$

Finally, $s_c = \jmath 2\pi f_c$.

Sol: The solution is a repeat what is summarized above: the system in Fig. 4.8.2 represents a transmission line having a wave speed of $c_o = 1/\sqrt{MC}$ and characteristic impedance $r_o = \sqrt{M/C}$. Each cell, composed of 2 masses M connected by one spring K, has length Δ.

We wish to define the *Nyquist frequency* f_c such that the wavelength $\lambda > 2\Delta$, where Δ is the cell length. Using the formula for the wavelength in terms of the wave velocity and frequency we find

$$\lambda = c_o/f_c = 2\Delta,$$

thus we conclude that

$$f < f_c = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}}. \tag{DE-3.6}$$

If we wish to have the system be accurate for a given frequency we may make the cell length Δ smaller, while keeping the velocity constant (MC is held constant). Thus the characteristic resistance [ohms/unit length] r_o must change as $f_c \to \infty$ and $\Delta \to 0$. We can either let $M \to \infty$ and $C \to 0$ (their product remains constant), or the other way around. In one case $r_o \to \infty$ and in the other case it goes to 0. ■

[a] The history of this relation has been traced back to 1841, as discussed by (Brillouin, 1953, Chap. I,II, Eq. 4.7).

– 3.3: Use the property of the Nyquist sampling frequency *$\omega < \omega_c$ (Eq. DE-3.4) to remove higher order powers of frequency*

$$1 + \cancelto{0}{\left(\frac{s}{s_c}\right)^2} \approx 1 \tag{DE-3.7}$$

to determine a band-limited approximation of $\mathcal{T}(s)$.

Sol:

$$\begin{aligned}\mathcal{T} &= \begin{bmatrix} 1 + 2(s/s_c)^2 & sM(1 + (s/s_c)^2) \\ sC & 1 + 2(s/s_c)^2 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & sM \\ sC & 1 \end{bmatrix}\end{aligned}$$

The approximation is highly accurate below the Nyquist cutoff frequency $s < s_c$. Given any desired frequency f, we can always make the cell size Δ smaller by decreasing M and C, while keeping $f < f_c$ and the cell velocity constant ($c_o = 1/\sqrt{MC}$). Thus the Nyquist condition represents a computational bound, not a physical limitation. ■

Problem # 4: *(4 pts) Now consider the cascade of N such $\mathcal{T}(s)$ matrices and perform an eigenanalysis.*

– *4.1: (4 pts) Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$ as functions of s/s_c.*
Sol: Matrix $\mathcal{T}(s)$ has eigenvalues

$$\lambda_\pm = 1 \mp 2s/s_c \approx e^{\pm 2s/s_c} = e^{\mp sT_c}.$$

From this we can interpret the eigenvalues as the cell delay $T_c = 2/s_c$.
The corresponding unnormalized eigenvectors are

$$\boldsymbol{E}_\pm = \begin{bmatrix} \mp\sqrt{M/C} \\ 1 \end{bmatrix},$$

where the characteristic impedance defined is $r_o = \sqrt{M/C}$. ■

Problem # 5: *(14 pts) Find the velocity transferfunction $H_{12}(s) = V_2/V_1|_{F_2=0}$.*

– *5.1: (3 pts) Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\mathcal{T}$ matrix, find*

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

Express H_{12} in terms of a residue expansion.
Sol: From Eq. DE-3.4a, $V_1 = sCF_2 - (s^2MC/2 + 1)V_2$. Since $F_2 = 0$

$$\frac{V_2}{V_1} = \frac{-1}{s^2MC/2+1} = \left(\frac{c_+}{s-s_+} + \frac{c_-}{s-s_-}\right)$$

having eigenfrequencies $s_\pm = \pm\jmath\sqrt{\frac{2}{2MC}} = \pm s_c$ and residues $c_\pm = \pm\jmath/\sqrt{2MC} = \pm s_c$. ■

– *5.2: (2 pts) Find $h_{21}(t) \leftrightarrow H_{21}(s)$.*
Sol:

$$h(t) = \oint_{\sigma_0-j\infty}^{\sigma_0+j\infty} \frac{e^{st}}{s^2MC/2+1}\frac{ds}{2\pi j} = c_+e^{-s_+t}u(t) + c_-e^{-s_-t}u(t).$$

The integral follows from the Cauchy Residue theorem (CRT). ■

– *5.3: (2 pts) What is the input impedance $Z_2 = F_2/V_2$, assuming $F_3 = -r_0V_3$?*
Sol: Starting from Eq. DE-3.4a find Z_2

$$Z_2(s) = \frac{F_2}{V_2} = T\begin{bmatrix} F \\ -V \end{bmatrix}_2 = \frac{-(1+s^2CM/2)r_0\cancel{V_2} - sM(1+s^2CM/4)\cancel{V_2}}{-sCr_0\cancel{V_2} - (1+s^2CM/2)\cancel{V_2}}$$

■

– 5.4: (5 pts) Simplify the expression for Z_2 as follows:

1. Assuming the *characteristic impedance* $r_0 = \sqrt{M/C}$,
2. terminate the system in r_0: $F_2 = -r_0 V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance r_0, the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $\mathcal{T}(s)$, it should be equal to r_0. Show that this is true for this setup.
Sol: Applying the Nyquist approximation (i.e., ignore second order frequency terms $(s/s_c)^2 \approx 0$)

$$
\begin{aligned}
Z_1(s) &= \frac{r_o(1+\cancelto{0}{s^2CM/2}) + sM(1+\cancelto{0}{s^2CM/4})}{r_o sC + (1+\cancelto{0}{s^2CM/2})} \\
&\approx \frac{r_o + sM}{1 + r_o sC} = \frac{MC}{MC} \cdot \frac{r_o + sM}{1 + r_o sC} = \frac{M}{C} \cdot \frac{r_o C + sMC}{M + r_o sMC} = r_o^2 \frac{r_o C + s/s_c}{M + r_o s/s_c} \\
&\approx r_o^2 \frac{r_o C + \cancelto{0}{s/s_c}}{M + r_o \cancelto{0}{s/s_c}} = r_o^3 \frac{C}{M} \\
&= r_o.
\end{aligned}
$$

We conclude that below the Nyquist cutoff frequency, as $N \to \infty$ the system equals a transmission line terminated by its characteristic impedance thus $Z_1(s) = r_o$. ■

– 5.5: (1 pts) State the ABCD matrix relationship between the first and Nth nodes in terms of the cell matrix. Write out the transfer function for one cell, H_{21}.
Sol:

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$$

Now use the formulae for the eigenvalues and vectors to obtain $\mathcal{T}$ for $N = 1$:

$$\mathcal{T} = E\Lambda E^{-1} = E \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} E^{-1}.$$

■

– 5.6: (1 pts) What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?
Sol:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \mathcal{T}^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}$$

along with the eigenvalue expansion

$$\mathcal{T}^N = E\Lambda^N E^{-1} = E \begin{bmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{bmatrix} E^{-1}.$$

where $\lambda_\pm^N = e^{\mp sNT_o}$. Recall that NT_o is the one way delay.
We conclude that as we add more cells, the delay linearly increases with N, since each eigenvalue represents the delay of one cell, and delay adds. ■