1.3 Problems NS-3

**Topic of this homework:** Pythagorean triplets, Pell’s equation, Fibonacci sequence

**Pythagorean triplets**

**Problem # 1** (: Euclid’s formula for the Pythagorean triplets $a, b, c$ is $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.)

- **1.1** (: What condition(s) must hold for $p$ and $q$ such that $a$, $b$, and $c$ are always positive and nonzero? )
  
  **Ans:**

- **1.2** (: Solve for $p$ and $q$ in terms of $a$, $b$, and $c$.)
  
  **Ans:**

**Pell’s equation:**

**Problem # 3** (: Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of )

$$x^2 - Ny^2 = 1.$$  

As shown in the text, the solutions $x_n, y_n$ for the case of $N = 2$ are given by the linear $2 \times 2$ matrix recursion

$$
\begin{bmatrix}
  x_{n+1} \\
  y_{n+1}
\end{bmatrix}
= 
1_j 
\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\begin{bmatrix} x_n \\ y_n \end{bmatrix}
$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1_j = \sqrt{-1} = e^{j\pi/2}$. It follows that the general solution to Pell’s equation for $N = 2$ is

$$
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{j\pi/2})^n 
\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.
$$
To calculate solutions to Pell's equation using the matrix equation above, we must calculate

\[ A^n = e^{\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \]

which becomes tedious for \( n > 2 \).

3.1 (*): Find the companion matrix and thus the matrix \( A \) that has the same eigenvalues as Pell's equation. Hint: Use Matlab's function \([E, \text{Lambda}] = \text{eig}(A)\) to check your results!

Ans:

3.2 (*): Solutions to Pell's equation were used by the Pythagoreans to explore the value of \( \sqrt{2} \). Explain why Pell's equation is relevant to \( \sqrt{2} \).

Ans:

3.3 (*): Find the first three values of \((x_n, y_n)^T\) by hand and show that they satisfy Pell's equation for \( N = 2 \). Ans: By hand, find the eigenvalues \( \lambda_\pm \) of the \( 2 \times 2 \) Pell's equation matrix.

Ans:
3.4 (By hand, show that the matrix of eigenvectors, \( E \), is ).

\[
E = [\vec{e}_+ \quad \vec{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}.
\]

**Ans:**

3.5 (Using the eigenvalues and eigenvectors you found for \( A \), verify that ).

\[
E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}
\]

**Ans:**

**The Fibonacci sequence**

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let \( x_n \) represent the Fibonacci sequence,

\[
x_{n+1} = x_n + x_{n-1},
\]

where the current input sample \( x_n \) is equal to the sum of the previous two inputs. This is a “discrete time” recurrence relationship. To solve for \( x_n \), we require some initial conditions. In this exercise, let us define \( x_0 = 1 \) and \( x_{n<0} = 0 \). This leads to the Fibonacci sequence \( \{1, 1, 2, 3, 5, 8, 13, \ldots \} \) for \( n = 0, 1, 2, 3, \ldots \).

Equation NS-3.1 is equivalent to the \( 2 \times 2 \) matrix equations

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

**Problem # 4** (Here we seek the general formula for \( x_n \). Like Pell’s equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast \( x_n \) as a \( 2 \times 2 \) matrix relationship and then proceed, as we did for the Pell case.).

3.4 (Show that the Fibonacci sequence \( x_n = x_{n-1} + x_{n-2} \) may be generated by ).

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

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**Ans:**

- **4.2** (What is the relationship between $y_n$ and $x_n$?)  
  **Ans:**

- **4.3** (Write a Matlab/Octave program to compute $x_n$ using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is $x_{40}$? *Note: Consider using the eigenanalysis of $A$ (text p. 60-61).*).  
  **Ans:**

- **4.4** (Using the eigenanalysis of the matrix $A$ (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence).  
  
  \[
  x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].
  \]  
  (NS-3.4)

- **4.5** (What are the eigenvalues $\lambda_\pm$ of the matrix $A$?).  
  **Ans:**

- **4.6** (How is the formula for $x_n$ related to these eigenvalues? *Hint: Find the eigenvectors.*).  
  **Ans:**
- **4.7**: What happens to each of the two terms.

\[
\left(\frac{1 \pm \sqrt{5}}{2}\right)^{n+1}
\]

**Ans:**

- **4.8**: What happens to the ratio \(x_{n+1}/x_n\).  **Ans:**

**Problem # 5**: Replace the Fibonacci sequence with.

\[
x_n = \frac{x_{n-1} + x_{n-2}}{2},
\]

such that the value \(x_n\) is the average of the previous two values in the sequence.

- **5.1**: What matrix \(A\) is used to calculate this sequence?  **Ans:**

- **5.2**: Modify your computer program to calculate the new sequence \(x_n\). What happens as \(n \to \infty\)?  **Ans:**

- **5.3**: What are the eigenvalues of your new \(A\)? How do they relate to the behavior of \(x_n\) as \(n \to \infty\)? *Hint: Use the same method as with the Fibonacci example.*  **Ans:**