1.3 Problems NS-3

Topic of this homework: Pythagorean triplets, Pell’s equation, Fibonacci sequence

Pythagorean triplets

Problem # 1  (Euclid’s formula for the Pythagorean triplets \( a, b, c \) is \( a = p^2 - q^2, b = 2pq, \) and \( c = p^2 + q^2 \).)

– 1.1 (What condition(s) must hold for \( p \) and \( q \) such that \( a, b, \) and \( c \) are always positive and nonzero? ).

Sol: \( p > q > 0 \) (strictly greater than)

– 1.2 (Solve for \( p \) and \( q \) in terms of \( a, b, \) and \( c \)).

Sol:

Method 1: Given \( a, c \), one may find \( p, q \) via matrix operations by solving the nonlinear system of equations for \( p, q \).

First solve linear system of equations for \( p^2, q^2 \):

\[
\begin{bmatrix}
  a \\
  c \\
\end{bmatrix} = \begin{bmatrix}
  1 & -1 \\
  1 & 1 \\
\end{bmatrix} \begin{bmatrix}
  p^2 \\
  q^2 \\
\end{bmatrix}
\]

Inverting this 2x2 matrix gives (the determinant \( \Delta = 2 \))

\[
\begin{bmatrix}
  p^2 \\
  q^2 \\
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
  1 & 1 \\
  -1 & 1 \\
\end{bmatrix} \begin{bmatrix}
  a \\
  c \\
\end{bmatrix}.
\]

Thus \( p = \pm \sqrt{(a + c)/2}, q = \pm \sqrt{(c - a)/2} \).

Method 2: The algebraic approach is:

\[
\begin{align*}
  a + c &= (p^2 - q^2) + (p^2 + q^2) = 2p^2 \\
  -a + c &= -(p^2 - q^2) + (p^2 + q^2) = 2q^2,
\end{align*}
\]

Thus \( p = \sqrt{(a + c)/2}, q = \sqrt{(c - a)/2}, \) where \( p, q \in \mathbb{N} \).

Method 1 seems more “transparent” than Method 2.
Pell’s equation:

Problem # 3 (: Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that \( \sqrt{2} \in \mathbb{I} \). We seek integer solutions of ).

\[ x^2 - Ny^2 = 1. \]

As shown in the text, The solutions \( x_n, y_n \) for the case of \( N = 2 \) are given by the linear \( 2 \times 2 \) matrix recursion

\[
\begin{bmatrix}
  x_{n+1} \\
  y_{n+1}
\end{bmatrix} = 1 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\
  y_n \end{bmatrix}
\]

with \( [x_0, y_0]^T = [1, 0]^T \) and \( 1j = \sqrt{-1} = e^{j\pi/2} \). It follows that the general solution to Pell’s equation for \( N = 2 \) is

\[
\begin{bmatrix} x_n \\
  y_n \end{bmatrix} = (e^{j\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\
  y_0 \end{bmatrix}.
\]

To calculate solutions to Pell’s equation using the matrix equation above, we must calculate

\[
A^n = e^{j\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{j\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},
\]

which becomes tedious for \( n > 2 \).

– 3.1 (: Find the companion matrix and thus the matrix \( A \) that has the same eigenvalues as Pell’s equation. Hint: Use Matlab’s function \([E,\Lambda] = eig(A)\) to check your results! ). Sol: The companion matrix is

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]

– 3.2 (: Solutions to Pell’s equation were used by the Pythagoreans to explore the value of \( \sqrt{2} \). Explain why Pell’s equation is relevant to \( \sqrt{2} \). ). Sol: As discussed in Sec. 2.5.2, as the iteration \( n \) increases, the ratio of the \( x_n/y_n \) approaches \( \sqrt{2} \).

– 3.3 (: Find the first three values of \( (x_n, y_n)^T \) by hand and show that they satisfy Pell’s equation for \( N = 2 \). Sol: See class notes (slide 9.4.2) for this calculation. ) By hand, find the eigenvalues \( \lambda_{\pm} \) of the \( 2 \times 2 \) Pell’s equation matrix.

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.
\]

Sol: The eigenvalues are given by the roots of the equation \( (1 - \lambda)^2 = 2 \). Thus \( \lambda_{\pm} = 1 \pm \sqrt{2} = \{2.4142, -0.4142\} \)

– 3.4 (: By hand, show that the matrix of eigenvectors, \( E \), is ).

\[
E = [\vec{e}_+ \mid \vec{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}.
\]

Sol:
– 3.5 (Using the eigenvalues and eigenvectors you found for $A$, verify that).

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

**Sol:** Using the formula for a matrix inverse, we find

$$E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

Thus

$$E^{-1}AE = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

$$= \frac{-1}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} (-\sqrt{2} + 2) & (\sqrt{2} + 2) \\ (-\sqrt{2} + 1) & (\sqrt{2} + 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \sqrt{2} & 0 \\ 0 & 1 + \sqrt{2} \end{bmatrix} = \Lambda$$

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let $x_n$ represent the Fibonacci sequence,

$$x_{n+1} = x_n + x_{n-1},$$

where the current input sample $x_n$ is equal to the sum of the previous two inputs. This is a “discrete time” recurrence relationship. To solve for $x_n$, we require some initial conditions.

In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence \{1, 1, 2, 3, 5, 8, 13, \ldots\} for $n = 0, 1, 2, 3, \ldots$

Equation NS-3.1 is equivalent to the $2 \times 2$ matrix equations

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad (\text{NS-3.2})$$

**Problem # 4:** Here we seek the general formula for $x_n$. Like Pell’s equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast $x_n$ as a $2 \times 2$ matrix relationship and then proceed, as we did for the Pell case.

– 4.1 (Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by).

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  \quad (\text{NS-3.3})

**Sol:** Given the Matrix Eigenequation, powers of the eigen equation $A^n = E\Lambda^n E^{-1}$. The final solution is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = E \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n E^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$  \quad (\text{NS-3.4})
4.2 (What is the relationship between \( y_n \) and \( x_n \)?)  
**Sol:** This equation says that 
\[
x_n = x_{n-1} + y_{n-1} \quad \text{and} \quad y_n = x_{n-1}.
\]
The latter equation may be rewritten as \( y_{n-1} = x_{n-2} \). Thus 
\[
x_n = x_{n-1} + x_{n-2}
\]
as requested.

4.3 (Write a Matlab/Octave program to compute \( x_n \) using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is \( x_{40} \)? 
**Note:** Consider using the eigenanalysis of \( A \) (text p. 60-61).)  
**Sol:** You can try something like: 
```matlab
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D
\( n \);
xn = xy(1);
```
Given the initial conditions we defined, \( x_{40} = 165, 580, 141 \).

4.4 (Using the eigenanalysis of the matrix \( A \) (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence.) 
\[
x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].
\]
(NS-3.5)

4.5 (What are the eigenvalues \( \lambda \pm \) of the matrix \( A \)?)  
**Sol:** The eigenvalues of the Fibonacci matrix are given by 
\[
det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,
\]
thus \( \lambda \pm = \frac{1 \pm \sqrt{5}}{2} = [1.618, -0.618] \).

4.6 (How is the formula for \( x_n \) related to these eigenvalues? Hint: Find the eigenvectors.)  
**Sol:** The eigenvectors (determined from the equation \( (A - \lambda \pm I)\vec{\epsilon} = \vec{0} \), and normalized to 1) are given by 
\[
\vec{\epsilon}_+ = \begin{bmatrix} \sqrt{\lambda_+^2 + 1} \\ 1 \end{bmatrix}, \quad \vec{\epsilon}_- = \begin{bmatrix} \sqrt{\lambda_-^2 + 1} \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} \vec{\epsilon}_+ & \vec{\epsilon}_- \end{bmatrix}
\]

From the eigenanalysis, we find that 
\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = E \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Solving for \( x_n \) we find that 
\[
x_n = \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \left( \lambda_+^n e_{11}e_{22} - \lambda_-^n e_{12}e_{21} \right)
= \frac{1}{\sqrt{\lambda_+^2 + 1} \sqrt{\lambda_-^2 + 1}} \left( \lambda_+^n \frac{\lambda_+}{\sqrt{\lambda_+^2 + 1}} - \lambda_-^n \frac{\lambda_-}{\sqrt{\lambda_-^2 + 1}} \right)
= \frac{1}{\sqrt{5}} \left( \lambda_+^n - \lambda_-^n \right)
\]
1.3. PROBLEMS NS-3

- 4.7 (What happens to each of the two terms).

\[
\left[\left(1 \pm \sqrt{5}\right)/2\right]^{n+1}?
\]

**Sol:** \( [(1 + \sqrt{5})/2]^{n+1} \rightarrow 0 \) and \( [(1 + \sqrt{5})/2]^{n+1} \rightarrow \infty \)

- 4.8 (What happens to the ratio \( x_{n+1}/x_n \).) **Sol:** \( x_{n+1}/x_n \rightarrow (1 + \sqrt{5})/2, \) because \( (1 - \sqrt{5})/2)^n \rightarrow 0 \) as \( n \rightarrow \infty \) thus for large \( n \), \( x_n \approx [(1 + \sqrt{5})/2]^{n+1} \).

**Problem # 5** (Replace the Fibonacci sequence with).

\[
x_n = \frac{x_{n-1} + x_{n-2}}{2},
\]

such that the value \( x_n \) is the average of the previous two values in the sequence.

- 5.1 (What matrix \( A \) is used to calculate this sequence?). **Sol:**

\[
A = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
1 & 0
\end{bmatrix}
\]

- 5.2 (Modify your computer program to calculate the new sequence \( x_n \).) What happens as \( n \rightarrow \infty \)? **Sol:** As \( n \rightarrow \infty \), \( x_n \rightarrow 2/3 \)

- 5.3 (What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \rightarrow \infty \)?) **Hint:** Use the same method as with the Fibonacci example. **Sol:** The eigenvalues are \( \lambda_+ = 1 \) and \( \lambda_- = -0.5 \). From Eq. ??, the expression for \( A^n \) is

\[
A^n = (E \Lambda E^{-1})^n = E \Lambda^n E^{-1} = \begin{bmatrix}
\lambda_+ & 0 \\
0 & \lambda_-
\end{bmatrix}^n = \begin{bmatrix}
\lambda_+^n & 0 \\
0 & \lambda_-^n
\end{bmatrix}.
\]

The solution is the sum of two sequences, one a constant and the other an oscillation that quickly fades. As \( n \rightarrow \infty \), \( \lambda_+^n = 1^n \rightarrow 1 \) and \( \lambda_-^n = (-1/2)^n \rightarrow 0 \). The solution becomes

\[
x_n = \frac{2}{3} \left[ \lambda_+^n - \lambda_-^n \right] = \frac{2}{3} \left[ 1^n - (-1)^n \right] \rightarrow \frac{2}{3}.
\]

*