Sol: Note: The number 1 should not be circled since it is *not* a prime. ■

– *8.2: What is the largest number you need to consider before only primes remain? Look up the definition of the Matlab/Octave* floor *function (e.g,* $\lfloor \pi \rfloor = 3$*).*
Sol: $\lfloor \sqrt{50} \rfloor = \lfloor 7.0711 \rfloor = 7$. ■

– *8.3: Generalize: For* $n = 1, \ldots, N$*, what is the largest number you need to consider before only the primes remain?*
Sol: floor($\sqrt{N}$) ■

– *8.4: Write each of these numbers as a product of primes: 22, 30, 34, 43, 44, 48, 49.*
Sol: *22*= $2 \cdot 11 = \pi_1 \pi_5$
30= $2 \cdot 3 \cdot 5 = \pi_1 \pi_2 \pi_3$
34= $2 \cdot 17 = \pi_1 \pi_7$
43= π_{14}
44= $4 \cdot 11 = \pi_1^2 \pi_5$
48= $4 \cdot 12 = 4^2 \cdot 3 = \pi_1^4 \pi_2$
49= $7^2 = \pi_4^2$
■

– *8.5: Find the largest prime* $\pi_k \leq 100$*. Do not use Matlab/Octave other than to check your answer.* Hint: Write the numbers starting with 100 and count backward: 100, 99, 98, 97, ... Cross off the even numbers, leaving $99, 97, 95, \ldots$. Pull out a factor (only one is necessary to show that it is not prime).
Sol: 99=11*9, $\pi_{25} = 97$. ■

– *8.6: Find the largest prime* $\pi_k \leq 1000$*. Do not use Matlab/Octave other than to check your answer.*
Sol: Write out the numbers starting with 1000 and counting backwards: 1000, 999, 998, 997, $\cdots$. Cross off the even numbers, leaving $999, 997, 995, \cdots$. Pull out a factor (only one is necessary to show that it is not prime). $9 \cdot 111, 997 = \pi_{168}, 5 \cdot 199 = \pi_3 \cdot \pi_{46}$. ■

– *8.7: Explain why* $\pi_k^{-s} = e^{-s \ln \pi_k}$.
Sol: This follows from the identify $z^a = e^{a \ln z}$ with $a, z \in \mathbb{C}$. ■

Problem # *9:CFA of ratios of large primes*

– *9.1:* ***(4pts)*** *Expand 23/7 as a continued fraction. Express your answer in bracket notation* (e.g., $\pi = [3., 7, 16, \cdots]$). Show your work. **Sol:** $23/7 = (21+2)/7 = 3 + 2/7 = 3 + 1/(6+1)/2 = 3 + 1/(6+1/2)$. In bracket notation $23/7 = [3., 6, 2]$. Matlab gives `rat(23/7)` $= 3 + 1/(4 + 1/(-2))$, or [1., 4, -2] because rounding $7/2$ can be taken as either 3+1/2 or 4-1/2. ■

– 9.2:Starting from the primes below 10^6*, form the CFA of* π_j/π_k *with* $j = 78498$ *and* $k < j$*.*
Sol: First generate 10^6 primes with the matlab command π=`primes(11+1e6)`.
The length of π is $j = 78499$, $\pi(j) = 1,000,003$, $\pi(j-1) = 999,983$ and $\pi(j-2) = 999,979$.
Let the target fraction be

$$T = \frac{\pi(end-1)}{\pi(end-2)} = \frac{999983}{999979} = 1.000004000084002.$$

Finding the CFA of T gives

$$rat(T) = 1 + 1/249995 = [1; 249995].$$

Factoring this integer gives `factor(249995)=5*49999`. ■

– 9.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? **Sol:** *The CFA terminates in only one term, as in the above example.* ■

– 9.4: ***(1pts)*** *Try the Matlab/Octave functions* `rats(23/7)`*,* `rats(3.2857)`*, and* `rats(3.2856)`*. What an you conclude?*
Sol: This function is similar to the CFA but uses rounding rather than truncation arithmatic. `rats(3.2857)`=32857/10000 but `rats(23/7)=23/7` because it rounds to 23/7, whereas `rats(3.2856)` = 4107/1250 because it does not. ■

– 9.5: ***(2pts)*** *Can* $\sqrt{2}$ *be represented as a finite continued fraction? Why or why not?*
Sol: No, because it is irrational. ■

– 9.6: ***(2pts)*** *What is the CFA for* $\sqrt{2} - 1$*?*

$$\text{Hint:} \qquad \sqrt{2} + 1 = \frac{1}{\sqrt{2} - 1} = [2; 2, 2, 2, \cdots].$$

Sol: $1 + \sqrt{2} = 2 + 1/(2 + 1/(2 + \cdots))$ or $[2., 2, 2, 2, \cdots]$, thus

$$\sqrt{2} - 1 = [2., 2, 2, 2, \cdots] - 2 = 0 + 1/(2 + 1/(2 + 1/(2 + \cdots))).$$

■

– 9.7: Show that

$$\frac{1}{1 - \sqrt{a}} = a^{\frac{11}{2}} + a^{\frac{9}{2}} + a^{\frac{7}{2}} + a^{\frac{5}{2}} + a^{\frac{3}{2}} + \sqrt{a} + a^5 + a^4 + a^3 + a^2 + a + 1 = 1 - a^6$$

```
syms a,b
b= taylor(1/( 1-sqrt(a) ))
simplify((1-sqrt(a))*b) = 1-a^6
```

Use symbolic analysis to show this, then explain. **Sol:** This seems like a very unlikely relationship. Unexpectedly the coefficents of this expansion are all 1, leading to a is a sixth degree polynomial. It is obvioiusly related to the six complex roots of unity. Thus we may find the companion matrix, followed by an eigen solution. This seems to be a Taylor expansion of six roots of unity, expressed in terms of removable singularities. See Cotes Theorem (1716) (Stillwell, 2010, p. 289). ■

1.3 Problems NS-3

Topic of this homework: Pythagorean triplets, Pell's equation, Fibonacci sequence

Pythagorean triplets

Problem # 1: *Euclid's formula for the Pythagorean triplets a, b, c is $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.*

– 1.1: What condition(s) must hold for p and q such that a, b, and c are always positive and nonzero?

Sol: $p > q > 0$ (strictly greater than) ■

– 1.2: Solve for p and q in terms of a, b, and c.

Sol:

Method 1: Given a, c, one may find p, q via matrix operations by solving the *nonlinear system of equations* for p, q.

First solve linear system of equations for p^2, q^2:

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ q^2 \end{bmatrix}$$

Inverting this 2x2 matrix gives (the determinant $\Delta = 2$)

$$\begin{bmatrix} p^2 \\ q^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.$$

Thus $p = \pm\sqrt{(a+c)/2}$, $q = \pm\sqrt{(c-a)/2}$.

Method 2: The algebraic approach is:

$$a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2$$

$$-a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2,$$

Thus $p = \sqrt{(a+c)/2}$, $q = \sqrt{(c-a)/2}$, where $p, q \in \mathbb{N}$.

Method 1 seems more "transparent" than Method 2. ■

Problem # *2: The ancient Babylonians (ca. 2000 BCE) cryptically recorded (a, c) pairs of numbers on a clay tablet, archeologically denoted* Plimpton-322 *(see 2.8).*

– 2.1: Find p and q for the first five pairs of a and c shown here from Plimpton-322.

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

Find a formula for a in terms of p and q.
Sol:

$$(a, c) = (119, 169) \qquad (p, q) = \pm(12, 5)$$
$$(a, c) = (3367, 4825) \qquad (p, q) = \pm(64, 27)$$
$$(a, c) = (4601, 6649) \qquad (p, q) = \pm(75, 32)$$
$$(a, c) = (12709, 18541) \qquad (p, q) = \pm(125, 54)$$
$$(a, c) = (65, 97) \qquad (p, q) = \pm(9, 4)$$

■

– 2.2: Based on Euclid's formula, show that $c > (a, b)$.

Sol: $c - a = (p^2 + q^2) - (p^2 - q^2) = 2q^2$
Because $2q^2$ is always positive, $c > a$
$c - b = (p^2 + q^2) - 2pq = (p - q)^2 > 0$
Note that by the definition of $p, q \in \mathbb{N}$, $p > q$. ■

– 2.3: What happens when $c = a$?

Sol: Then its not a triangle since $b = 0$. The triangle is degenerate. ■

– 2.4: Is $b + c$ a perfect square? Discuss.

Sol: $b + c = p^2 + 2pq + q^2 = (p + q)^2$. Since p and q are integers, $b + c$ will always be a perfect square ($\sqrt{b + c}$ will always be an integer).
■

Pell's equation:

Problem # *3: Pell's equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of*

$$x^2 - Ny^2 = 1.$$

As shown in Sec. 2.5.2, the solutions x_n, y_n for the case of $N = 2$ are given by the linear 2×2 matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1\jmath = \sqrt{-1} = e^{\jmath\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\jmath\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{\jmath\pi n/2}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}^n = e^{\jmath\pi n/2}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix},$$

which becomes tedious for $n > 2$.

– 3.1: Find the companion matrix and thus the matrix A that has the same eigenvalues as Pell's equation. Hint: Use Matlab's function `[E,Lambda] = eig(A)` to check your results!

Sol: The companion matrix is

$$A = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$$

■

– 3.2: Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.

Sol: As discussed in Sec. 2.5.2, as the iteration n increases, the ratio of the x_n/y_n approaches $\sqrt{2}$. ■

– 3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell's equation for $N = 2$. **Sol:** *See class notes (slide 9.4.2) for this calculation.* ■ *By hand, find the eigenvalues $\lambda_\pm$ of the 2×2 Pell's equation matrix*

$$A = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}.$$

Sol: The eigenvalues are given by the roots of the equation $(1 - \lambda_\pm)^2 = 2$. Thus $\lambda_\pm = 1 \pm \sqrt{2} = \{2.1412, -.4142\}$ ■

– 3.4: By hand, show that the matrix of eigenvectors, E, is

$$E = [\vec{e}_+ \quad \vec{e}_-] = \frac{1}{\sqrt{3}}\begin{bmatrix}-\sqrt{2} & \sqrt{2}\\ 1 & 1\end{bmatrix}.$$

Sol:

The eigenvectors $\vec{e}_\pm$ may be found by solving

$$A\begin{bmatrix}e_1\\ e_2\end{bmatrix} = \lambda_\pm\begin{bmatrix}e_1\\ e_2\end{bmatrix} \to (A - \lambda_\pm I)\begin{bmatrix}e_1\\ e_2\end{bmatrix} = 0$$

For λ_+, this gives

$$0 = \begin{bmatrix}1-(1+\sqrt{2}) & 2\\ 1 & 1-(1+\sqrt{2})\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} = \begin{bmatrix}-\sqrt{2} & 2\\ 1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}$$

which gives the relation between the elements of $\vec{e}_+$, e_1, e_2, as $e_1 = \sqrt{2}e_2$.
The eigenvectors are defined to be unit length and orthogonal, namely

1. $||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1$
2. $\vec{e}_+ \cdot \vec{e}_- = 0$.

Once we normalize $\vec{e}_+$ to have unit length, we obtain the first eigenvector

$$\vec{e}_+ = \frac{1}{\sqrt{3}}\begin{bmatrix}-\sqrt{2}\\ 1\end{bmatrix}$$

Repeating this for λ_- gives

$$\vec{e}_- = \frac{1}{\sqrt{3}}\begin{bmatrix}\sqrt{2}\\ 1\end{bmatrix}$$

Thus, the matrix of eigenvalues is

$$E = \frac{1}{\sqrt{3}}\begin{bmatrix}-\sqrt{2} & \sqrt{2}\\ 1 & 1\end{bmatrix}$$

■

– *3.5: Using the eigenvalues and eigenvectors you found for A, verify that*

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Sol: Using the formula for a matrix inverse, we find

$$E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

Thus

$$\begin{aligned} E^{-1}AE &= \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix} \\ &= \frac{-1}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} (-\sqrt{2}+2) & (\sqrt{2}+2) \\ (-\sqrt{2}+1) & (\sqrt{2}+1) \end{bmatrix} \\ &= \begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix} = \Lambda \end{aligned}$$

■

– *3.6: Once you have diagonalized A, use your results for E and Λ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell's equation with $N = 2$.*

Sol: $x_{10} = -3363$ and $y_{10} = -2378$. Note this formulation gives the negative solution, but since the values for $n = 10$ are real, when they are squared in Pell's equation, it makes no difference whether they are negative or positive. ■

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let x_n represent the Fibonacci sequence,

$$x_{n+1} = x_n + x_{n-1}, \qquad \text{(NS-3.1)}$$

where the current input sample x_n is equal to the sum of the previous two inputs. This is a "discrete time" recurrence relationship. To solve for x_n, we require some initial conditions. In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ for $n = 0, 1, 2, 3, \ldots$.

Equation NS-3.1 is equivalent to the 2×2 matrix equations

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \qquad \text{(NS-3.2)}$$

Problem # *4: Here we seek the general formula for x_n. Like Pell's equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast x_n as a 2×2 matrix relationship and then proceed, as we did for the Pell case.*

– *4.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by*

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \qquad \text{(NS-3.3)}$$

Sol: Given the Matrix Eigenequation, powers of the eigen equation $\mathbf{A}^n = \mathbf{E}\Lambda^n\mathbf{E}^{-1}$. The final solution is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \boldsymbol{E} \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n \boldsymbol{E}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \qquad \text{(NS-3.4)}$$

■

– *4.2: What is the relationship between y_n and x_n?*

Sol: This equation says that $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_{n-1}$. The latter equation may be rewritten as $y_{n-1} = x_{n-2}$. Thus

$$x_n = x_{n-1} + x_{n-2}$$

as requested. ■

– *4.3: Write a Matlab/Octave program to compute x_n using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is x_{40}?* Note: Consider using the eigenanalysis of A, described by Eq. NS-2 2.18 of the text.

Sol: You can try something like:

```
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D^n*inv(E)*[1; 0];
xn = xy(1);
```

Given the initial conditions we defined, $x_{40} = 165,580,141$. ■

– *4.4: Using the eigenanalysis of the matrix A (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence*

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{NS-3.5}$$

– *4.5: What are the eigenvalues $\lambda_\pm$ of the matrix A?*

Sol: The eigenvalues of the Fibonacci matrix are given by

$$det\begin{bmatrix}1-\lambda & 1\\ 1 & -\lambda\end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda-1/2)^2 - 5/4 = 0,$$

thus $\lambda_\pm = \frac{1\pm\sqrt{5}}{2} = [1.618, -0.618]$. ■

– *4.6: How is the formula for x_n related to these eigenvalues?* Hint: Find the eigenvectors.

Sol: The eigenvectors (determined from the equation $(A - \lambda_\pm I)\vec{e}_\pm = \vec{0}$, and normalized to 1) are given by

$$\vec{e}_+ = \begin{bmatrix}\frac{\lambda_+}{\sqrt{\lambda_+^2+1}}\\ \frac{1}{\sqrt{\lambda_+^2+1}}\end{bmatrix} \qquad \vec{e}_- = \begin{bmatrix}\frac{\lambda_-}{\sqrt{\lambda_-^2+1}}\\ \frac{1}{\sqrt{\lambda_-^2+1}}\end{bmatrix} \qquad E = [\vec{e}_+ \quad \vec{e}_-]$$

From the eigenanalysis, we find that

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = E\begin{bmatrix}\lambda_+^n & 0\\ 0 & \lambda_-^n\end{bmatrix}E^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{bmatrix}\begin{bmatrix}\lambda_+^n & 0\\ 0 & \lambda_-^n\end{bmatrix}\frac{1}{(e_{11}e_{22} - e_{12}e_{21})}\begin{bmatrix}e_{22} & -e_{12}\\ -e_{21} & e_{11}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}.$$

Solving for x_n we find that

$$\begin{aligned} x_n &= \frac{1}{(e_{11}e_{22} - e_{12}e_{21})}\left(\lambda_+^n e_{11}e22 - \lambda_-^n e_{12}e_{21}\right)\\ &= \frac{1}{\frac{\sqrt{5}}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}}\left[\lambda_+^n\left(\frac{\lambda_+^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}\right) - \lambda_-^n\left(\frac{\lambda_-^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}\right)\right]\\ &= \frac{1}{\sqrt{5}}\left[\lambda_+^{n+1} - \lambda_-^{n+1}\right] \end{aligned}$$

■

– *4.7: What happens to each of the two terms*

$$\left[\left(1\pm\sqrt{5}\right)/2\right]^{n+1}?$$

Sol: $[(1+\sqrt{5})/2]^{n+1} \to 0$ and $[(1+\sqrt{5})/2]^{n+1} \to \infty$ ■