CHAPTER 1. NUMBER SYSTEMS

- 8.2: What is the largest number you need to consider before only primes remain? Look up the definition of the Matlab/Octave floor function (e.g., \( \lfloor \pi \rfloor = 3 \)).

\[ \lfloor \sqrt{50} \rfloor = \lfloor 7.0711 \rfloor = 7. \]

- 8.3: Generalize: For \( n = 1, \ldots, N \), what is the largest number you need to consider before only the primes remain?

\[ \text{sol:} \ \lfloor \sqrt{N} \rfloor. \]

- 8.4: Write each of these numbers as a product of primes: 22, 30, 34, 43, 44, 48, 49.

\[ \begin{align*}
22 &= 2 \cdot 11 = \pi_1\pi_5 \\
30 &= 2 \cdot 3 \cdot 5 = \pi_1\pi_2\pi_3 \\
34 &= 2 \cdot 17 = \pi_1\pi_7 \\
43 &= \pi_{14} \\
44 &= 4 \cdot 11 = \pi_2^2\pi_5 \\
48 &= 4 \cdot 12 = 4^2 \cdot 3 = \pi_1^4\pi_2 \\
49 &= 7^2 = \pi_4^2 \\
\end{align*} \]

- 8.5: Find the largest prime \( \pi_k \leq 100 \). Do not use Matlab/Octave other than to check your answer. Hint: Write the numbers starting with 100 and count backward: 100, 99, 98, 97, \ldots. Cross off the even numbers, leaving 99, 97, 95, \ldots. Pull out a factor (only one is necessary to show that it is not prime).

\[ \text{sol:} \ 99 = 11 \cdot 9, \ \pi_{25} = 97. \]

- 8.6: Find the largest prime \( \pi_k \leq 1000 \). Do not use Matlab/Octave other than to check your answer.

\[ \text{sol:} \ \text{Write out the numbers starting with 1000 and counting backwards: 1000, 999, 998, 997, \ldots. Cross off the even numbers, leaving 999, 997, 995, \ldots. Pull out a factor (only one is necessary to show that it is not prime).} \ 9 \cdot 111, 997 = \pi_{168}, 5 \cdot 199 = \pi_3 \cdot \pi_{46}. \]

- 8.7: Explain why \( \pi_k^{-s} = e^{-s \ln \pi_k} \).

\[ \text{sol:} \ \text{This follows from the identity} \ z^a = e^{a \ln z} \text{ with} \ a, z \in \mathbb{C}. \]

Problem # 9: CFA of ratios of large primes

- 9.1: (4pts) Expand 23/7 as a continued fraction. Express your answer in bracket notation (e.g., \( \pi = [3, 7, 16, \ldots] \)). Show your work. \( \text{sol:} \ 23/7 = (21 + 2)/7 = 3 + 2/7 = 3 + 1/(6 + 1/2) = 3 + 1/(6 + 1/2). \) In bracket notation \( 23/7 = [3, 6, 2] \). Matlab gives \( \text{rat}(23/7) = 3 + 1/(4 + 1/(-2)), \) or \([1, 4, -2]\) because rounding \( 7/2 \) can be taken as either \( 3+1/2 \) or \( 4-1/2 \).
1.2. PROBLEMS NS-2

9.2: Starting from the primes below $10^6$, form the CFA of $\pi_j/\pi_k$ with $j = 78498$ and $k < j$.

**Sol:** First generate $10^6$ primes with the matlab command $\pi =$ primes $(11 + 1e6)$.
The length of $\pi$ is $j = 78499$, $\pi(j) = 1,000,003$, $\pi(j-1) = 999,983$ and $\pi(j-2) = 999,979$.
Let the target fraction be
$$T = \frac{\pi(\text{end} - 1)}{\pi(\text{end} - 2)} = \frac{999983}{999979} = 1.000004000084002.$$ Finding the CFA of $T$ gives
$$\text{rat}(T) = 1 + 1/249995 = [1; 249995].$$
Factoring this integer gives $\text{factor}(249995) = 5 \times 49999$.

9.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? **Sol:** The CFA terminates in only one term, as in the above example.

9.4: (1pts) Try the Matlab/Octave functions $\text{rats}(23/7)$, $\text{rats}(3.2857)$, and $\text{rats}(3.2856)$. What an you conclude? **Sol:** This function is similar to the CFA but uses rounding rather than truncation arithmetic. $\text{rats}(3.2857) =$ 32857/10000 but $\text{rats}(23/7) = 23/7$ because it rounds to 23/7, whereas $\text{rats}(3.2856) = 4107/1250$ because it does not.

9.5: (2pts) Can $\sqrt{2}$ be represented as a finite continued fraction? Why or why not? **Sol:** No, because it is irrational.

9.6: (2pts) What is the CFA for $\sqrt{2} - 1$?

**Hint:** $\sqrt{2} + 1 = \frac{1}{\sqrt{2} - 1} = [2; 2, 2, 2, \cdots].$

**Sol:** $1 + \sqrt{2} = 2 + 1/(2 + 1/(2 + \cdots))$ or $[2, 2, 2, 2, \cdots]$, thus
$$\sqrt{2} - 1 = [2, 2, 2, 2, \cdots] - 2 = 0 + 1/(2 + 1/(2 + 1/(2 + \cdots))).$$

9.7: Show that
$$\frac{1}{1 - \sqrt{a}} = a^\frac{1}{2} + a^\frac{3}{2} + a^2 + a^\frac{5}{2} + a^3 + \sqrt{a} + a^5 + 2a^2 + a + 1 = 1 - a^6$$

**Sol:** This seems like a very unlikely relationship. Unexpectedly the coefficients of this expansion are all 1, leading to a is a sixth degree polynomial. It is obviously related to the six complex roots of unity. This may be found by an eigen solution. It seems to be a Taylor expansion of six roots of unity, expressed in terms of removable singularities. See Cotes Theorem (1716) (Stillwell, 2010, p. 289).
CHAPTER 1. NUMBER SYSTEMS

1.3 Problems NS-3

**Topic of this homework:** Pythagorean triplets, Pell’s equation, Fibonacci sequence

**Pythagorean triplets**

**Problem # 1: Euclid’s formula for the Pythagorean triplets** \(a, b, c\) is \(a = p^2 - q^2\), \(b = 2pq\), and \(c = p^2 + q^2\).

- 1.1: What condition(s) must hold for \(p\) and \(q\) such that \(a\), \(b\), and \(c\) are always positive and nonzero?

  **Sol:** \(p > q > 0\) (strictly greater than)

- 1.2: Solve for \(p\) and \(q\) in terms of \(a\), \(b\), and \(c\).

  **Sol:**

  **Method 1:** Given \(a, c\), one may find \(p, q\) via matrix operations by solving the nonlinear system of equations for \(p, q\).

  First solve linear system of equations for \(p^2, q^2\):

  \[
  \begin{bmatrix}
  a \\
  c
  \end{bmatrix} = \begin{bmatrix}
  1 & -1 \\
  1 & 1
  \end{bmatrix} \begin{bmatrix}
  p^2 \\
  q^2
  \end{bmatrix}
  \]

  Inverting this 2x2 matrix gives (the determinant \(\Delta = 2\))

  \[
  \begin{bmatrix}
  p^2 \\
  q^2
  \end{bmatrix} = \frac{1}{2} \begin{bmatrix}
  1 & 1 \\
  -1 & 1
  \end{bmatrix} \begin{bmatrix}
  a \\
  c
  \end{bmatrix}.
  \]

  Thus \(p = \pm \sqrt{(a + c)/2}\), \(q = \pm \sqrt{(c-a)/2}\).

  **Method 2:** The algebraic approach is:

  \[
  a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2
  \]

  \[
  -a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2,
  \]

  Thus \(p = \sqrt{(a + c)/2}\), \(q = \sqrt{(c-a)/2}\), where \(p, q \in \mathbb{N}\).

  Method 1 seems more “transparent” than Method 2. ■
Problem # 2: The ancient Babylonians (ca. 2000 BCE) cryptically recorded \((a, c)\) pairs of numbers on a clay tablet, archeologically denoted Plimpton-322 (see 2.8).

– 2.1: Find \(p\) and \(q\) for the first five pairs of \(a\) and \(c\) shown here from Plimpton-322.

<table>
<thead>
<tr>
<th>(a)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
<tr>
<td>65</td>
<td>97</td>
</tr>
</tbody>
</table>

Find a formula for \(a\) in terms of \(p\) and \(q\).

**Sol:**

\[
(a, c) = (119, 169) \quad (p, q) = \pm (12, 5)
\]

\[
(a, c) = (3367, 4825) \quad (p, q) = \pm (64, 27)
\]

\[
(a, c) = (4601, 6649) \quad (p, q) = \pm (75, 32)
\]

\[
(a, c) = (12709, 18541) \quad (p, q) = \pm (125, 54)
\]

\[
(a, c) = (65, 97) \quad (p, q) = \pm (9, 4)
\]

– 2.2: Based on Euclid’s formula, show that \(c > (a, b)\).

**Sol:**

\[
c - a = (p^2 + q^2) - (p^2 - q^2) = 2q^2
\]

Because \(2q^2\) is always positive, \(c > a\)

\[
c - b = (p^2 + q^2) - 2pq = (p - q)^2 > 0
\]

Note that by the definition of \(p, q \in \mathbb{N}\), \(p > q\).

– 2.3: What happens when \(c = a\)?

**Sol:** Then it's not a triangle since \(b = 0\). The triangle is degenerate.

– 2.4: Is \(b + c\) a perfect square? Discuss.

**Sol:** \(b + c = p^2 + 2pq + q^2 = (p + q)^2\). Since \(p\) and \(q\) are integers, \(b + c\) will always be a perfect square (\(\sqrt{b + c}\) will always be an integer).

Pell’s equation:

Problem # 3: Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that \(\sqrt{2} \in \mathbb{I}\). We seek integer solutions of

\[
x^2 - Ny^2 = 1.
\]

As shown in Sec. 2.5.2, the solutions \(x_n, y_n\) for the case of \(N = 2\) are given by the linear 2 \(\times\) 2 matrix recursion

\[
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} = 1_j \begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
x_n \\
y_n
\end{bmatrix}
\]

with \([x_0, y_0]^T = [1, 0]^T\) and \(1_j = \sqrt{-1} = e^{j\pi/2}\). It follows that the general solution to Pell’s equation for \(N = 2\) is

\[
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix} = (e^{j\pi/2})^n \begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}.
\]
To calculate solutions to Pell’s equation using the matrix equation above, we must calculate

\[ A^n = e^{\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = e^{\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \]

which becomes tedious for \( n > 2 \).

**3.1:** Find the companion matrix and thus the matrix \( A \) that has the same eigenvalues as Pell’s equation. Hint: Use Matlab’s function \([E, \text{Lambda}] = \text{eig}(A)\) to check your results!

**Sol:** The companion matrix is

\[ A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \]

**3.2:** Solutions to Pell’s equation were used by the Pythagoreans to explore the value of \( \sqrt{2} \). Explain why Pell’s equation is relevant to \( \sqrt{2} \).

**Sol:** As discussed in Sec. 2.5.2, as the iteration \( n \) increases, the ratio of the \( x_n/y_n \) approaches \( \sqrt{2} \).

**3.3:** Find the first three values of \((x_n, y_n)^T\) by hand and show that they satisfy Pell’s equation for \( N = 2 \).

**Sol:** See class notes (slide 9.4.2) for this calculation.

**3.4:** By hand, find the eigenvalues \( \lambda_{\pm} \) of the \( 2 \times 2 \) Pell’s equation matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \]

The eigenvalues are given by the roots of the equation \((1 - \lambda_{\pm})^2 = 2\). Thus \( \lambda_{\pm} = 1 \pm \sqrt{2} = \{2.1412, -0.4142\} \).

**3.5:** By hand, show that the matrix of eigenvectors, \( E \), is

\[ E = \begin{bmatrix} \vec{e}_+ & \vec{e}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}. \]

The eigenvectors \( \vec{e}_{\pm} \) may be found by solving

\[ A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda_{\pm} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \rightarrow (A - \lambda_{\pm} I) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \]

For \( \lambda_+ \), this gives

\[ 0 = \begin{bmatrix} 1 - (1 + \sqrt{2}) & 2 \\ 1 & 1 - (1 + \sqrt{2}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \]

which gives the relation between the elements of \( \vec{e}_+, e_1, e_2 \), as \( e_1 = \sqrt{2}e_2 \).

The eigenvectors are defined to be unit length and orthogonal, namely

1. \( ||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1 \)
2. \( \vec{e}_+ \cdot \vec{e}_- = 0 \).

Once we normalize \( \vec{e}_+ \) to have unit length, we obtain the first eigenvector

\[ \vec{e}_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \]

Repeating this for \( \lambda_- \) gives

\[ \vec{e}_- = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \]

Thus, the matrix of eigenvalues is

\[ E = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}. \]
1.3. PROBLEMS NS-3

– 3.5: Using the eigenvalues and eigenvectors you found for \( A \), verify that

\[
E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}
\]

**Sol:** Using the formula for a matrix inverse, we find

\[
E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}
\]

Thus

\[
E^{-1}AE = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]

\[
= \frac{-1}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} (-\sqrt{2} + 2) & (\sqrt{2} + 2) \\ (-\sqrt{2} + 1) & (\sqrt{2} + 1) \end{bmatrix}
\]

\[
= \frac{1 - \sqrt{2}}{1 + \sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \Lambda
\]

– 3.6: Once you have diagonalized \( A \), use your results for \( E \) and \( \Lambda \) to solve for the \( n = 10 \) solution \((x_{10}, y_{10})^T\) to Pell’s equation with \( N = 2 \).

**Sol:** \( x_{10} = -3363 \) and \( y_{10} = -2378 \). Note this formulation gives the negative solution, but since the values for \( n = 10 \) are real, when they are squared in Pell’s equation, it makes no difference whether they are negative or positive.

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let \( x_n \) represent the Fibonacci sequence,

\[
x_{n+1} = x_n + x_{n-1}, \quad (NS-3.1)
\]

where the current input sample \( x_n \) is equal to the sum of the previous two inputs. This is a “discrete time” recurrence relationship. To solve for \( x_n \), we require some initial conditions. In this exercise, let us define \( x_0 = 1 \) and \( x_{n<0} = 0 \). This leads to the Fibonacci sequence \( \{1, 1, 2, 3, 5, 8, 13, \ldots\} \) for \( n = 0, 1, 2, 3, \ldots \).

Equation NS-3.1 is equivalent to the \( 2 \times 2 \) matrix equations

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad (NS-3.2)
\]

**Problem # 4:** Here we seek the general formula for \( x_n \). Like Pell’s equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast \( x_n \) as a \( 2 \times 2 \) matrix relationship and then proceed, as we did for the Pell case.

– 4.1: Show that the Fibonacci sequence \( x_n = x_{n-1} + x_{n-2} \) may be generated by

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (NS-3.3)
\]

**Sol:** Given the Matrix Eigenequation, powers of the eigen equation \( A^n = \Lambda \eta^n E^{-1} \). The final solution is

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = E \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n E^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \quad (NS-3.4)
\]
4.2: What is the relationship between \( y_n \) and \( x_n \)?

**Sol:** This equation says that \( x_n = x_{n-1} + y_{n-1} \) and \( y_n = x_{n-1} \). The latter equation may be rewritten as \( y_{n-1} = x_{n-2} \). Thus

\[
x_n = x_{n-1} + x_{n-2}
\]
as requested.

4.3: Write a Matlab/Octave program to compute \( x_n \) using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is \( x_{40} \)? Note: Consider using the eigenanalysis of \( A \), described by Eq. NS-2.18 of the text.

**Sol:** You can try something like:

```matlab
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D
xn = xy(1); 
```

Given the initial conditions we defined, \( x_{40} = 165,580,141 \).

4.4: Using the eigenanalysis of the matrix \( A \) (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence

\[
x_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}
\]

(NS-3.5)

4.5: What are the eigenvalues \( \lambda \) of the matrix \( A \)?

**Sol:** The eigenvalues of the Fibonacci matrix are given by

\[
\det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,
\]

thus \( \lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2} = [1.618, -0.618] \).

4.6: How is the formula for \( x_n \) related to these eigenvalues? Hint: Find the eigenvectors.

**Sol:** The eigenvectors (determined from the equation \( (A - \lambda I) \vec{e}_{\pm} = 0 \), and normalized to 1) are given by

\[
\vec{e}_+ = \begin{bmatrix} \lambda^+ \\ \sqrt{\lambda^+_1^2 + \lambda^+_2^2} \end{bmatrix} \quad \vec{e}_- = \begin{bmatrix} \lambda^- \\ \sqrt{\lambda^-_1^2 + \lambda^-_2^2} \end{bmatrix} \quad E = [\vec{e}_+ \quad \vec{e}_-]
\]

From the eigenanalysis, we find that

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = E \begin{bmatrix} \lambda_n^+ & 0 \\ 0 & \lambda_n^- \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \lambda_n^+ & 0 \\ 0 & \lambda_n^- \end{bmatrix} \frac{1}{(e_{11} e_{22} - e_{12} e_{21})} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Solving for \( x_n \), we find that

\[
x_n = \frac{1}{(e_{11} e_{22} - e_{12} e_{21})} \left( \lambda_n^+ e_{11} e_{22} - \lambda_n^- e_{12} e_{21} \right)
\]

\[
= \frac{1}{\sqrt{5}} \lambda_n^+ \left( \frac{\lambda_n^+}{\sqrt{(\lambda_n^+)^2 + 1}} \right) - \lambda_n^- \left( \frac{\lambda_n^-}{\sqrt{(\lambda_n^-)^2 + 1}} \right)
\]

\[
\approx \frac{1}{\sqrt{5}} \left( (\lambda_n^+)^{n+1} - (\lambda_n^-)^{n+1} \right)
\]

4.7: What happens to each of the two terms

\[
\left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} \quad \text{and} \quad \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}
\]

**Sol:** \( [(1 + \sqrt{5})/2]^{n+1} \to 0 \) and \( [(1 - \sqrt{5})/2]^{n+1} \to \infty \).