1.3 Problems NS-3

Topic of this homework: Pythagorean triplets, Pell's equation, Fibonacci sequence

Pythagorean triplets

Problem # *1: Euclid's formula for the Pythagorean triplets a, b, c is $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.*

– 1.1: What condition(s) must hold for p and q such that a, b, and c are always positive and nonzero?

Sol: $p > q > 0$ (strictly greater than) ■

– 1.2: Solve for p and q in terms of a, b, and c.

Sol:

Method 1: Given a, c, one may find p, q via matrix operations by solving the *nonlinear system of equations* for p, q.

First solve linear system of equations for p^2, q^2:

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ q^2 \end{bmatrix}$$

Inverting this 2x2 matrix gives (the determinant $\Delta = 2$)

$$\begin{bmatrix} p^2 \\ q^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.$$

Thus $p = \pm\sqrt{(a + c)/2}$, $q = \pm\sqrt{(c - a)/2}$.

Method 2: The algebraic approach is:

$$a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2$$

$$-a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2,$$

Thus $p = \sqrt{(a + c)/2}$, $q = \sqrt{(c - a)/2}$, where $p, q \in \mathbb{N}$.

Method 1 seems more "transparent" than Method 2. ■

Problem # 2: *The ancient Babylonians (ca. 2000 BCE) cryptically recorded (a, c) pairs of numbers on a clay tablet, archeologically denoted* Plimpton-322 *(see 2.7).*

– *2.1: Find p and q for the first five pairs of a and c shown here from Plimpton-322.*

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

Find a formula for a in terms of p and q.
Sol:

$$(a,c) = (119, 169) \qquad (p,q) = \pm(12,5)$$
$$(a,c) = (3367, 4825) \qquad (p,q) = \pm(64,27)$$
$$(a,c) = (4601, 6649) \qquad (p,q) = \pm(75,32)$$
$$(a,c) = (12709, 18541) \qquad (p,q) = \pm(125,54)$$
$$(a,c) = (65, 97) \qquad (p,q) = \pm(9,4)$$

■

– *2.2: Based on Euclid's formula, show that $c > (a, b)$.*

Sol: $c - a = (p^2 + q^2) - (p^2 - q^2) = 2q^2$
Because $2q^2$ is always positive, $c > a$
$c - b = (p^2 + q^2) - 2pq = (p - q)^2 > 0$
Note that by the definition of $p, q \in \mathbb{N}$, $p > q$. ■

– *2.3: What happens when $c = a$?*

Sol: Then its not a triangle since $b = 0$. The triangle is degenerate. ■

– *2.4: Is $b + c$ a perfect square? Discuss.*

Sol: $b + c = p^2 + 2pq + q^2 = (p + q)^2$. Since p and q are integers, $b + c$ will always be a perfect square ($\sqrt{b + c}$ will always be an integer).
■

Pell's equation:

Problem # 3: *Pell's equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of*

$$x^2 - Ny^2 = 1.$$

As shown in Sec. 2.5.2, the solutions x_n, y_n for the case of $N = 2$ are given by the linear 2×2 matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1\jmath = \sqrt{-1} = e^{\jmath\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\jmath\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{\jmath\pi n/2}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}^n = e^{\jmath\pi n/2}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix},$$

which becomes tedious for $n > 2$.

– 3.1: Find the companion matrix and thus the matrix A that has the same eigenvalues as Pell's equation. Hint: Use Matlab's function `[E,Lambda] = eig(A)` to check your results!

Sol: The companion matrix is

$$A = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$$

■

– 3.2: Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.

Sol: As discussed in Sec. 2.5.2, as the iteration n increases, the ratio of the x_n/y_n approaches $\sqrt{2}$. ■

– 3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell's equation for $N = 2$. **Sol:** *See class notes (slide 9.4.2) for this calculation.* ■ *By hand, find the eigenvalues $\lambda_\pm$ of the 2×2 Pell's equation matrix*

$$A = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}.$$

Sol: The eigenvalues are given by the roots of the equation $(1-\lambda_\pm)^2 = 2$. Thus $\lambda_\pm = 1 \pm \sqrt{2} = \{2.1412, -.4142\}$ ■

– 3.4: By hand, show that the matrix of eigenvectors, E, is

$$E = [\vec{e}_+ \quad \vec{e}_-] = \frac{1}{\sqrt{3}}\begin{bmatrix}-\sqrt{2} & \sqrt{2}\\ 1 & 1\end{bmatrix}.$$

Sol:

The eigenvectors $\vec{e}_\pm$ may be found by solving

$$A\begin{bmatrix}e_1\\e_2\end{bmatrix} = \lambda_\pm\begin{bmatrix}e_1\\e_2\end{bmatrix} \rightarrow (A-\lambda_\pm I)\begin{bmatrix}e_1\\e_2\end{bmatrix} = 0$$

For λ_+, this gives

$$0 = \begin{bmatrix}1-(1+\sqrt{2}) & 2\\ 1 & 1-(1+\sqrt{2})\end{bmatrix}\begin{bmatrix}e_1\\e_2\end{bmatrix} = \begin{bmatrix}-\sqrt{2} & 2\\ 1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}e_1\\e_2\end{bmatrix}$$

which gives the relation between the elements of $\vec{e}_+$, e_1, e_2, as $e_1 = \sqrt{2}e_2$.
The eigenvectors are defined to be unit length and orthogonal, namely

1. $||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1$
2. $\vec{e}_+ \cdot \vec{e}_- = 0.$

Once we normalize $\vec{e}_+$ to have unit length, we obtain the first eigenvector

$$\vec{e}_+ = \frac{1}{\sqrt{3}}\begin{bmatrix}-\sqrt{2}\\ 1\end{bmatrix}$$

Repeating this for λ_- gives

$$\vec{e}_- = \frac{1}{\sqrt{3}}\begin{bmatrix}\sqrt{2}\\ 1\end{bmatrix}$$

Thus, the matrix of eigenvalues is

$$E = \frac{1}{\sqrt{3}}\begin{bmatrix}-\sqrt{2} & \sqrt{2}\\ 1 & 1\end{bmatrix}$$

■

– 3.5: Using the eigenvalues and eigenvectors you found for A, verify that

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Sol: Using the formula for a matrix inverse, we find

$$E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

Thus

$$\begin{aligned} E^{-1}AE &= \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix} \\ &= \frac{-1}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} (-\sqrt{2}+2) & (\sqrt{2}+2) \\ (-\sqrt{2}+1) & (\sqrt{2}+1) \end{bmatrix} \\ &= \begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix} = \Lambda \end{aligned}$$

■

– 3.6: Once you have diagonalized A, use your results for E and Λ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell's equation with $N = 2$.

Sol: $x_{10} = -3363$ and $y_{10} = -2378$. Note this formulation gives the negative solution, but since the values for $n = 10$ are real, when they are squared in Pell's equation, it makes no difference whether they are negative or positive. ■

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let x_n represent the Fibonacci sequence,

$$x_{n+1} = x_n + x_{n-1}, \tag{NS-3.1}$$

where the current input sample x_n is equal to the sum of the previous two inputs. This is a "discrete time" recurrence relationship. To solve for x_n, we require some initial conditions. In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ for $n = 0, 1, 2, 3, \ldots$.

Equation NS-3.1 is equivalent to the 2×2 matrix equations

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{NS-3.2}$$

Problem # 4: *Here we seek the general formula for x_n. Like Pell's equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast x_n as a 2×2 matrix relationship and then proceed, as we did for the Pell case.*

– 4.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{NS-3.3}$$

Sol: Given the Matrix Eigenequation, powers of the eigen equation $\mathbf{A}^n = \mathbf{E}\Lambda^n\mathbf{E}^{-1}$. The final solution is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \boldsymbol{E} \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n \boldsymbol{E}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{NS-3.4}$$

■

– *4.2: What is the relationship between y_n and x_n?*
Sol: This equation says that $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_{n-1}$. The latter equation may be rewritten as $y_{n-1} = x_{n-2}$. Thus

$$x_n = x_{n-1} + x_{n-2}$$

as requested. ■

– *4.3: Write a Matlab/Octave program to compute x_n using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is x_{40}?* Note: Consider using the eigenanalysis of A, described by Eq. NS-2 2.18 of the text.
Sol: You can try something like:

```
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D^n*inv(E)*[1; 0];
xn = xy(1);
```

Given the initial conditions we defined, $x_{40} = 165,580,141$. ■

– *4.4: Using the eigenanalysis of the matrix A (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence*

$$x_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]. \tag{NS-3.5}$$

– *4.5: What are the eigenvalues $\lambda_\pm$ of the matrix A?*

Sol: The eigenvalues of the Fibonacci matrix are given by

$$det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,$$

thus $\lambda_\pm = \frac{1 \pm \sqrt{5}}{2} = [1.618, -0.618]$. ■

– *4.6: How is the formula for x_n related to these eigenvalues?* Hint: Find the eigenvectors.

Sol: The eigenvectors (determined from the equation $(A - \lambda_\pm I)\vec{e}_\pm = \vec{0}$, and normalized to 1) are given by

$$\vec{e}_+ = \begin{bmatrix} \frac{\lambda_+}{\sqrt{\lambda_+^2+1}} \\ \frac{1}{\sqrt{\lambda_+^2+1}} \end{bmatrix} \qquad \vec{e}_- = \begin{bmatrix} \frac{\lambda_-}{\sqrt{\lambda_-^2+1}} \\ \frac{1}{\sqrt{\lambda_-^2+1}} \end{bmatrix} \qquad E = [\vec{e}_+ \quad \vec{e}_-]$$

From the eigenanalysis, we find that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = E \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving for x_n we find that

$$\begin{aligned} x_n &= \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \left(\lambda_+^n e_{11} e22 - \lambda_-^n e_{12} e_{21} \right) \\ &= \frac{1}{\frac{\sqrt{5}}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}} \left[\lambda_+^n \left(\frac{\lambda_+^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}} \right) - \lambda_-^n \left(\frac{\lambda_-^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}} \right) \right] \\ &= \frac{1}{\sqrt{5}} \left[\lambda_+^{n+1} - \lambda_-^{n+1} \right] \end{aligned}$$

■

– *4.7: What happens to each of the two terms*

$$\left[\left(1 \pm \sqrt{5} \right) /2 \right]^{n+1}?$$

Sol: $[(1+\sqrt{5})/2]^{n+1} \to 0$ and $[(1+\sqrt{5})/2]^{n+1} \to \infty$ ■

– *4.8: What happens to the ratio x_{n+1}/x_n?*
Sol: $x_{n+1}/x_n \to (1+\sqrt{5})/2$, because $\left((1-\sqrt{5})/2\right)^n \to 0$ as $n \to \infty$ thus for large n, $x_n \approx [(1+\sqrt{5})/2]^{n+1}$. ■

Problem # *5: Replace the Fibonacci sequence with*

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

such that the value x_n is the average of the previous two values in the sequence.

– *5.1: What matrix A is used to calculate this sequence?*
Sol:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

■

– *5.2: Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?*
Sol: As $n \to \infty$, $x_n \to 2/3$ ■

– *5.3: What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$?* Hint: You can expect the closed-form expression for x_n to be similar to Eq. NS-3.4.
Sol: The eigenvalues are $\lambda_+ = 1$ and $\lambda_- = -0.5$. From Eq. NS-2 2.18, the expression for A^n is

$$A^n = (E\Lambda E^{-1})^n = E\Lambda^n E^{-1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n = \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix}.$$

The solution is the sum of two sequences, one a constant and the other an oscillation that quickly fades. As $n \to \infty$, $\lambda_+^n = 1^n \to 1$ and $\lambda_-^n = (-1/2)^n \to 0$. The solution becomes

$$x_n = \frac{2}{3}\left[\lambda_+^n - \lambda_-^n\right] = \frac{2}{3}\left[1^n - (-1)^n\right] \to \frac{2}{3}.$$

■

Problem # *6: Consider the expression*

$$\sum_1^N f_n^2 = f_N f_{N+1}.$$

– *6.1: Find a formula for f_n that satisfies this relationship. Hint: It holds for only the Fibonacci recursion formula.*
Sol: Write this out for N and $N-1$:

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_{N-1}^2 + f_N^2 &= f_N f_{N+1} \\ f_1^2 + f_2^2 + \cdots + f_{N-1}^2 \qquad &= f_{N-1} f_N \end{aligned}$$

Subtracting gives

$$\begin{aligned} f_N^{\not 2} &= \not{f_N} f_{N+1} - f_{N-1}\not{f_N} = \not{f_N}(f_{N+1} - f_{N-1}) \\ f_N &= f_{N+1} - f_{N-1} \end{aligned}$$

Thus the relation only holds for the Fibonacci recursion formula. ■

CFA as a matrix recursion

Problem # 7: *The CFA may be writen as a matrix recursion. For this we adopt a special notation, unlike other matrix notations,[a] with $k \in \mathbb{N}$:*

$$\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & \lfloor x_k \rfloor \\ 0 & \frac{1}{x_k - \lfloor x_k \rfloor} \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k .$$

This equation says that $n_{k+1} = \lfloor x_k \rfloor$ and $x_{k+1} = 1/(x_k - \lfloor x_k \rfloor)$. It does *not* mean that $n_{k+1} = \lfloor x_k \rfloor x_k$, as would be implied by standard matrix notation. The lower equation says that $r_k = x_k - \lfloor x_k \rfloor$ is the *remainder*—namely, $x_k = \lfloor x - k \rfloor + r_k$ (Octave/Matlab's `rem(x,floor(x))` function), also known as `mod(x,y)`.

– *7.1: Start with $n_0 = 0 \in \mathbb{N}$, $x_0 \in \mathbb{I}$, $n_1 = \lfloor x_0 \rfloor \in \mathbb{N}$, $r_1 = x - \lfloor x \rfloor \in \mathbb{I}$, and $x_1 = 1/r_1 \in \mathbb{I}$, $r_n \neq 0$. For $k = 1$ this generates on the left the next CFA parameter $n_2 = \lfloor x_1 \rfloor$ and $x_2 = 1/r_2 = 1/(x_0 - \lfloor x_0 \rfloor)$ from n_0 and x_0. Find $[n, x]^T_{k+1}$ for $k = 2, 3, 4, 5$.*

Sol: If $x_0 = \pi$, then $n_1 = \lfloor \pi \rfloor = 3$, $r_1 = \pi - n_1 = 0.14159 \cdots$, and $x_1 = 1/r_1 \approx 7.06$:

$$\begin{bmatrix} 3 \\ 7.06251 \end{bmatrix}_1 = \begin{bmatrix} 0 & \lfloor \pi \rfloor \\ 0 & \frac{1}{\pi - \lfloor \pi \rfloor} \end{bmatrix} \begin{bmatrix} 0 \\ \pi \end{bmatrix}_0$$

and for $n = 2$

$$\begin{bmatrix} 7 \\ 15.99659 \end{bmatrix}_2 = \begin{bmatrix} 7 \\ \frac{1}{0.06251} \end{bmatrix}_2 = \begin{bmatrix} 0 & 7 \\ 0 & \frac{1}{7.0625-7} \end{bmatrix} \begin{bmatrix} 3 \\ 7.06251 \end{bmatrix}_1$$

For $n = 3$, $\pi_3 = [n_1; n_2, n_3] = [3; 7, 15]$. Continuing $n_4 = \lfloor 1.003418 \rfloor = 1$ and $n_5 = 292$. ■

[a] This notation is highly nonstandard due to the nonlinear operations. The matrix elements are *derived* from the vector rather than multiplying them. These calculation may be done with the help of Matlab/Octave.