Chapter 4

Vector differential equations

4.1 Problems VC-1

4.1.1 Topics of this homework:

Vector algebra and fields in $\mathbb{R}^3$, gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss's (divergence) and Stokes's (curl) laws, system classification (postulates).

4.1.2 Scalar fields and the ∇ operator

Problem # *1: Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).*

– 1.1: Find the gradient of $T(\boldsymbol{x})$ and make a sketch of T and the gradient.

Sol: $\nabla(x^2 + y) = 2x\hat{\mathbf{x}} + \hat{\mathbf{y}}$. The temperature is quadratic in x and linear in y, which has the shape of a trough in x, linearly increasing in y. In the y ($\hat{\mathbf{y}}$) direction the gradient is constant, and in the $\hat{\mathbf{x}}$ direction, it is linear, and goes through zero at $x = 0$, with $T(0) = 0$. Skiing in the y direction would be a constant ride of slope 1. If the snow had no friction, you would accelerate, but the terminal velocity would be due to the friction of the snow on the skis. Along the x direction, you would accelerate, at first, coming down, and at $x = 0$ you would stop accelerating, and begin slow down. This would be a more interesting problem if you treated it in terms of the forces on the skis and included friction as well as gravity. ■

– 1.2: Compute $\nabla^2 T(\boldsymbol{x})$ to determine whether $T(\boldsymbol{x})$ satisfies Laplace's equation.

Sol: Forming this operation we find that

$$\frac{\partial^2}{\partial x^2}x^2 + \cancel{\frac{\partial^2}{\partial y^2}}y = 2.$$

So $T(\boldsymbol{x})$ does not satisfy laplace's equation, rather it satisfies the Poisson equation $\nabla^2 T(\boldsymbol{x}) = 2$. ■

– 1.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.

Sol: The iso-potential contours are the concave parabolas $y = T_0 - x^2$. ■

– 1.4: The heat flux[1] *is defined as $\boldsymbol{J}(x, y) = -\kappa(x, y)\nabla T$, where $\kappa(x, y)$ is a constant that denotes thermal conductivity at the point (x, y). Given that $\kappa = 1$ everywhere (the medium is homogeneous), plot the vector $\boldsymbol{J}(x, y) = -\nabla T$ at $x = 2$, $y = 1$. Be clear about the origin,*

[1] The heat flux is proportional to the change in temperature times the thermal conductivity κ of the medium.

direction, and length of your result.

Sol: $\mathbf{J} = \nabla T = -2x\hat{\mathbf{x}} + -\hat{\mathbf{y}}$ thus $-\kappa\nabla T(2,1) = \mathbf{J} = -(4\hat{\mathbf{x}} + \hat{\mathbf{y}})$, which has a length of $\sqrt{17}$ and is pointed $1\sqrt{1}$ unit down and $4/\sqrt{17}$ units to the left. ■

– 1.5: Find the vector $\perp$ to $\nabla T(x, y)$—that is, tangent to the iso-temperature contours. Hint: Sketch it for one (x, y) point (e.g., $2, 1$) and then generalize.

Sol: We may invoke the third dimension $\hat{\mathbf{z}}$ to generate this vector: $\pm\hat{\mathbf{z}} \times \nabla T = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \pm 1 \\ 2x & 1 & 0 \end{bmatrix} = \mp(1\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}})$. Alternatively, rotate ∇T by $\pm\pi/2$ in the (x, y) plane. ■

– 1.6: The thermal resistance R_T is defined as the potential drop ΔT over the magnitude of the heat flux $|\boldsymbol{J}|$. At a single point the thermal resistance is

$$R_T(x, y) = -\nabla T/|\boldsymbol{J}|.$$

How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?

Sol: $R_T(x, y) = 1/\kappa(x, y)$. In general, resistance is the reciprocal of conductivity (conductance). This is true for electrical and acoustic systems as well. ■

Problem # *2: Acoustic wave equation*

Note: In this problem, we will work in the frequency domain.

– 2.1: The basic equations of acoustics in one dimension are

$$-\frac{\partial}{\partial x}\mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\frac{\partial}{\partial x}\mathcal{V} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

Here $\mathcal{P}(x, \omega)$ is the pressure (in the frequency domain), $\mathcal{V}(x, \omega)$ is the volume velocity (the integral of the velocity over the wavefront with area A), $s = \sigma + \omega \jmath$, $\rho_o = 1.2$ is the specific density of air, $\eta_o = 1.4$, and P_o is the atmospheric pressure (i.e., 10^5 Pa). Note that the pressure field $\mathcal{P}$ is a scalar (pressure does not have direction), while the volume velocity field $\mathcal{V}$ is a vector (velocity has direction).

We can generalize these equations to three dimensions using the ∇ operator

$$-\nabla\mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\nabla \cdot \mathcal{V} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

– 2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure $\mathcal{P}$,

$$\nabla^2\mathcal{P} = \frac{s^2}{c_0^2}\mathcal{P},$$

where c_0 is a constant representing the speed of sound.

Sol: We wish to remove $\mathcal{V}$ from the two equations, to obtain a single equation in pressure. If we take the partial wrt x of the pressure equation, and then substitute the velocity equation, to remove the velocity:

$$\nabla^2\mathcal{P} = -\rho_o s\nabla \cdot \mathcal{V} = \frac{s^2\rho_o}{\eta_o P_o}\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}$$

■

– 2.3: What is c_0 in terms of η_0, ρ_0, and P_0?

Sol: Comparing the last two terms from the previous solution we see that

$$c_o = \sqrt{\eta_o P_o/\rho_o}.$$

■

– 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., $dx/dt \leftrightarrow sX(s)$]. For your notation, define the time–domain signal using a lowercase letter, $p(x,y,z,t) \leftrightarrow \mathcal{P}$.

Sol:

$$\nabla^2 p(x,y,z,t) = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} p(x,y,z,t)$$

■

4.1.3 Vector fields and the ∇ operator

4.1.4 Vector algebra

Problem # 3: *Let $\boldsymbol{R}(x,y,z) \equiv x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$.*

– 3.1: If a, b, and c are constants, what is $\boldsymbol{R}(x,y,z)\cdot\boldsymbol{R}(a,b,c)$?

Sol: Using the formula for a scalar dot product:

$$\mathbf{R}(x,y,z)\cdot\mathbf{R}(a,b,c) \equiv [x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}]\cdot[a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}]$$

$$= x(t)a + y(t)b + z(t)c.$$

■

– 3.2: If a, b, and c are constants, what is $\frac{d}{dt}\left(\boldsymbol{R}(x,y,z)\cdot\boldsymbol{R}(a,b,c)\right)$?

Sol: $\left(a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) + c\frac{d}{dt}z(t)\right)$. ■

Problem # 4: *Find the divergence and curl of the following vector fields:*

– 4.1: $\boldsymbol{v} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$

Sol: $\nabla\cdot\mathbf{v} = 0$, $\nabla\times\mathbf{v} = 0$ ■

– 4.2: $\boldsymbol{v}(x,y,z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$

Sol: $\nabla\cdot\mathbf{v} \equiv \partial_x x + \partial_y xy + \partial_z z^2 = 1 + x + 2z$ $\nabla\times\mathbf{v} \equiv \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial_x & \partial_y & \partial_z\\ x & xy & z^2\end{vmatrix} = (0-0)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (y-0)\hat{\mathbf{z}} = y\hat{\mathbf{z}}$

■

– 4.3: $\boldsymbol{v}(x,y,z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$

Sol: Divergence: $\partial_x x + \partial_y xy + \partial_z\log(z) = 1 + x + 1/z$, Curl: $\hat{\mathbf{x}}\left(\partial_y\log(z) - \partial_z xy\right) + \hat{\mathbf{y}}\left(\partial_z x - \partial_x\log(z)\right) + \hat{\mathbf{z}}\left(\partial_x xy - \partial_y x\right) = \hat{\mathbf{z}}y$ ■

– 4.4: $\boldsymbol{v}(x,y,z) = \nabla(1/x + 1/y + 1/z)$

Sol: First find $\mathbf{v} = -(\hat{\mathbf{x}}/x^2 + \hat{\mathbf{y}}/y^2 + \hat{\mathbf{z}}/z^2)$. Divergence of $\mathbf{v}$: $-(\partial_x 1/x^2 + \partial_y 1/y^2 + \partial_z 1/z^2) = 2(1/x^3 + 1/y^3 + 1/z^3)$, Curl of $\mathbf{v}$: 0, because the curl of the gradient is always zero. ■

4.1.5 Vector and scalar field identities

Problem # 5: *Find the divergence and curl of the following vector fields:*

– 5.1: $\boldsymbol{v} = \nabla\phi$, where $\phi(x,y) = xe^y$

Sol: $\nabla\times\nabla\phi = 0$, and $\nabla^2\phi = xe^y$ ■

– 5.2: $\boldsymbol{v} = \nabla\times\boldsymbol{A}$, where $\boldsymbol{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Sol: $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, and $\nabla\times(\nabla\times\mathbf{A}) = 0$ ■

– 5.3: $\boldsymbol{v} = \nabla\times\boldsymbol{A}$, where $\boldsymbol{A} = y\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Sol: $\nabla\cdot(\nabla\times\mathbf{A}) = 0$, and $\nabla\times(\nabla\times\mathbf{A}) = -2\hat{\mathbf{y}}$ ■

– *5.4: For any differentiable vector field $\boldsymbol{V}$, write two vector calculus identities that are equal to zero.*
Sol: *Curl of the gradient* $\nabla \times \nabla\Phi(x,y,z) = 0$ and the *divergence of the curl* $\nabla \cdot \nabla \times \mathbf{V}(x,y,z) = 0$ are both zero. (Page 780, Stillwell) ■

– *5.5: What is the most general form a vector field may be expressed in, in terms of scalar Φ and vector $\boldsymbol{A}$ potentials?*
Sol: $\mathbf{V} = \nabla\Phi(x,y,z) + \nabla \times \mathbf{A}(x,y,z)$, where Φ is the scalar potential and $\mathbf{A}$ is the vector potential. ■

Problem # *6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.*

– *6.1: Let $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Find $\nabla \cdot (\nabla \times \boldsymbol{v})$.*
Sol: 0 ■

– *6.2: Let $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Find $\nabla \times (\nabla\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}})$*
Sol: 0 ■

– *6.3: Let $\boldsymbol{v}(x,y,z) = \nabla(x + y^2 + \sin(\log(z)))$. Find $\nabla \times \boldsymbol{v}(x,y,z)$.*
Sol: It is zero because $\nabla \times \nabla f(x,y,z)$ is *always* zero. ■

4.1.6 Integral theorems

Problem # *7: For each of the following problems, in a few words, identify either Gauss's or Stokes's law, define what it means, and explain the formula that follows the question.*

– *7.1: What is the name of this formula?*

$$\int_{\mathcal{S}} \hat{\mathbf{n}} \cdot \boldsymbol{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v}\, dV.$$

Sol: This is the integral form of *Gauss' law*. The unit normal vector is $\perp$ to the surface $\mathcal{S}$ having area $A \equiv \int_{\mathcal{S}} dA$ The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field $\nabla \cdot \mathbf{v}$ over the volume contained by the surface, and defined as $\mathcal{V}$. ■

– *7.2: What is the name of this formula?*

$$\int_S (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{V} \cdot d\boldsymbol{R}$$

Give one important application. **Sol:** Stokes Theorem, which relates the differential to the integral form of Maxwell's equations. ■

– *7.3: Describe a key application of the vector identity*

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \boldsymbol{\nabla}^2\mathbf{V}.$$

Sol: When we wish to reduce Maxwell's two curl equations to the vector wave equation, we must use this identity. ■