4.2 Problems VC-2

4.2.1 Topics of this homework:

Partial differential equations; fundamental theorem of vector calculus (Helmholtz's theorem); wave equation; Maxwell's equations (ME) and variables ($\boldsymbol{E}, \boldsymbol{D}; \boldsymbol{B}, \boldsymbol{H}$); Second-order vector differentials; Webster horn equation.

Notation: The following notation is used in this homework:

1. $s = \sigma + j\omega$ is the Laplace frequency, as used in the Laplace transform.
2. A Laplace transform pair is indicated by the symbol $\leftrightarrow$: for example, $f(t) \leftrightarrow F(s)$.
3. π_k is the kth prime; for example, $\pi_k \in \mathbb{P}$, $\pi_k = [2, 3, 5, 7, 11, 13, \ldots]$ for $k = 1, \ldots, 6$).

4.2.2 Partial differential equations (PDEs): Wave equation

Problem # *1: Solve the wave equation in one dimension by defining $\xi = t \mp x/c$.*

– 1.1: Show that d'Alembert's solution, $\varrho(x,t) = f(t - x/c) + g(t + x/c)$, is a solution to the acoustic pressure wave equation in one dimension:

$$\frac{\partial^2 \varrho(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \varrho(x,t)}{\partial t^2},$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions. **Sol:**

$$\frac{\partial}{\partial x}\varrho(x,t) = \frac{\partial}{\partial x} f(t - x/c) + \frac{\partial}{\partial x} g(t + x/c) = \frac{-1}{c} f'(t - x/c) + \frac{1}{c} g'(t + x/c) \tag{VC-2.1}$$

$$\frac{\partial^2}{\partial x^2}\varrho(x,t) = \frac{\partial^2}{\partial x^2} f(t - x/c) + \frac{\partial^2}{\partial x^2} g(t + x/c) = \frac{1}{c^2} f''(t - x/c) + \frac{1}{c^2} g''(t + x/c) \tag{VC-2.2}$$

$$\frac{\partial^2}{\partial t^2}\varrho(x,t) = \frac{\partial^2}{\partial t^2} f(t - x/c) + \frac{\partial^2}{\partial t^2} g(t + x/c) = f''(t - x/c) + g''(t + x/c) \tag{VC-2.3}$$

■

Problem # *2: Solving the wave equation in spherical coordinates (i.e., three dimensions)*

– 2.1: Write the wave equation in spherical coordinates $\varrho(r, \theta, \phi, t)$. Consider only the radial term r (i.e., dependence on angles θ and ϕ is assumed to be zero). Hint: The form of the Laplacian as a function of the number of dimensions is given in Eq. NS-3.9 (page 157). Alternatively, look it up on the internet or in a calculus book.

Sol: Given the formula for the Laplacian in spherical coordinates, the wave equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\varrho(r,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varrho(r,t)$$

■

– 2.2: Show that this equation is true:

$$\nabla_r^2 \varrho(r) \equiv \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\varrho(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2} r\varrho(r). \tag{VC-2.4}$$

Hint: Expand both sides of the equation. **Sol:** Both sides of the equation expand to

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}$$

■

– 2.3: Use the results from Eq. VC-2.1 to show that the solution to the spherical wave equation is

$$\nabla_r^2\, \varrho(r,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,\varrho(r,t) \tag{VC-2.5}$$

$$\varrho(r,t) = \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r}. \tag{VC-2.6}$$

Sol: This proceed exactly as in the rectangular case (see above) except one must first recognize that the Laplacian in spherical coordinates may be written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r\varrho(r). \tag{VC-2.7}$$

One then may proceed to use the solution for the rectangular case, but for $r\varrho(r)$, and then divide that solution by r. ■

– 2.4: Using $f(\xi) = \sin(\xi)u(\xi)$ and $g(\xi) = e^{\xi}u(\xi)$, write the solutions to the spherical wave equation, where $u(\xi)$ is the Heaviside step function.

Sol: In each case replace $\xi = t - x/c$ to obtain the solution to the wave equation for 1 dimensional waves. Thus

$$\begin{aligned}\varrho(r,t) &= \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r} \\ &= \frac{\sin(t-r/c)}{t-r/c}u(t-r/c) + \frac{e^{(t+r/c)}u(t+r/c)}{t+r/c}\end{aligned}$$

■

– 2.5: Sketch this $f(\xi)$ and $g(\xi)$ for several times (e.g., 0, 1, and 2 seconds), and describe the behavior of the pressure $\varrho(r,t)$ as a function of time t and radius r.

Sol: Plot the functions at several times (e.g., 0, 1 2 seconds), as a function of x. The first function becomes smaller as the radius grows. The second function becomes larger as the inbound waves approaches $r = 0$. ■

– 2.6: What happens when the inbound wave reaches the center at $r = 0$?

Sol: Stand back. It blows up. The equations fail when the solution becomes so large that the linearity assumption fails. I'm not sure what actually happens, in practice. This seems to be how they detonate nuclear weapons. ■

4.2.3 Helmholtz's formula

Every differentiable vector field may be written as the sum of a scalar potential ϕ and a vector potential $\boldsymbol{w}$. This relationship is best known as the fundamental theorem of vector calculus (also called Helmholtz's formula):

$$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{w}. \tag{VC-2.8}$$

This formula seems to be a natural extension of the algebraic products $\boldsymbol{A}\cdot\boldsymbol{B} \perp \boldsymbol{A}\times\boldsymbol{B}$, since $\boldsymbol{A}\cdot\boldsymbol{B} \propto \|\boldsymbol{A}\|\|\boldsymbol{B}\|\cos(\theta)$ and $\boldsymbol{A}\times\boldsymbol{B} \propto \|\boldsymbol{A}\|\|\boldsymbol{B}\|\sin(\theta)$, as developed in Appendix A.3.1, page 203. Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler's identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

As shown in Table 5.1 (p. 155), Helmholtz's formula separates a vector field (i.e., $\mathbf{v}(\mathbf{x})$) into compressible and rotational parts:

1. The rotational (e.g., angular) part is defined by the vector potential $\boldsymbol{w}$, which requires that $\nabla\times\nabla\times\boldsymbol{w} \neq 0$. A field is irrotational (conservative) when $\nabla\times\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a scalar potential, $\boldsymbol{v} = \nabla\phi$ (note that this is how a conservative field is usually defined, by saying there exists some ϕ such that $\boldsymbol{v} = \nabla\phi$).[2]

2. The compressible (e.g., radial) part of a field is defined by the scalar potential ϕ, which requires that $\nabla\cdot\nabla\phi = \nabla^2\phi \neq 0$. A field is incompressible (solenoidal) when $\nabla\cdot\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a vector potential, $\boldsymbol{v} = \nabla\times\boldsymbol{w}$.

[2] A note about the relationship between the generating function and the test: You might imagine special cases where $\nabla\times\boldsymbol{w} \neq 0$ but $\nabla\times\nabla\times\boldsymbol{w} = 0$ (or $\nabla\phi \neq 0$ but $\nabla^2\phi = 0$). In these cases, the vector (or scalar) potential can be recast as a scalar (or vector) potential. For example, consider a field $\boldsymbol{v} = \nabla\phi_0 + \boldsymbol{b}$, where $\boldsymbol{b} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Note that $\boldsymbol{b}$ can actually be generated by either a scalar potential ($\phi_1 = \frac{1}{2}[x^2+y^2+z^2]$, such that $\nabla\phi_1 = \boldsymbol{b}$) or a vector potential ($\boldsymbol{w}_0 = \frac{1}{2}[z^2\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}]$, such that $\nabla\times\boldsymbol{w}_0 = \boldsymbol{b}$). We find that $\nabla\times\boldsymbol{v} = 0$; therefore $\boldsymbol{v}$ must be irrotational. We say this irrotational field is generated by $\nabla\phi = \nabla(\phi_0 + \phi_1)$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities: (1) $\nabla \cdot (\nabla \times \boldsymbol{w}) = 0$ and (2) $\nabla \times (\nabla \phi) = 0$.

Problem # *3: Define the following:*

– *3.1: A conservative vector field*

Sol: A conservative vector field is defined as the gradient of a scalar potential $\mathbf{v} = \nabla \phi(x, y, z)$. Every conservative field is necessarily *irrotational* (the test for an irrotational field is $\nabla \times \mathbf{v} = 0$). ■

– *3.2: An irrotational vector field*

Sol: The vector field $\mathbf{v}$ is rotational if there exists a vector potential $\mathbf{w}$ such that $\mathbf{v} = \nabla \times \mathbf{w}(x, y, z)$. The for *irrotational* is $\nabla \times \mathbf{v} = 0$. A purely rotational field is not conservative. ■

– *3.3: An incompressible vector field*

Sol: A field $\mathbf{v}$ is incompressible if $\nabla \cdot \mathbf{v} = 0$. ■

– *3.4: A solenoidal vector field*

Sol: A rotational field is one having a divergence of zero, i.e., $\nabla \cdot \mathbf{v} = 0$, or alternatively, $\mathbf{v} \equiv \nabla \times \mathbf{w}(x, y, z)$, since any field defined by a curl is rotational, since the divergence of the curl is always zero. ■

– *3.5: When is a conservative field irrotational?*

Sol: Always! ■

– *3.6: When is an incompressible field irrotational?*

Sol: A field is incompressible if $\nabla \cdot v = 0$ and irrotational if $\nabla \times v = 0$. So, almost never. The only case is the trivial solution $\mathbf{v} = 0$, or a constant field $\mathbf{v} = x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}} + z_0\hat{\mathbf{z}}$. ■

Problem # *4: For each of the following, (i) compute $\nabla \cdot \boldsymbol{v}$, (ii) compute $\nabla \times \boldsymbol{v}$, and (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.).*

– *4.1:* $\boldsymbol{v}(x, y, z) = -\nabla(3yx^3 + y \log(xy))$

Sol: The field is conservative (or irrotational) because it is defined by a gradient. To test for irrotational, show that the curl is zero. But $\nabla \times \nabla \phi(x, y, z) = 0$ for any $\phi(x, y, z)$. Thus you do not need to do any computation, just state the answer. ■

– *4.2:* $\boldsymbol{v}(x, y, z) = xy\hat{\mathbf{x}} - z\hat{\mathbf{y}} + f(z)\hat{\mathbf{z}}$

Sol: To test for a irrotational field, take the curl, to see if it is zero:

$$\nabla \times \mathbf{v} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ xy & -z & f(z) \end{vmatrix} = \hat{\mathbf{x}} - x\hat{\mathbf{z}}, \tag{VC-2.9}$$

which is not zero. We can also see by inspection that $\nabla \cdot \mathbf{v} \neq 0$. Thus the vector field is rotational and compressible. ■

– *4.3:* $\boldsymbol{v}(x, y, z) = \nabla \times (x\hat{\mathbf{x}} - z\hat{\mathbf{y}})$

Sol: $\mathbf{v} = \hat{\mathbf{x}}$. Therefore, $\nabla \times \mathbf{v} = 0$, and $\nabla \cdot \mathbf{v} = 0$. This field is technically incompressible and irrotational, but it is also very boring, since it is a constant. ■

4.2.4 Maxwell's Equations

The variables have the following names and defining equations (see Table 5.4, p. 185):

Symbol	Equation	Name	Units
$\boldsymbol{E}$	$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$	Electric field strength	[volts/m]
$\boldsymbol{D} = \epsilon_o \boldsymbol{E}$	$\nabla \cdot \boldsymbol{D} = \rho$	Electric displacement (flux density)	[coul/m^2]
$\boldsymbol{H}$	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dot{\boldsymbol{D}}$	Magnetic field strength	[amps/m]
$\boldsymbol{B} = \mu_o \boldsymbol{H}$	$\nabla \cdot \boldsymbol{B} = 0$	Magnetic induction (flux density)	[webers/m^2]

Note that $\boldsymbol{J} = \sigma \boldsymbol{E}$ is the *current density* (which has units of [amps/m^2]). Furthermore, the *speed of light in vacuo* is $c_o = 3 \times 10^8 = 1/\sqrt{\mu_o \epsilon_o}$ [m/s], and the *characteristic resistance* of light $r_o = 377 = \sqrt{\mu_o/\epsilon_o}$ [Ω (i.e., ohms)].

4.2.5 Speed of light

Problem # *5: The* speed of light *in vacuo is* $c_o = 1/\sqrt{\mu_o\epsilon_o} \approx 3 \times 10^8$ *[m/s]. The* characteristic resistance *in vacuo is* $r_o = \sqrt{\mu_o/\epsilon_o} \approx 377$ [Ω].

– 5.1: Find a formula for the in-vacuo permittivity ϵ_o *and* permeability *in terms of* c_o *and* r_o. **Sol:** $\epsilon_o = 1/r_oc_o$ *and* $\mu_o = r_o/c_o$. ■ *Based on your formula, what are the numeric values of* ϵ_o *and* μ_o*?*
Sol: $\epsilon_o \approx 10^{-8}/3 \cdot 377 = 8.84 \cdot 10^{-12}$ and $\mu_o \approx 377/3 \cdot 10^8 = 1.26 \cdot 10^{-6}$. ■

– 5.2: In a few words, identify the law given by this equation, define what it means, and explain the formula:

$$\int_{\mathcal{S}} \hat{\mathbf{n}} \cdot \boldsymbol{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v}\, dV.$$

Sol: This is the integral form of *Gauss' law*. The unit normal vector is $\perp$ to the surface $\mathcal{S}$ having area $A \equiv \int_{\mathcal{S}} dA$ The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field $\nabla \cdot \mathbf{v}$ over the volume contained by the surface, and defined as $\mathcal{V}$. ■

4.2.6 Application of Maxwell's equations

Problem # *6: The electric Maxwell equation is* $\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$*, where* $\boldsymbol{E}$ *is the* electric field strength *and* $\dot{\boldsymbol{B}}$ *is the time rate of change of the* magnetic induction field*, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface* S*, where* $\hat{\mathbf{n}}$ *is a unit vector normal to the surface (you may also find it useful to define the closed path* C *around the surface):*

$$\iint_S [\nabla \times \boldsymbol{E}] \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \iint_S \boldsymbol{B} \cdot \hat{\mathbf{n}} dS.$$

– 6.1: Apply Stokes's theorem to the left-hand side of the equation.
Hint: view this relation in therms of the "integral forms" of the curl. **Sol:** The surface S must be open, with its edge C defining the path for the line integral.

$$\text{emf} \equiv \iint_S \nabla \times E \cdot \hat{\mathbf{n}} dS = \oint_C \mathbf{E} \cdot d\mathbf{R}. \qquad \text{(VC-2.10)}$$

From Stokes' theorem: the *electromotive force* (emf) is the line integral of $\mathbf{E}$ around the rim of the open surface. Think of the flux change as the Thévenin source driving the voltage. ■

– 6.2: Consider the right-hand side of the equation. How is it related to the magnetic flux Ψ *through the surface* S*?*
Sol: It is equal to the negative time rate of change of the flux, $-\dot{\Psi}$. From Gauss' Law the total magnetic flux Ψ is the surface integral over the normal component of the magnetic flux density $\boldsymbol{B}$. After applying Gauss' Laws, the surface integral becomes

$$\Psi = -\iint_S \boldsymbol{B} \cdot \hat{\mathbf{n}} dS \qquad \text{(VC-2.11)}$$

■

– 6.3: Assume the right-hand side of the equation is zero. Can you relate your answer in question 6.1 to one of Kirchhoff's laws?
Sol: This result is well know as Kirchhoff's first (voltage) law (KVL), emf $= \sum_k V_k = -\dot{\Psi}$. When the flux induced into the loop may be ignored (e.g., it is very small), the sum of the voltages around the loop is zero. In rectangular coordinates with a plane surface this is simply $\Phi = B_nA$, where A is the area and B_n the normal component of B ($\perp$ to the surface S). ■

Problem # *7: The magnetic Maxwell equation is* $\nabla \times \boldsymbol{H} = \boldsymbol{C} \equiv \boldsymbol{J} + \dot{\boldsymbol{D}}$*, where* $\boldsymbol{H}$ *is the* magnetic field strength*,* $\boldsymbol{J} = \sigma\boldsymbol{E}$ *is the conductive (resistive) current density, and the* displacement current $\dot{\boldsymbol{D}}$ *is the time rate of change of the* electric flux density $\boldsymbol{D}$*. Here we defined a new variable* $\boldsymbol{C}$ *as the total current density.*

– *7.1: First consider the equation over a two-dimensional surface S:*

$$\iint_S [\nabla \times \boldsymbol{H}] \cdot \hat{\mathbf{n}} dS = \iint_S [\boldsymbol{J} + \dot{\boldsymbol{D}}] \cdot \hat{\mathbf{n}} dS = \iint_S \boldsymbol{C} \cdot \hat{\mathbf{n}} dS.$$

Then apply Stokes's theorem to the left-hand side of this equation. In a sentence or two, explain the meaning of the resulting equation. Hint: What is the right-hand side of the equation? **Sol:** The surface S must be open, with its edge C prescribing the line integral, and its surface of $\boldsymbol{C}$ defines the total current $I(t)$. The normal component of the surface integral over the total current $\boldsymbol{C}$ gives total current $I(t)$. By Stokes theorem:

$$\text{mmf} \equiv \iint_S \nabla \times \mathbf{H} \cdot \hat{\mathbf{n}} dS = \oint_C \mathbf{H} \cdot d\mathbf{R} = \iint_S \mathbf{C} \cdot \hat{\mathbf{n}} dS = I(t)$$

This is Ampere's Law. ■

Problem # *8: Consider the next equation in three dimensions. Take the divergence of both sides and integrate over a volume V (closed surface S):*

$$\iiint_V \nabla \cdot [\nabla \times \mathbf{H}] dV = \iiint_V \nabla \cdot \mathbf{C} dV.$$

– *8.1: What happens to the left-hand side of this equation?* Hint: Can you apply a vector identity? **Sol:** *It is 0.* ■ *Apply the divergence theorem (sometimes known as Gauss's theorem) to the right-hand side of the equation, and interpret your result.* Hint: Can you relate your result to one of Kirchhoff's laws?
Sol: We get

$$\iiint_V \nabla \cdot \mathbf{C} dV = \iint_S \mathbf{C} \cdot \hat{\mathbf{n}} dS = 0$$

This result is Kirchhoff's second (current) law (KCL), $\sum_k I_k = \iint \dot{\boldsymbol{D}}(t) \cdot d\boldsymbol{S}$. When the stray capacitance ($\dot{\mathbf{D}}$) can be ignored the sum of the currents into the 'node' is zero. Generalizing, a 'node' to a volume V, the total current $I(t)$ flowing in/out of the volume is the integral of the normal component of the current density over the cross-sectional closed surface area, which equals 0. ■

4.2.7 Second-order differentials

Problem # *9: This problem is about second-order vector differentials.*

– *9.1: If $\boldsymbol{v}(x, y, z) = \nabla\phi(x, y, z)$, then what is $\nabla \cdot \boldsymbol{v}(x, y, z)$?*
Sol: Since $\nabla \cdot \nabla = \nabla^2$ this is $\nabla^2\phi(x, y, z)$. ■

– *9.2: Evaluate $\nabla^2\phi$ and $\nabla \times \nabla\phi$ for $\phi(x, y) = xe^y$.*
Sol: CoG =0 $\nabla \times \nabla\phi$=0, $\nabla^2\phi = xe^y$ ■

– *9.3: Evaluate $\nabla \cdot (\nabla \times \boldsymbol{v})$ and $\nabla \times (\nabla \times \boldsymbol{v})$ for $\boldsymbol{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.*
Sol: $\nabla \cdot (\nabla \times \mathbf{v}) = 0$, $\nabla \times (\nabla \times \mathbf{v}) = 0$ ■

– *9.4: When $\boldsymbol{V}(x, y, z) = \nabla(1/x + 1/y + 1/z)$, what is $\nabla \times \boldsymbol{V}(x, y, z)$?*
Sol: This is always zero. ■

– *9.5: When was Maxwell born and when did he die? How long did he live (within ±10 years)?*
Sol: He lived 48 years, from 1831 to 1879. ■

4.2.8 Capacitor analysis

Problem # *10: Find the solution to the Laplace equation between two infinite[3] parallel plates separated by a distance d. Assume that the left plate at $x = 0$ is at voltage $V(0) = 0$ and the right plate at $x = d$ is at voltage $V_d \equiv V(d)$.*

[3] We study plates that are infinite because this means the electric field lines are perpendicular to the plates, running directly from one plate to the other. However, we solve for per-unit-area characteristics of the capacitor.

– *10.1: Write Laplace's equation in one dimension for $V(x)$.*
Sol: This is the Laplace equation for rectangular coordinates

$$\frac{\partial^2 V(x)}{\partial x^2} = 0$$

■

– *10.2: Write the general solution to your differential equation for $V(x)$.*
Sol: Integration is trivial since the solution must be of the form $V(x) = A + Bx$. ■

– *10.3: Apply the boundary conditions $V(0) = 0$ and $V(d) = V_d$ to determine the constants in your equation from question 10.2.*
Sol: From the BC $A = 0$ and $B = V_d/d$. Thus $V(x) = \frac{V_d}{d}x$. ■

– *10.4: Find the charge density per unit area ($\sigma = Q/A$, where Q is charge and A is area) on the surface of each plate.* Hint: $\boldsymbol{E} = -\nabla V$, and Gauss's law states that $\iint_S \boldsymbol{D}\cdot\hat{\mathbf{n}}dS = Q_{\text{enc}}$.
Sol: To find the charge, we must first compute the electric field from the voltage using $\mathbf{E} = -\nabla V(x)$

$$-\mathbf{E} \equiv \nabla V(r) = \hat{\mathbf{x}}\frac{\partial}{\partial x}V(x) = \hat{\mathbf{x}}V_d$$

Since $\mathbf{D} = \epsilon_o\mathbf{E}$ we find the normal component of the $\mathbf{D}$ field

$$\mathbf{D} = \epsilon_o\mathbf{E} = -\epsilon_o\nabla V$$

is just a constant Thus using Gauss' law ($\sigma = -\frac{1}{A}\int_S D_x dA = D_r$), the surface charge density σ in *farads per square-meter* is

$$\sigma = \frac{\epsilon_o}{d}V_d$$

■

– *10.5: Determine the per-unit-area capacitance C of the system.*
Sol: Since $\sigma = CV_d$, the capacity C per unit area is

$$C = \frac{\epsilon_o}{d} \ [\text{F/m}^2].$$

The units are farads per square-meter. Note that the sign must work out so that $C > 0$.
https://en.wikipedia.org/wiki/Capacitance ■

4.2.9 Webster horn equation

Problem # *11: Horns illustrate an important generalization of the solution of the one dimensional wave equation in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns are in vocal tract acoustics, loudspeaker design, cochlear mechanics, and any case that has wave propagation. Write the formula for the Webster horn equation, and explain the variables.*
Sol: The horn equation may be written as

$$\frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial \varrho}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 \varrho}{\partial t^2}. \qquad \text{(VC-2.12)}$$

where $A(x)$ is the area of the horn at x (range variable). $\varrho(x,t)$ is the pressure and c is the wave speed. ■