An Invitation to Mathematical Physics *and Its History*

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http://jontalle.web.engr.illinois.edu/uploads/298/

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Abstract

An understanding of physics requires knowledge of mathematics. The converse is not true. By definition, pure mathematics contains no physics. Yet historically, mathematics has a rich history filled with physical applications. Mathematics was developed by people with the intent of making things work. As an engineer, I see these creators of early mathematics as budding engineers. This book is an attempt to tell their story, of the development of mathematical physics, as viewed by an engineer.

There are two distinct ways to learn mathematics: the traditional math way, by memorizing everything (the hard way), and the easy "physical" way, by associating every mathematical concept with its physical counterpart. These two methods have their pros and cons:

- Physicists and engineers choose to associate mathematical and physical concepts.
- Mathematicians choose to *not* make these associations.

We might conjecture why this is: Budding physicists and engineers may have come to mathematics while learning physics. Budding mathematicians may appreciate the beauty in the pure structures of mathematics.

Both methods have a chronological development, i.e., a history. Amazingly these histories heavily overlap. Why is that? Because most early mathematics evolved from attempts to understand the world, with the goal of navigating it. Pure mathematics followed when the corresponding physics was not yet discovered (or perhaps doesn't exist).

This is not strictly a book on mathematics, physics and history: rather, it is a marriage of math and physics, presented roughly in chronological order, via its history. To teach mathematical physics in an orderly way, our treatment requires a step backwards in terms of the mathematics, but a step forward in terms of the physics. Historically speaking, mathematics was created by individuals, who by modern standards, would be viewed as engineers. This book contains the basic information that a well-informed engineer needs to know.

The presentation is broken down around three streams, presented as five chapters: 1) Introduction, 2) Number Systems, 3) Algebra Equations, 4) Scalar Calculus, and 5) Vector Calculus. The material is delivered as 43 "Lectures," spread out over a semester of 15 weeks, 3 lectures per week, with a three-lecture time-out for administrative duties. Eleven problem sets are provided for weekly assignments. These problems are written out in LaTeX, with built-in solutions that may be compiled into the text by un-commenting one line of LaTeX. Once the assignments are turned in, each student is given the solution.

Many students have rated these assignments as the most important part of the course. There is a built-in interplay between these assignments and the lectures. On many occasions I solved the problems in class, as motivation to come to every class.

There are four exams, one at the end of each of the three sections, plus the final. The first exam is in class, two others and the final are evening exams. Each exam and assignment is a LaTeX file, with solutions encoded with a one line software switch. When the exam is returned by the student, the full solution is provided, while the exam is fresh in the students mind, providing a *teaching moment*. The exams are entirely based on the assignments. It is my philosophy that, in principle,

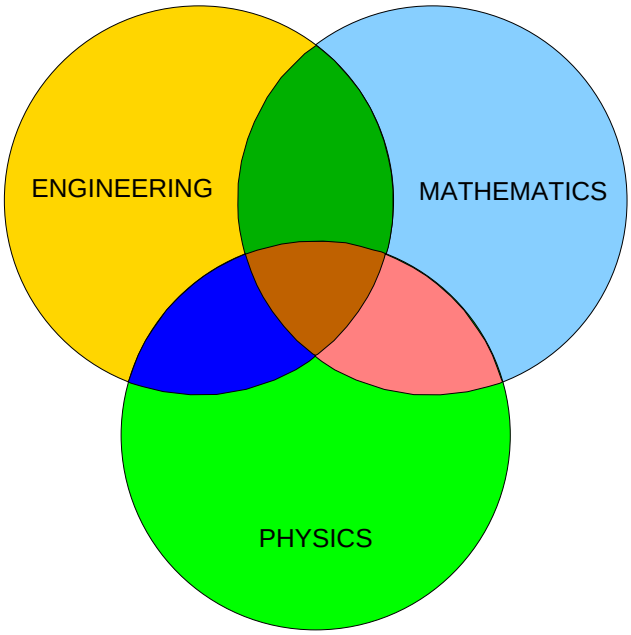

Figure 1: *There is a natural symbiotic relationship between Mathematics, Engineering, and Physics (MEP), depicted in the Venn diagram. Mathematics provides the method and rigor. Engineering transforms the method into technology. Physics explores the boundaries. While these three disciplines work well together, there is poor communication, in part due to the different vocabulary. But style may be more at issue. For example, Mathematics rarely uses a system of units, whereas Physics and Engineering depend critically on them. Mathematics strives to abstract the ideas into proofs. Physics rarely uses a proof. When they attempt rigor, physicists and Engineers typically get into difficulty.*

the students can see the exam in advance of taking it. In a real sense they do, since each exam is based on the assignments.

Author's Personal Statement

An expert is someone who has made all possible mistakes in a small field. I don't know if I would be called an expert, but I certainly have made my share of mistakes. I openly state that "I love making mistakes, because I learn so much from them." One might call that the "expert's corollary."

This book has been written out of both my love for the topic of mathematical physics, and a desire to share many key concepts, and many new ideas on these basic concepts. Over the years I have developed a certain *physical sense* of math, along with a related mathematical sense of physics. While doing my research,[1] I have come across what I feel are certain conceptual holes that need filling, and sense many deep relationships between math and physics that remain unidentified. While what we presently teach is not wrong, it is missing these relationships. What is lacking is an intuition for how math "works." Good scientists "listen" to their data. In the same way we need to start listening to the language of mathematics. We need to let mathematics guide us toward our engineering goals.

[1] https://auditorymodels.org/index.php/Main/Publications

As summarized in Fig. 1, this marriage of math, engineering and physics (MEP)[2] will help us make progress in understanding the physical world. We must turn to mathematics and physics when trying to understand the universe. My views follow from a lifelong attempt to understand human communication, i.e., the perception and decoding of human speech sounds. This research arose from my 32 years at Bell Labs in the Acoustics Research Department. There such lifelong pursuits were not only possible, they were openly encouraged. The idea was that if you are successful at something, take it as far as you can. But on the other hand "you should not do something well that's not worth doing." People got fired for the latter. I should have left for a university after a mere 20 years,[3] but the job was just too cushy.

In this text it is my goal to clarify conceptual errors while telling the story of physics and mathematics. My views have been inspired by classic works, as documented in the bibliography. This book was inspired by my reading of Stillwell (2002), through Chapter 21 (Fig. 2, p. 11). Somewhere in Chapter 22 I switched to the third edition (Stillwell, 2010), at which point I realized I had much more to master. It became clear that by teaching this material to first year engineers, I could absorb the advanced material at a reasonable pace. This book soon followed.

Summary

This is foremost a math book, but not the typical math book. *First*, this book is for the engineering minded, for those who need to understand math to do engineering, to learn how things work. In that sense the book is more about physics and engineering than mathematics. Math skills are essential for making progress in building things, be it pyramids or computers, as clearly shown by the many great civilizations of the Chinese, Egyptians, Arabs (people of Mesopotamia), and Greeks and Romans.

Second, this is a book about the math that developed to explain physics, to allow people to engineer complex things. To sail around the world one needs to know how to navigate. This requires a model of the planets and stars. You can only know where you are, on earth, once you understand where earth is, relative to the sun, planets, Milky Way and the distant stars. The answer to such a cosmic questions depends strongly on who you ask. Who is qualified to answer such a question? It is best answered by those who study mathematics applied to the physical world. The utility and accuracy of that answer depends critically on the depth of understanding of the physics of the cosmic clock.

The English astronomer Edmond Halley (1656–1742) asked Newton (1643–1727) for the equation that describes the orbit of the planets. Halley was obviously interested in comets. Newton immediately answered "an ellipse." It is said that Halley was stunned by the response (Stillwell, 2010, p. 176), as this was what had been experimentally observed by Kepler (c1619), and thus he knew Newton must have some deeper insight. Both were eventually Knighted.

When Halley asked Newton to explain how he knew, Newton responded "I calculated it." But when challenged to show the calculation, Newton was unable to reproduce it. This open challenge eventually led to Newton's grand treatise, *Philosophiae Naturalis Principia Mathematica* (July 5, 1687). It had a humble beginning, as a letter to Halley, explaining how to calculate the orbits of the planets. To do this Newton needed mathematics, a tool he had mastered. It is widely accepted that Isaac Newton and Gottfried Leibniz invented calculus. But the early record shows that perhaps Bhāskara II (1114–1185 CE) had mastered the art well before Newton.[4]

[2]MEP is a focused alternative to STEM.

[3]I started around December 1970, fresh out of graduate school, and retired in December 5 2002.

[4]`http://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch8_5.html`

Third, the main goal of this book is to teach motivated engineers mathematics, in a way that it can be understood, mastered and remembered. How can this impossible goal be achieved? The answered is to fill in the gaps with *Who did what, and when?* Compared with the math, the historical record is easily mastered.

To be an expert in a field, one must know its history. This includes who the people were, what they did, and the credibility of their story. Do you believe the Pope or Galileo on the roles of the sun and the earth? The observables provided by science are clearly on Galileo's side. Who were those first engineers? They are names we all know: Archimedes, Pythagoras, Leonardo da Vinci, Galileo, Newton, etc. All of these individuals had mastered mathematics. This book presents the tools taught to every engineer. Rather than memorizing complex formulas, make the relations "obvious" by mastering each simple underlying concept.

Fourth, when most educators look at this book, their immediate reactions are: *Each lecture is a topic we spend a week on (in our math/physics/engineering class).* And: *You have too much material crammed into one semester.* The first sentence is correct, the second is not. Tracking the students who have taken the course, looking at their grades, and interviewing them personally, demonstrate that the material presented here is appropriate for one semester.[5]

To write this book I had to master the language of mathematics (John D'Angelo language). I had already mastered the language of engineering, and a good part of physics. One of my secondary goals is to build this scientific Tower of Babel, by unifying the terminology and removing the jargon.

Acknowledgments

Besides thanking my parents, I would like to credit John Stillwell for his constructive, historical summary of mathematics. My close friend and colleague Steve Levinson somehow drew me into this project, without my even knowing it. My brilliant graduate student Sarah Robinson was constantly at my side, grading homeworks and exams, and tutoring the students. Without her, I would never have survived the first semester the material was taught. Her proofreading skills are amazing. Thank you Sarah for your infinite help. Finally I would like to thank John D'Angelo for putting up with my many silly questions. When it comes to the heavy hitting, John was always there to provide a brilliant explanation that I could easily translate into Engineerese (Matheering?) (i.e., engineer language).

My delightful friend Robert Fossum, emeritus professor of mathematics from the University of Illinois, who kindly pointed out my flawed use of mathematics. James (Jamie) Hutchinson's precise use of the English language, dramatically raised the bar on my more than occasionally-casual writing style. To each of you, thank you!

Finally I would like to thank my wife Sheau Feng Jeng, aka Patricia Allen, for her unbelievable support and love. She delivered constant peace of mind, without which this project could never have been started, much less finished. Many others played important roles, but they must remain anonymous.

–Jont Allen, Mahomet IL, Dec. 24, 2015 (Jan 1, 2018)

[5] http://www.istem.illinois.edu/news/jont.allen.html

Mathematics and its History (Stillwell, 2002)

Contents

Figure 2: *Table of contents of Stillwell (2002)*

Preface

It is widely acknowledged that interdisciplinary science is the backbone of modern scientific inquiry as embodied in the STEM (Science, Technology, Engineering, and Mathematics) programs. Cross-disciplinary research is about connecting different areas of knowledge. However, while STEM is being taught, interdisciplinary science is not, due to its inherent complexity and breadth. There are few people to teach it. As diagrammed in Fig. 1 (p. 8), Mathematics, Engineering and Physics (MEP) define the core of these studies.

STEM vs. MEP

Mathematics is about rigor. Mathematicians specifically attend to the definitions of increasingly general concepts. Thus mathematics advances slowly, as these complex definitions must be collectively agreed upon. Since it embraces rigor, mathematics shuns controversy. Physics explores the fringes of uncertainty, since physicists love controversy. Engineering addresses the advancement of technology. Engineers, much like mathematicians, are uncomfortable with uncertainty, but are trained to deal with it. Richard Hamming expressed this thought succinctly:

> Great scientists tolerate ambiguity very well. They believe the theory enough to go ahead; they doubt it enough to notice the errors and faults so they can step forward and create the new replacement theory (Hamming, 1986).

To create an interdisciplinary STEM program, the MEP core is critical. In my view, this core should be based on mathematics, from the historical perspective, starting with Euclid or before (i.e., Chinese mathematics), up to modern information theory and logic. As a bare minimum, the *fundamental theorems of mathematics* (arithmetic, algebra (real, complex and matrix), calculus (real, complex and matrix) etc.) should be taught to every MEP student. The core of this curriculum is outlined in Fig. 1.5 (p. 27). Again quoting Hamming:

> Well, that book I'm trying to write [on Mathematics, Calculus, Probability and Statistics] says clearly ... we can no longer tell you what mathematics you need, we must teach you how to find it for yourself. ... We must give up the results, and teach the methods. You are only shown finished theorems and proofs. That is just plain wrong for teaching.

If first year students are taught a common MEP methodology and vocabulary, presented in terms of the history of mathematics, they will be equipped to

1. Exercise interdisciplinary science (STEM)

2. Communicate with other MEP trained (STEM) students and professors.

The goal should be a *comprehensive understanding of the fundamental concepts of mathematics*, defined in terms of the needs of the student. The goal should *not* to be drilling on surface integrals, at least not in the first year. As suggested by Hamming, these details must be self-taught, at the time they are needed. Furthermore, start with the students who place out of early math courses: they love math and are highly motivated to learn as much as possible. I contend that if first or second year students are given a comprehensive early conceptual understanding, they will end up at the top of their field.

As identified by Hamming, a key problem is the traditional university approach, a five to eight semester sequence of: Calc I, II, III, Linear Algebra IV, DiffEq V, Real analysis VI, Complex analysis VII and given near-infinite stamina, Number theory VIII, over a time frame of three or more years (six semesters). This was the way I learned math. The process simply took too long, and the concepts spread too thin. After following this regime, I felt I had not fully mastered the material, so I started over. I consider myself to be largely self-taught.

We need a more effective teaching method. I am not suggesting we replace the standard six semester math curriculum of Calc I, II, III, etc. Rather, I am suggesting a broad unified introduction to all these topics, based on an historical approach. The present approach is driving the talent away from science and mathematics, by focusing too much on the details (as clearly articulated by Hamming). One needs more than a high school education to succeed in college engineering courses. The key missing element in our present education system is teaching critical thought. Drilling facts does not do that.

By learning mathematics in the context of history, the student will fully and easily appreciate the underlying concepts. The history provides a uniform terminology for understanding the fundamentals of MEP. The present teaching method, using abstract proofs, with no (or few) figures or physical principles and units, by design, removes intuition and the motivation that was available to the creators of these fundamentals. The present six-semester regime serves many students poorly, leaving them with little insight (i.e., intuition) and an aversion to mathematics.

Postscript Dec 5, 2017 How to cram five semesters of math into one semester, and leave the students with something they can remember? Here are some examples:

Maxwell's equations (MEs) are a fundamental challenging topic, presented in one lecture (Section 1.5.15, p. 165). Here is how it works:

1. The development starts with Sections 1.4.1-1.4.9 (pp. 105-123), which develop complex integration (p. 107) and the Laplace transform (pp. 121-122).

2. Kennelly's (1893) 1893 complex impedance, as defined by Ohm's law, is the ratio of the force over the flow, the key elements being 1) capacitance (e.g., compliance) per unit area (ϵ_o [Fd/m^2]), and 2) inductance (e.g., mass) per unit area (μ_o [H/m^2]).

3. Section 1.5.1 (p. 126) develops analytic field theory, while Sect. 1.5.2 (p. 128) introduces Grad $\nabla(\cdot)$, Div $\boldsymbol{\nabla}\cdot(\cdot)$, and Curl $\boldsymbol{\nabla}\times(\cdot)$, starting from the scalar $\boldsymbol{A}\cdot\boldsymbol{B}$ and vector product $\boldsymbol{A}\times\boldsymbol{B}$ of two vectors.

4. On p. 160, second-order operators are introduced and given physical meanings (based on the physics of fluids), but most important they are given memorable names (DoC, CoG, Dog, God, and CoC: p. 128). Thanks to this somewhat quaint and gamy innovation, the students can both understand and easily remember the relationships between these confusing second-order vector calculus operations.

5. Exercises and examples are interspersed throughout the lectures.

6. The foregoing carefully sets the stage for ME (p. 165), introduced using proper names *and units* of the electrical and magnetic field intensity (strengths, and forces seen by charge) $\boldsymbol{E}, \boldsymbol{H}$, and electric and magnetic flux (flow) $\dot{\boldsymbol{D}}, \dot{\boldsymbol{B}}$, as summarized on page 166. The meanings of ME equations are next explored in integral form. After Sect. 1.5.2-1.5.15, the students are fully conversant with MEs. The conformation of this is in the final exam grade distributions.

Postscript Dec 15, 2017 As this book comes to completion, I'm reading and appreciating the Feynman lectures. We all know (I hope you know) that Feynman had a special lecture style, that was both entertaining, and informative. His communication skill was a result of his depth of understanding, and he was not afraid to question the present understanding of physics. He was always on a quest. He died in 1988, at the age of 70. Let us all be on his quest. Any belief, that we have figured out the ways of the universe, is absurd. We have a lot to learn. Major errors in our understanding must to be corrected. We cannot understand the world around us until we understand its creation. That is, we cannot understand where we are going until we understand from where we came.

– Jont Allen

Chapter 1

Introduction

Much of early mathematics dating before 1600 BCE, centered around the love of art and music, due to the sensations of light and sound. Our psychological sense a of color and pitch are determined by the frequencies (i.e., wavelengths) of light and sound. The Chinese and later the Pythagoreans are well known for their early contributions to early music theory. We are largely ignorant of exactly what the Chinese scholars knew. The best record comes from Euclid, who lived the 3rd century, after Pythagoras. Thus we can only trace the early mathematics back to the Pythagoreans in the 6th century (580-500 BCE), which is centered around the Pythagorean theorem and early music theory.

Pythagoras strongly believed that "all is number," meaning that every number, and every mathematical and physical concept, could be explained by integral (integer) relationships, mostly based on either ratios, or the Pythagorean theorem. It is likely that his belief was based on Chinese mathematics from thousands of years earlier. It is also believed that his ideas about the importance of integers followed from the theory of music. The musical notes (pitches) obey natural integral ratio relationships, based on the octave (a factor of two in frequency). The western 12-tone scale breaks the octave into 12 ratios. Today this has been rationalized to be the 12 root of 2, which is approximately equal to 18/17. This innate sense of frequency ratios comes from the physiology the auditory organ (the cochlea).

As acknowledged by Stillwell (2010, p. 16), the Pythagorean view is relevant today:

> *With the digital computer, digital audio, and digital video coding everything, at least approximately, into sequences of whole numbers, we are closer than ever to a world in which "all is number."*

1.1 Early Science and Mathematics

While early Asian mathematics is not fully documented, it clearly defined the course for math for at least several thousand years. The first recorded mathematics were those of the Chinese (5000-1200 BCE) and the Egyptians (3,300 BCE). Some of the best early records were left by the people of Mesopotamia (Iraq, 1800 BCE).[1]

The first 5,000 years of math are not well documented, but the basic record is clear, as outlined in Fig. 1.1. Thanks to Euclid and later Diophantus (c250 CE), we have some vague understanding of Chinese mathematics. For example, *Euclid's formula* (Eq. 1.9, p. 48; Sec. G.2.2, Fig. G.3,

[1] See Fig. 1.10, p. 49.

p. 233) provides a method for computing Pythagorean triplets, a formula believed to be due to the Chinese (Stillwell, 2010, pp. 4-9).[2]

Chinese bells and stringed musical instruments were exquisitely developed with tonal quality, as documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was so rich that one must ask why the Chinese failed to initiate the industrial revolution. Specifically, why did Europe eventually dominate with its innovation when it was the Chinese who did the extensive early invention?

It could have been for the wrong reasons, but perhaps our best insight into the scientific history from China may have come from an American chemist and scholar from Yale, Joseph Needham, who learned to speak Chinese after falling in love with a Chinese woman, and ended up researching early Chinese science and technology for the US government.

Verify!

According to Lin (1995) this is known as the *Needham question*:

> "Why did modern science, the mathematization of hypotheses about Nature, with all its implications for advanced technology, take its meteoric rise only in the West at the time of Galileo[, but] had not developed in Chinese civilization or Indian civilization?"

Needham cites the many developments in China:[3]

> "Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon considered as the three most important inventions facilitating the West's transformation from the Dark Ages to the modern world, were invented in China." (Lin, 1995)

> "Needham's works attribute significant weight to the impact of Confucianism and Taoism on the pace of Chinese scientific discovery, and emphasizes what it describes as the 'diffusionist' approach of Chinese science as opposed to a perceived independent inventiveness in the western world. Needham held that the notion that the Chinese script had inhibited scientific thought was 'grossly overrated' " (Grosswiler, 2004).

Lin was focused on military applications, missing the importance of non-military contributions. A large fraction of mathematics was developed to better understand the solar system, acoustics, musical instruments and the theory of sound and light. Eventually the universe became a popular topic, as it still is today.

Regarding the "Needham question," I suspect the resolution is now clear. In the end, China withdrew from its several earlier expansions (Menzies, 2004, 2008).

1.1.1 Lec 1 The Pythagorean theorem

Thanks to Euclid's Elements (c323 BCE) we have an historical record, tracing the progress in geometry, as established by the Pythagorean theorem, which states that *for any right triangle*

$$c^2 = a^2 + b^2, \tag{1.1}$$

having sides of lengths $(a, b, c) \in \mathbb{R}$ that are either positive real numbers, or more interesting, integers, such that $c > [a, b]$ and $a + b > c$. Early integer solutions were likely found by trial and error rather than by Euclid's formula.

figure?

If a, b, c are lengths, then a^2, b^2, c^2 are each the area of a square. Equation 1.1 says that the

[2] One might view Euclid significant role as a mathematical messenger.

[3] https://en.wikipedia.org/wiki/Joseph_Needham\#cite_note-11

Chronological history pre 16^{th} century

20^{th} BCE *Chinese (Primes; quadratic equation; Euclidean algorithm (GCD))*

18^{th} BCE *Babylonia (Mesopotamia/Iraq) (quadratic equation)*

6^{th} BCE *Pythagoras (Thales) and the Pythagorean "tribe"*

4^{th} BCE *Euclid (quadratic equation); Archimedes*

3^{d} BCE *Eratosthenes 276-194BCE*

3^{d} CE *Diophantus c250CE*

4^{th} *Alexandria Library destroyed 391CE*

6^{th} *Brahmagupta (negative numbers; quadratic equation) 598-670CE*

10^{th} *al-Khwārizmī (algebra) 830CE Hasan Ibn al-Haytham (Alhazen) 965-1040CE*

12^{th} *Bhaskara (calculus) 1114-1183 Marco Polo 1254-1324*

15^{th} *Leonardo da Vinci 1452-1519; Michelangelo 1475-1564 Copernicus 1473-1543*

16^{th} *Tartaglia (cubic eqs); Bombelli 1526-1572 Galileo Galilei 1564-1642*

Time-Line

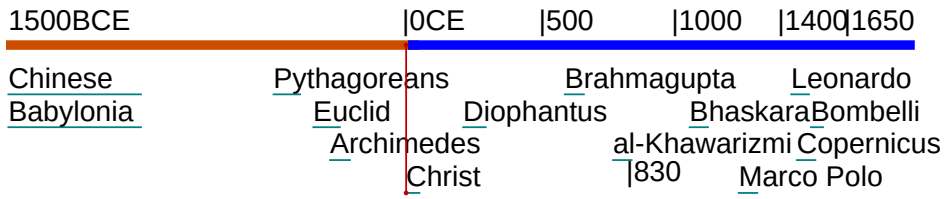

Figure 1.1: *Mathematical time-line between 1500* BCE *and 1650* CE*. The western renaissance is considered to have occurred between the 15-17 centuries. However the Asian "renaissance" was likely well before the 1st century (i.e., 1500* BCE*?). There is significant evidence that a Chinese 'treasure ship' visited Italy in 1434, initiating the Italian renaissance (Menzies, 2008). This was not the first encounter between the Italians and the Chinese, as documented in 'The travels of Marco Polo' (c1300* CE*).* [fig:TimeLineBCE]

area a^2 plus the area b^2 equals the area c^2. Today a simple way to prove this is to compute the magnitude of the complex number $c = a + b\jmath$, which forces the right angle

$$|c|^2 = (a + b\jmath)(a - b\jmath) = a^2 + b^2. \tag{1.2}$$

However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers and algebra had yet to be discovered.

Almost 700 years after Euclid's *Elements*, the Library of Alexandria was destroyed by fire (391 CE), taking with it much of the accumulated Greek knowledge. As a result, one of the best technical records remaining is Euclid's Elements, along with some sparse mathematics due to Archimedes (c300 BCE) on geometrical series, computing the volume of a sphere, the area of the parabola, and elementary hydrostatics. In c1572 a copy of a book by Diophantus *Arithmetic* was discovered by Bombelli in the Vatican library (Stillwell, 2010, p. 51). This book became an inspiration for Galileo, Descartes, Fermat and Newton.

Early number theory: Well before Pythagoras, the Babylonians (c1,800 BCE) had tables of triplets of integers $[a, b, c]$ that obey Eq. 1.1, such as $[3, 4, 5]$. However the triplets from the Babylonians were larger numbers, the largest being $a = 12709, c = 18541$. A stone tablet (Plimpton-322)

dating back to 1800 BCE (Fig. 1.10, p. 49) was found with integers for $[a, c]$. Given such sets of two numbers, which determined a third positive integer $b = 13500$ such that $b = \sqrt{c^2 - a^2}$, this table is more than convincing that the Babylonians were well aware of *Pythagorean triplets* (PTs), but less convincing that they had access to Euclid's formula (Eq. 1.9 p. 48).

It seems likely that Euclid's Elements was largely the source of the fruitful era due to the Greek mathematician Diophantus (215-285) (Fig. 1.1), who developed the concept of *discrete mathematics*, now known as *Diophantine analysis*. The term means that the solution, not the equation, are integers. The work of Diophantus was followed by fundamental change in mathematics, possibly leading to the development of algebra, but at least including the discovery of

1. negative numbers,
2. quadratic equation (Brahmagupta, 7^{th} CE),
3. algebra (al-Khwārizmī, 9^{th} CE), and
4. complex arithmetic (Bombelli, 15^{th} CE).

These discoveries overlapped with the European middle (aka, dark) ages. While Europe went "dark," presumably European intellectuals did not stop working during these many centuries.[4]

1.1.2 What is mathematics?

Mathematics is a language, not so different from other languages. Today's mathematics is a written language with an emphasis on symbols and glyphs, biased toward Greek letters. The specific evolution of these symbols is interesting (Mazur, 2014). Each symbol is dynamically assigned a meaning, appropriate for the problem being described. These symbols are then assembled to make sentences. It is similar to Chinese in that the spoken and written versions are different across dialects. Like Chinese, the sentences may be read out loud in any language (dialect), while the mathematical sentence (like Chinese characters) is universal.

There is a second answer to this question, told by studying its history, which starts with the earliest record. This chronological view starts, of course, with the study of numbers. First there is the taxonomy of numbers. It took thousands of year to realize that numbers are more than the counting numbers $\mathbb{N}$, and to create a symbol for nothing (i.e., zero), and negative numbers. With the invention of the abacus, a memory aid for the manipulation of complex sets of real integers, one could do very detailed calculations. But this required the discovery of *algorithms* (procedures) to add, subtract, multiply (many adds of the same number) and divide (many subtracts of the same number), such as the *Euclidean algorithm.* Eventually it became clear to the experts (early mathematicians) that there were natural rules to be discovered, thus books (e.g., *Euclid's Elements*) were written to summarize this knowledge.

Learning mathematics is just like building a house. If you learn how to use all the tools needed to build a house, you can build it. But you must master the tool chain, from the bottom up. If you are missing even one of the tools (if the chain is broken at any point), you will fail. There are many concepts that must be learned, and they form a long chain. All the concepts must be mastered, not simply learned. You need to learn about nails, screws, hammers, drills, lime, sand, water, concrete, you must know how to read drawings, down to the little marks, shovels, work with the labor force,

[4] It would be interesting to search the archives of the monasteries, where the records were kept, to determine exactly what happened during this religious blackout.

lay a floor, bricks, ceramics, a compass, straight edge, Euclidean geometry, simple mathematics, ..., framing, siding, gutters, ..., drive ways, plumbing, sewers, and the list goes on.

A short list for mathematics is numbers ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{C}$), algebra, derivatives, anti-derivatives (i.e., integration), differential equations, vectors and the spaces they define, matrices, matrix algebra, eigen-values and vectors, solutions of systems of equations, matrix differential equations and their eigen solution. If you try to memorize all these concepts, it is hopeless. They need to be learned in order, just as the case of building the house. Context is everything. You can't build a house if you don't know about screws or cement (plaster). Likewise in mathematics, you will not learn to integrate if you have failed to understand the difference between integers, complex numbers, and roots of polynomials.

Context is everything, and the most important context is physics. Without a physical problem to solve, at least at the beginning, there can be no mathematics, which began because people wanted to navigate the earth, or weigh things, namely understand the concept of gravity. Many questions were deep, such as "Where was the center of the universe?" Actually it was simple, ask the Pope and he will tell you. But church dogma only goes to the end of the block. Mathematics, along with a heavy dose of physics, finally answered this huge question. Someone needed to perfect the telescope, and put satellites into space, and view the cosmos. Without mathematics, none of this would, or could have happened.

Like any language, the more mathematics you learn, the easier it is to understand, because mathematics is built from the bottom up. It's a continuous set of concepts, much like the construction of a house. If you try to learn calculus and differential equations, while skipping simple number theory, the lessons will be more difficult to understand. You will end up memorizing instead of understanding, and as a result you will forget all of it. When you truly understand some thing, it can never be forgotten. A nice example is the solution to a quadratic equation: If you learn how to *complete the square* (p. 59), you will never forget the quadratic formula. Complex numbers naturally follow.

Math (the syntax) is a language: It seems strange when people complain that they "can't learn math,"[5] but then claim to be good at languages. Pre-high-school students tend to confuse arithmetic with math. One does not need to be good at arithmetic to be good at math (but it doesn't hurt).

Math is a language, with the symbols taken from various languages, with a bias toward Greek, obviously due to the popularity of Euclid's Elements. Learning languages is an advanced social skill. However the social outcomes are very different between learning a language and math. Learning a new language is fun because it opens doors to other cultures.

Math is different due to the rigor of the rules of the language, along with the way it is taught (e.g., not as a language). A third difference between math and language is that math evolved from physics, with important technical applications.

The rules of math are defined by algebra. For example, the sentence $a = b$ means that the number a has the same value as the number b. The sentence is spoken as "a equals b." The numbers are nouns and the equal sign says they are equivalent, playing the role of a verb, or action symbol. Following the rules of algebra, this sentence may be rewritten as $a - b = 0$. Here the symbols for minus and equal indicate two types of actions.

Sentences can become arbitrarily complex, such as the definition of the integral of a function, or a differential equation. But in each case, the mathematical sentence is written down, may be read out loud, has a well defined meaning, and may be manipulated into equivalent forms following the

[5] "It looks like Greek to me."

rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, known as theorems.

The writer of an equation should always translate (explicitly summarize the meaning of the expression), so the reader will not miss the main point, as a simply a matter of clear writing.

Just as math is a language, so may language be thought of as mathematics. To properly write correct English it is necessary to understand the construction of the sentence. It is important to identify the subject, verb, object, and various types of modifying phrases. Look up the interesting distinction between *that* and *which*.[6] Thus, like math, language has rules. Most individuals use what "sounds right," but if you're learning English as a second language, it is necessary to understand the rules, which are arguably easier to master than the foreign speech sounds.

1.1.3 Early physics as mathematics: Back to Pythagoras

The role of mathematics is to summarize algorithms (i.e., sets of rules), and formalize the idea as a theorem. Pythagoras and his followers, the Pythagoreans, believed that there was a fundamental relationship between mathematics and the physical world. The Pythagoreans may have been the first to capitalize on the relationship between science and mathematics, to use mathematics to design things for profit.[7] This may have been the beginning of capitalizing technology, based on the relationship between physics and math. This impacted commerce in many ways, such as map making, tools, implements of war (the wheel, gunpowder), art (music), water transport, sanitation, secure communication, food, etc. Of course the Chinese were the first to master many of these technologies.

Why is Eq. 1.1 called a *theorem*, and what exactly needs to be proved? We do not need to prove that (a, b, c) obey this relationship, since this is a condition that is observed. We do not need to prove that a^2 is the area of a square, as this is the definition of the area of a square. What needs to be proved is that the relation $c^2 = a^2 + b^2$ holds *if, and only if,* the angle between the two shorter sides is $90°$. The Pythagorean theorem (Eq. 1.1) did not begin with Euclid or Pythagoras, rather they appreciated its importance, and documented it.

In the end the Pythagoreans, who instilled fear in the neighborhood, were burned out, and murdered. This may be the fate of mixing technology with politics:

> "Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Mesopotamia in 497 BCE."
>
> –Stillwell (2010, p. 16)

1.1.4 Modern mathematics is born

Modern mathematics (what we practice today) was born in the 15-16th centuries, in the minds of Leonardo da Vinci, Bombelli, Galileo, Descartes, Fermat, and many others (Stillwell, 2010). Many of these early masters were, like the Pythagoreans, secretive to the extreme about how they solved problems. This soon changed due to Galileo, Mersenne, Descartes and Newton, causing mathematics to blossom. During this time the developments were hectic, seemingly disconnected.

[6] https://en.oxforddictionaries.com/usage/that-or-which

[7] It seems likely that the Chinese and Egyptians also did this, but it is more difficult to document.

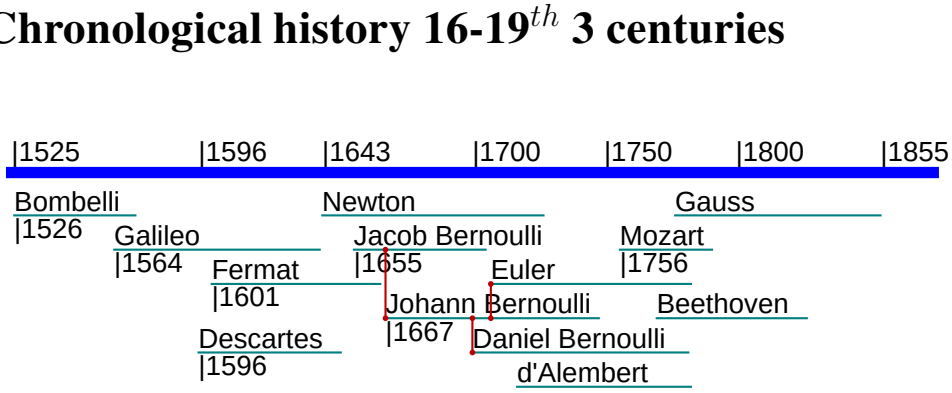

Figure 1.2: *Time-line for the development of the theory of calculus, between 16^{th} and 19^{th} CE. Given the time-line, it seems likely that Bombelli's discovery of Diophantus's book "Arithmetic" in the Vatican library triggered many of the ideas presented by Galileo, Descartes and Fermat, followed by others (i.e., Newton). Bombelli's discovery might be considered as a magic moment in mathematics. The vertical red lines indicate mentor-student relationships. See Fig. 1.1 (p. 19) gives the time-line from 1500BCE to 1650CE, Fig. 1.13 (p. 56) presents the 17-20 CE (Newton–Einstein) view from 1640–1950, and Fig. 1.24 (p. 103) outlines the full (Bombelli–Einstein, 1525-1925) period.*

This is a wrong impression, as the development was dependent on new technologies such as the telescope (optics) and more accurate time and frequency measurements, due to Galileo's studies of the pendulum, and a better understanding of the relation $f\lambda = c$, between frequency f and wavelength λ.

1.1.5 Science meets mathematics

Galileo: In 1589 Galileo famously conceptualized an experiment where he suggested dropping two different weights from the Leaning Tower of Pisa, and he suggested that they must take the same time to hit the ground. Conceptually this is a mathematically sophisticated experiment, driven by a mathematical argument in which he considered the two weights to be connected by an elastic cord (a spring). His studies resulted in the concept of *conservation of energy*, one of the cornerstones of physical theory since that time.

By joining them with an elastic cord, the masses become one. If the velocity were proportional to the mass, as believed by Archimedes, the sum of the two weights would necessarily fall even faster. This results in a logical fallacy: How can two masses fall faster than either? This also violates the concept of conservation of energy, as the total energy of two masses would be greater than that of the parts. In fact Galileo's argument may have been the first time that the principle of *conservation of energy* was clearly stated.

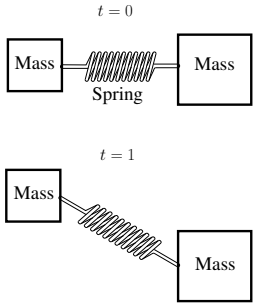

Figure 1.3: *Depiction of the argument of Galileo (unpublished book of 1638) as to why weights of different masses (i.e., weight) must fall with the same velocity, contrary to what Archimedes had proposed c250 BCE.*

It seems likely that Galileo was attracted to this model of two masses connected by a spring because he was also interested in planetary motion, which consist of masses (sun, earth, moon), also mutually attracted by gravity.

Galileo also performed related experiments on pendulums, where he varied the length l, mass m, and and angle θ of the swing. By measuring the period (periods/unit time) he was able to formulate precise rules between the variables. This experiment also measured the force exerted by gravity, so the experiments were related, but in very different ways. The pendulum served as the ideal clock, as it needed very little energy to keep it going, due to its very low friction (energy loss).

In a related experiment, Galileo measured the duration of a day by counting the number of swings of the pendulum in 24 hours, measured precisely by daily period of a star as it crossed the tip of a church steeple. The number of seconds in a day is precisely an integer (rounded to the nearest integer). It is the product of 24 [hr/day] $\times 60 \times 60$ [s/hr] = 86400 = $2^7 3^3 5^2$ [s/day]. This number may be reduced many ways, and remain precise, starting with seven factors of 2, 2 factors of 3 or 1 factor of 5. Factoring the number of days in a year (5*73) is impractical, thus cannot be easily treated as an integer.

Galileo also extended work on the relationship of wavelength and frequency of a sound wave in musical instruments. On top of these impressive accomplishments, Galileo greatly improved the telescope, which he needed for his observations of the planets.

Many of Galileo's contributions resulted in new mathematics, leading to Newton's discovery of the wave equation (c1687), followed 60 years later by its one-dimensional general solution by d'Alembert (c1747).

Mersenne: Mersenne (1588–1648) also contributed to our understanding of the relationship between the wavelength and the dimensions of musical instruments. At first Mersenne strongly disagreed with Galileo, partially due to errors in Galileo's reports of his results. But once Mersenne saw the significance of Galileo's conclusion, he being Galileo's strongest advocate, helping to spread the word (Palmerino, 1999).

Newton: With the closure of Cambridge University due to the plague of 1665, Newton returned home to Woolsthorpe-by-Colsterworth (95 [mi] north of London), to worked by himself, for over a year. It was during this time he did his most creative work.

While Newton (1642–1726) may be best known for his studies on light, he was the first to predict the speed of sound. However his theory was in error[8] by $\sqrt{c_p/c_v} = \sqrt{1.4} = 1.183$. This famous error would not be resolved for over two hundred years, awaiting the formulation of thermodynamics and the equi-partition theorem, by Maxwell and Boltzmann, and others.

Just 11 years prior to Newton's 1687 Principia, there was a basic understanding that sound and light traveled at very different speeds, due to the experiments of Ole Rømer.[9] [10]

> Ole Rømer first demonstrated in 1676 that light travels at a finite speed (as opposed to instantaneously) by studying the apparent motion of Jupiter's moon Io. In 1865, James Clerk Maxwell proposed that light was an electromagnetic wave, and therefore traveled at the speed c_o appearing in his theory of electromagnetism.[11]

[8] The square root of the ratio of the *specific heat capacity* at constant pressure c_p to that at constant volume c_v.

[9] `https://www.youtube.com/watch?v=b9F8Wn4vf5Y`

[10] `https://www.youtube.com/watch?v=rZ0wx3uD2wo`

[11] `https://en.wikipedia.org/wiki/Speed_of_light`

Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Figure 13.11: Johann Bernoulli

Figure 10.4: Leonhard Euler

Figure 1.4: *Above: Jakob (1655–1705) and Johann (1667–1748) Bernoulli. Below: Leonhard Euler (1707-1783) and Jean le Rond d'Alembert (1717-1783). Euler was blind in his right eye, hence the left portrait view. The figure numbers are from Stillwell (2010).*

The idea behind Rømer's discovery was that due to the large distance between earth and Io, there was a difference between the period of the moon when Jupiter was closest to earth vs. when it was furthest from earth. This difference in distance caused a delay or advance in the observed eclipse of Io as it went behind Jupiter, delayed by the difference in time due to the difference in distance. It is like watching a video of a clock, delayed or speed up. When the video is slowed down, the time will be inaccurate (it will indicate an earlier time).

Studies of vision and hearing: Since light and sound (music) played such a key role in the development of the early science, it was important to fully understand the mechanism of our perception of light and sound. There are many outstanding examples where physiology impacted mathematics. Leonardo da Vinci (1452–1519) is well known for his early studies of the human body. Exploring our physiological senses requires a scientific understanding of the physical processes of vision and hearing, first considered by Newton (1687) (1643–1727), but first properly researched much later by Helmholtz (1863a)[12] (Stillwell, 2010, p. 261). Helmholtz's (1821–1894) studies and theories of music and the perception of sound are fundamental scientific contributions (Helmholtz, 1863a). His best known mathematical contribution is today known as the *fundamental theorem of vector calculus*, or simply *Helmholtz theorem*.

The amazing Bernoulli family: The first individual who seems to have openly recognized the importance of mathematics, enough to actually teach it, was Jacob Bernoulli (1654–1705) (Fig. 1.4). Jacob worked on what is now viewed as the standard package of analytic "circular" (i.e., periodic) functions: $\sin(x)$, $\cos(x)$, $\exp(xj)$, $\log(x)$.[13] Eventually the full details were developed (for real variables) by Euler (Sections 1.3.1 p. 72 and H.1.1, p. 240).

From Fig. 1.2 we see that Jacob was contemporary with Descartes, Fermat, and Newton. Thus it seems likely that he was strongly influenced by Newton, who in turn was influenced by Descartes,[14] Vi\`{e}te and Wallis (Stillwell, 2010, p. 175).

Jacob Bernoulli, like all successful mathematicians of the day, was largely self-taught. Yet Jacob was in a new category of mathematicians, because he was an effective teacher. Jacob taught his sibling Johann (1667–1748), who then taught his sibling Daniel (1700–1782). But most importantly, Johann taught Leonhard Euler (1707–1783) (Figs. 1.4 and 1.24, p. 103), the most prolific (thus influential) of all mathematicians. This teaching resulted in an explosion of new ideas and understanding. It is most significant that all four mathematicians published their methods and findings. Much later, Jacob studied with students of Descartes[15] (Stillwell, 2010, p. 268-9).

Euler: Euler went far beyond all the Bernoulli family (Stillwell, 2010, p. 315). A special strength of Euler was the degree to which he published. First he would master a topic, and then he would publish. His papers continued to appear long after his death (Calinger, 2015). It is also interesting that Leonhard Euler was a contemporary of Mozart (and James Clerk Maxwell of Abraham Lincoln).

d'Alembert: Another individual of that time of special note, who also published extensively, was d'Alembert (Fig. 1.4). Some of the most creative ideas were first proposed by d'Alembert.

[12] `https://en.wikipedia.org/wiki/Acoustics`

[13] The $\log$ and $\tan$ functions are related by Eq. 1.81 p. 104.

[14] `https://en.wikipedia.org/wiki/Early_life_of_Isaac_Newton`

[15] It seems clear that Descartes was also a teacher.

Unfortunately, and perhaps somewhat unfairly, his rigor was criticized by Euler, and later by Gauss (Stillwell, 2010). But once the tools of mathematics were finally openly published, largely by Euler, mathematics grew exponentially.[16]

Gauss: Figure 1.24 (p. 103) shows the time-line of the most famous mathematicians. It was one of the most creative times in mathematics. Gauss was born at the end of Euler's long and productive life. I suspect that Gauss owed a great debt to Euler: surely he must have been a scholar of Euler. One of Gauss's most important achievement may have been his contribution to solving the open question about the density of prime numbers. Likely the best summary of Gauss's work is `http://www-history.mcs.st-andrews.ac.uk/Biographies/Gauss.html`.

- *The Pythagorean theorem is the mathematical spring which bore the three streams.*
- *Several centuries per stream:*

1) **Numbers:**

6^{th} *BCE* $\mathbb{N}$ *counting numbers,* $\mathbb{Q}$ *rationals,* $\mathbb{P}$ *primes*

5^{th} *BCE* $\mathbb{Z}$ *common integers,* $\mathbb{I}$ *irrationals*

7^{th} *CE* *zero* $\in \mathbb{Z}$

2) **Geometry:** *(e.g., lines, circles, spheres, toroids, . . .)*

17^{th} *CE* *Composition of polynomials (Descartes, Fermat)*
Euclid's geometry & algebra $\Rightarrow$ *analytic geometry*

18^{th} *CE* *Fundamental theorem of algebra*

3) **Infinity:** *(*$\infty \rightarrow$ *Sets)*

17-18^{th} *CE* *Taylor series, functions, calculus (Newton, Leibniz)*

19^{th} *CE* $\mathbb{R}$ *real,* $\mathbb{C}$ *complex 1851*

20^{th} *CE* *Set theory*

Figure 1.5: *Three streams followed from Pythagorean theorem: number systems (Stream 1), geometry (stream 2) and infinity (stream 3).*

1.1.6 Three Streams from the Pythagorean theorem

From the outset of his presentation Stillwell (2010, p. 1) defines "three great *streams* of mathematical thought: *Numbers, Geometry and Infinity*" that flow from the Pythagorean theorem, as summarized in Table 1.5. This is a useful concept, based on reasoning not as obvious as one might think. Many factors are in play here. One of these is the strongly held opinion of Pythagoras that all mathematics should be based on integers. The rest are tied up in the long, necessarily complex history of mathematics, as best summarized by the fundamental theorems (Table 1.1, p. 38), each of which is discussed in detail in a relevant chapter. Stillwell's concept of three streams, following from the Pythagorean theorem, is the organizing principle behind the this book:

[16] There are at least three useful exponential scales: Factors of 2, factors of $e \approx 2.7$, and factors of 10. The decibel uses factors of 2 (6 [dB]) and factors of 10 (20 [dB]). Information theory uses powers of 2 (1 [bit]), 4 (2 [bits]). Circuit theory uses factors of both e and 10.

1. *Introduction* Chapter 1 is intended to be a self-contained survey of basic pre-college mathematics, as a detailed overview of the fundamentals, presented as three streams:

   1.2 Number systems (Stream 1)

   1.3 Algebraic equations (Stream 2)

   1.4 Scalar calculus (Stream 3a)

   1.5 Vector calculus (Stream 3b)

   If you're a student, stick to Chapter 1.

   Chapters 2-5 are disorganized rambling research ideas that have not yet found a home. Students, please stay out of these chapters.

2. *Number Systems* (Chapter G: Stream 1) Some uncertain ideas of number systems, starting with prime numbers, through complex numbers, vectors and matrices.

3. *Algebraic Equations* (Chapter H: Stream 2) Algebra and its development, as we know it today. The theory of real and complex equations and functions of real and complex variables. Complex impedance $Z(s)$ of complex frequency $s = \sigma + \omega\jmath$ is covered with some care, developing the topic which is needed for engineering mathematics.

4. *Scalar Calculus* (Chapter I: Stream 3a) Ordinary differential equations. Integral theorems. Acoustics.

5. *Vector Calculus*: (Chapter J: Stream 3b) Vector partial differential equations. Gradient, divergence and curl differential operators. Stokes's and Green's theorems. Maxwell's equations.

1.2 Stream 1: Number Systems

Number theory (discrete, i.e., integer mathematics) was a starting point for many key ideas. For example, in Euclid's geometrical constructions the Pythagorean theorem for real $[a, b, c]$ was accepted as true, but the emphasis in the early analysis was on integer constructions, such as Euclid's formula for Pythagorean triplets (Eq. 1.9, Fig. 1.9, p. 48).

As we shall see, the Pythagorean theorem is a rich source of mathematical constructions, such as composition of polynomials, and solutions of Pell's equation by eigen-vector and recursive analysis methods. Recursive difference equation solutions predate calculus, going back at least to the Chinese (c2000 BCE). These are early (pre-limit) forms of differential equations, best analyzed using an eigen-vector or eigen-function expansion (a geometrical concept from linear algebra), as an orthogonal set of normalized (unit-length) vectors (Appendix D, p. 199),

The first use of zero and ∞: It is hard to imagine that one would not appreciate the concept of zero and negative numbers when using an abacus. It does not take much imagination to go from counting numbers $\mathbb{N}$ to the set of all integers $\mathbb{Z}$, including zero. On an abacus, subtraction is obviously the inverse of addition. Subtraction, to obtain zero abacus beads, is no different than subtraction from zero, giving *negative* beads. To assume the Romans who first developed counting

sticks, or the Chinese who then deployed the concept using beads, did not understand negative numbers, is impossible.

However, understanding the concept of zero (and negative numbers) is not the same as having a symbolic notation. The Roman number system has no such symbols. The first recorded use of a symbol for zero is said to be by Brahmagupta[17] in 628 CE.[18] [19] Defining zero (c628 CE) depends on the concept of subtraction, which formally requires the creation of algebra (c830 CE, Fig. 1.1, p. 19). But apparently it takes more than 600 years, i.e., from the time Roman numerals were put into use, without any symbol for zero, to the time when the symbol for zero is first documented. Likely this delay is more about the political situation, such as government rulings, than mathematics.

The concept that caused much more difficulty was ∞, first solved by Riemann in 1851 with the development of the *extended plane*, which mapped the plane to a sphere (Fig. 1.19 p. 93). His construction made it clear that the point at ∞ is simply another point on the open plane, since rotating the sphere (extended plane) moves the point at ∞ to a finite point on the plane, thereby closing the plane.

1.2.1 Lec 2: The Taxonomy of Numbers: $\mathbb{N}, \mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{I}, \mathbb{R}, \mathbb{C}$

Once symbols for zero and negative numbers were accepted, progress could be made. To fully understand numbers, a transparent notation was required. First one must differentiate between the different classes (genus) of numbers, providing a notation that defines each of these classes, along with their relationships. It is logical to start with the most basic *counting numbers*, which we indicate with the double-bold symbol $\mathbb{N}$. For easy access, double-bold symbols and set-theory symbols, i.e., $\{\cdot\}, \cup, \cap, \in, \notin, \perp$ etc., are summarized in Appendix A.

Counting numbers $\mathbb{N}$: These are known as the "natural numbers" $\mathbb{N} = 1, 2, 3, \cdots$, denoted by the double-bold symbol $\mathbb{N}$. For clarity we shall refer to the natural numbers as *counting numbers*, since *natural* here means *integer*. The mathematical sentence "$2 \in \mathbb{N}$" is read as *2 is a member of the set of counting numbers*. The word *set* is defined as *the collection of any objects that share a specific property*. Typically the set may be defined either as a sentence, or by example.

Primes $\mathbb{P}$: A number is prime ($\pi_n \in \mathbb{P}$) if its only factors are 1 and itself.[20] The set of Primes $\mathbb{P}$ is a *subset* of the counting numbers ($\mathbb{P} \subset \mathbb{N}$). A somewhat amazing fact, well known to the earliest mathematicians, is that every integer may be written as a unique product of primes. A second key idea is that the density of primes $\rho_\pi(N) \sim 1/\log(N)$ ($\rho_\pi(N)$ is inversely proportional to the log of N (Eq. G.1, p. 226), an observation first made by Gauss (Goldstein, 1973). A third is that there is a prime between every integer N and $2N$, excluding $2N$.

Exercise: Write out the first 10 to 20 integers in prime-factored form. **Solution:** $1, 2, 3, 2^2, 5, 2 \cdot 3, 7, 2^3, 3^2, 2 \cdot 5, 11, 3 \cdot 2^2, 13, 2 \cdot 7, 3 \cdot 5, 2^4, 17, 2 \cdot 3^2, 19, 2^2 \cdot 5$.

[17] http://news.nationalgeographic.com/2017/09/origin-zero-bakhshali-manuscript-video-spd/
[18] The fall of the Roman Empire has been established as Sept. 4, 476 CE.
[19] https://www.nytimes.com/2017/10/07/opinion/sunday/who-invented-zero.html
[20] 40 primes are generated by the quadratic $P_k = k^2 - k + 41$, $k = 1 : 40$. But why?

Exercise: Write integers 2 to 20 in terms of π_n. Here is a table to assist you:

n	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
π_n	2	3	5	7	11	13	17	19	23	29	31	$\cdots$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$
$\Pi\pi_n$	π_1	π_2	π_1^2	π_3	$\pi_1\pi_2$	π_4	π_1^3	π_2^2	$\pi_1\pi_3$	π_5	$\pi_1^2\pi_2$	π_6	$\pi_1\pi_4$	$\cdots$

We shall use the convenient notation π_n for the prime numbers, indexed by $\mathbb{N}$. The first 12 primes ($n = 1, \ldots, 12$) are π_n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37. Since $4 = 2^2$ and $6 = 2 \cdot 3$ may be factored, $4, 6 \notin \mathbb{P}$ (read as: *4 and 6 are not in the set of primes*). Given this definition, multiples of a prime, i.e., $[2, 3, 4, 5, \ldots] \cdot \pi_k$ of any prime π_k, cannot be prime. It follows that all primes except 2 must be odd and every integer N is unique in its factorization.

Coprimes are two numbers with no common prime factors. For example, $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$ are *not* coprime, as they have 2 as a common factor, whereas $21 = 3 \cdot 7$ and $10 = 2 \cdot 5$ *are*. By definition all pairs of primes are coprime. We shall use the notation $m \perp n$ to indicate that m, n are coprime. The ratio of two coprimes is reduced, as it has no factors to cancel. The ratio of two numbers that are not coprime may always be reduced by canceling the common factors. This is called the *reduced form*, or an *irreducible fraction*. When doing numerical work, for computational accuracy it is always beneficial to work with the reduced form.

The *fundamental theorem of arithmetic* states that each integer may be uniquely expressed as a product of primes. The *Prime Number Theorem* estimates the mean density of primes over $\mathbb{N}$.

Integers $\mathbb{Z}$: These include positive and negative counting numbers and zero. Notionally we might indicate this using *set notation* as $\mathbb{Z} = -\mathbb{N}, \{0\}, \mathbb{N}$. Read this as *The integers are in the set composed of the negative of the natural numbers* $-\mathbb{N}$*, zero, and* $\mathbb{N}$. Note that $\mathbb{N} \subset \mathbb{Z}$.

Rational numbers $\mathbb{Q}$: These are defined as numbers formed from the ratio of two integers. Given two numbers $n, d \in \mathbb{N}$, then $n/d \in \mathbb{Q}$. Since d may be 1, it follows that the rationals include the counting numbers as a subset. For example, the rational number 3/1 $\in \mathbb{N}$.

The main utility of rational numbers is that that they can efficiently approximate any number on the real line, to any precision. For example, the rational approximation $\pi \approx 22/7$, has a relative error of ≈0.04%.

Fractional number $\mathbb{F}$: A fractional number $\mathbb{F}$ is defined as the ratio of coprimes. If $n, d \in \mathbb{P}$, then $n/d \in \mathbb{F}$. Given this definition, $\mathbb{F} \subset \mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$. Because of the powerful approximating power of rational numbers, the fractional set $\mathbb{F}$ has special utility. For example, $\pi \approx 22/7$, $1/\pi \approx 7/22$ (to 0.04%), $e \approx 19/7$ to 0.15%, and $\sqrt{2} \approx 7/5$ to 1%.

Irrational numbers $\mathbb{I}$: Every real number that is not rational is *irrational* ($\mathbb{Q} \perp \mathbb{I}$). Irrational numbers include π, e and the square roots of primes. These are decimal numbers that never repeat, thus requiring infinite precision in their representation. Such numbers cannot be represented on a computer, as they would require an infinite number of bits (precision).

The rationals $\mathbb{Q}$ and irrationals $\mathbb{I}$ split the reals ($\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, $\mathbb{Q} \perp \mathbb{I}$), thus each is a subset of the reals ($\mathbb{Q} \subset \mathbb{R}$, $\mathbb{I} \subset \mathbb{R}$). This relation is analogous to that of the integers $\mathbb{Z}$ and fractionals $\mathbb{F}$, which split the rationals ($\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$, $\mathbb{Z} \perp \mathbb{F}$) (thus each is a subset of the rationals ($\mathbb{Z} \subset \mathbb{Q}$, $\mathbb{F} \subset \mathbb{Q}$)).

Irrational numbers ($\mathbb{I}$) were famously problematic for the Pythagoreans, who incorrectly theorized that all numbers were rational. Like ∞, irrational numbers required mastering a new and difficult concept before they could even be defined: It was essential to understand the factorization of counting numbers into primes (i.e., the fundamental theorem of arithmetic) before the concept of irrationals could be sorted out. Irrational numbers could only be understood once limits were mastered.

As discussed in Sect. Lec. 1.2.5 (p. 45), fractionals can approximate any irrational number with arbitrary accuracy. Integers are also important, but for a very different reason. All numerical computing today is done with $\mathbb{Q} = \mathbb{F} \cup \mathbb{Z}$. Indexing uses integers $\mathbb{Z}$, while the rest of computing (flow dynamics, differential equations, etc.) is done with fractionals $\mathbb{F}$ (i.e., IEEE-754). Computer scientists are trained on these topics, and computer engineers need to be at least conversant with them.

Real numbers $\mathbb{R}$: Reals are the union of rational and irrational numbers, namely $\mathbb{R} = \mathbb{Z} \cup \mathbb{F} \cup \mathbb{I}$. Lengths in Euclidean geometry are reals. Many people assume that *IEEE 754 floating point numbers* (c1985) are real (i.e., $\in \mathbb{R}$). In fact they are rational ($\mathbb{Q} = \{\mathbb{F} \cup \mathbb{Z}\}$) approximations to real numbers, designed to have a very large dynamic range. The hallmark of fractional numbers ($\mathbb{F}$) is their power in making highly accurate approximations of any real number.

Using Euclid's compass and ruler methods, one can make line length proportionally shorter or longer, or (approximately) the same. A line may be made be twice as long, an angle bisected. However, the concept of an integer length in Euclid's geometry was not defined.[21] Nor can one construct an imaginary or complex line, as all lines are assumed to be real.

Real numbers were first fully accepted only after set theory was developed by Cantor (1874) (Stillwell, 2010, pp. 461). At first blush, this seems amazing, given how widely accepted real numbers are today. In some sense they were accepted by the Greeks, as lengths of real lines.

Complex numbers $\mathbb{C}$: Complex numbers are best defined as *ordered pairs of real numbers*.[22] The word "complex," as used here, does *not* mean that the numbers are complicated or difficult. They are also know as "imaginary" numbers, but this does not mean the numbers disappear. Complex numbers are quite special in engineering mathematics, as roots of polynomials. The most obvious example is the quadratic formula, for the roots of polynomials of degree 2, having coefficients $\in \mathbb{C}$. All real numbers have a natural order on the real line. Complex numbers do not have a natural order. For example, $\jmath > 1$ makes no sense.

Today the common way to write a complex number is using the notation $z = a + b\jmath \in \mathbb{C}$, where $a, b \in \mathbb{R}$. Here $1\jmath = \sqrt{-1}$. We also define $1\imath = -1\jmath$ to account for the two possible signs of the square root. Accordingly $1\jmath^2 = 1\imath^2 = -1$.

Cartesian multiplication of complex numbers follows the basic rules of real algebra, for example, the rules of multiplying two polynomials. Multiplication of two first-degree polynomials (i.e., monomials) gives

$$(a + bx)(c + dx) = ac + (ad + bc)x + bdx^2$$

If we substitute $1\jmath$ for x, and use the definition $1\jmath^2 = -1$, we obtain the Cartesian product of the two complex numbers

$$(a + b\jmath)(c + d\jmath) = ac - bd + (ad + bc)\jmath.$$

[21] As best I know.

[22] A polynomial $a + bx$ and a 2-vector $[a, b]^T = \begin{bmatrix} a \\ b \end{bmatrix}$ are also examples of ordered pairs.

Thus multiplication and division of complex numbers obey the usual rules of algebra. However there is a critical extension: Cartesian multiplication only holds when the angles sum to less than $\pm\pi$, namely the range of the complex plane. When the angles add to more that $\pm\pi$, one must use polar coordinates, where the angles add, for angles beyond $\pm\pi$ (Boas, 1987, p. 8). This is particularly striking for the Laplace transform of a delay (Table F.3, p. 210).

Complex numbers can be challenging, providing unexpected results. For example, it is not obvious that $\sqrt{3+4\jmath} = \pm(2+\jmath)$.

Exercise: Verify. **Solution:** Squaring both sides $\sqrt{3+4\jmath}^2 = \pm(2+\jmath)^2 = 4-\jmath^2+4\jmath = 3+4\jmath$.

An alternative to Cartesian multiplication of complex numbers is to work in polar coordinates. The polar form of complex number $z = a + b\jmath$ is written in terms of its magnitude $\rho = \sqrt{a^2+b^2}$ and angle $\theta = \angle z = \tan^{-1} z = \arctan z$, as

$$z = \rho e^{\theta\jmath} = \rho(\cos\theta + \jmath\sin\theta).$$

From the definition of the complex natural log function

$$\ln z = \ln \rho e^{\theta\jmath} = \ln\rho + \theta\jmath,$$

which is important, even critical, in engineering calculations.[23] When the angles of two complex numbers are greater that $\pm\pi$, one must use polar coordinates. It follows that when computing the phase, this is much different than the single- and double-argument $\angle\theta = \arctan(z)$ function.

The polar representation makes clear the utility of a complex number: Its magnitude scales while its angle Θ rotates. The property of scaling and rotating is what makes complex numbers useful in engineering calculations. This is especially obvious when dealing with impedances, which have complex roots with very special properties, as discussed in Sect. 1.3.5 (p. 76).

Matrix representation: An alternative way to represent complex numbers is in terms of 2x2 matrices. This relationship is defined by the mapping from a complex number to a 2x2 matrix

$$a + b\jmath \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad 1 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad 1\jmath \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e^{\theta\jmath} \leftrightarrow \begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}. \tag{1.3}$$

The conjugate of $a + b\jmath$ is then defined as $a - b\jmath \leftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. By taking the inverse of the 2x2 matrix (assuming $|a + b\jmath| \neq 0$), one can define the ratio of one complex number by another. Until you try out this representation, it may not seem obvious, or even that it could work.

This representation proves that $1\jmath$ is not necessary when defining a complex number. What $1\jmath$ can do is to conceptually simplify the algebra. It is worth your time to become familiar with the matrix representation, to clarify any possible confusions you might have about multiplication and division of complex numbers. This matrix representation can save you time, heartache and messy algebra. Once you have learned how to multiply two matrices, it's a lot simpler than doing the complex algebra. In many cases we will leave the results of our analysis in matrix form, to avoid the algebra altogether.[24] Thus both representations are important. More on this topic may be found in Chapter G.

[23] Chapter 2 discusses the definition of the phase, i.e., how is it computed (i.e., $\arctan(e^{\theta\jmath})$, `arctan2(x,y)`), and the importance of the unwrapped phase, as in the example (Table F.3, p. 210) $\delta(t-\tau) \leftrightarrow e^{-\tau\jmath}$.

[24] Sometimes we let the computer do the final algebra, numerically, as 2x2 matrix multiplications.

History of complex numbers: It is notable how long it took for complex numbers to be accepted (1851), relative to when they were first introduced by Bombelli (16^{th} century CE). One might have thought that the solution of the quadratic, known to the Chinese, would have settled this question. It seems that complex integers (aka, *Gaussian integers*) were accepted before non-integral complex numbers. Perhaps this was because real numbers ($\mathbb{R}$) were not accepted (i.e., proved to exist, thus mathematically defined) until the development of *set theory* in the late 19^{th} century to sort out a proper definition of the real number, due to the existence of irrational numbers.

Exercise: Using Matlab/Octave, verify that

$$\frac{a+b\jmath}{c+d\jmath} = \frac{ab+bd+(bc-ad)\jmath}{c^2+d^2} \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2+d^2}. \quad (1.4)$$

Solution: The best way may be using numbers. Below is symbolic code, independent of numerics:

```
syms a b c d A B
A=[a -b;b a];
B=[c -d;d c];
C=A*inv(B)
```

Numerical taxonomy:

A simplified taxonomy of numbers is given by the mathematical sentence

$$\pi_k \in \mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

This sentence says:

1. Every prime number π_k is in the set of primes $\mathbb{P}$,
2. which is a subset of the set of counting numbers $\mathbb{N}$,
3. which is a subset of the set of integers $\mathbb{Z} = -\mathbb{N}, 0, \mathbb{N}$,
4. which is a subset of the set of rationals $\mathbb{Q}$ (ratios of counting numbers $\mathbb{N}$),
5. which is a subset of the set of reals $\mathbb{R}$,
6. which is a subset of the set of complex numbers $\mathbb{C}$.

The rationals $\mathbb{Q}$ may be further decomposed into the fractionals $\mathbb{F}$ and the integers $\mathbb{Z}$ ($\mathbb{Q} = \mathbb{F} \cup \mathbb{Z}$), and the reals $\mathbb{R}$ into the rationals $\mathbb{Q}$ and the irrationals $\mathbb{I}$ ($\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$). This classification nicely defines all the numbers used in engineering and physics.

The taxonomy structure may be summarize with the single compact sentence, starting with the prime numbers π_k and ending with complex numbers $\mathbb{C}$: Add Venn diag!

$$\pi_k \in \mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset (\mathbb{Z} \cup \mathbb{F} = \mathbb{Q}) \subset (\mathbb{Q} \cup \mathbb{I} = \mathbb{R}) \subset \mathbb{C}.$$

As discussed in Appendix A (p. 183), all numbers may be viewed as complex. Namely, every real number is complex if we take the imaginary part to be zero (Boas, 1987). For example, $2 \in \mathbb{P} \subset \mathbb{C}$. Likewise every purely imaginary number (e.g., $0 + 1\jmath$) is complex with zero real part.

Finally, note that complex numbers $\mathbb{C}$, much like vectors, do not have "rank-order," meaning one complex number cannot be larger or smaller than another. It makes no sense to say that $\jmath > 1$ or $\jmath = 1$ (Boas, 1987). The real and imaginary parts and the magnitude and phase, have order. If time t were complex, there could be no yesterday and tomorrow.[25]

Applications of integers

The most relevant question at this point is "Why are integers important?" First, we count with them, so we can keep track of "how much." But there is much more to numbers than counting: We use integers for any application where absolute accuracy is essential, such as banking transactions (making change), the precise computing of dates (Stillwell, 2010, p. 70) and location ("I'll meet you at 34 and Vine at noon on Jan. 1, 2034."), building roads or buildings out of bricks (objects built from a unit size).

To navigate we need to know how to predict the tides, the location of the moon and sun, etc. Integers are important because they are precise: Once a month there is a full moon, easily recognizable. The next day its slightly less than full. If one could represent our position as integers in time and space, we would know exactly where we are at all times. But such an integral representation of our position or time is not possible.

The Pythagoreans claimed that all was integer. From a practical point of view, it seems they were right. Today all computers compute floating point numbers as fractionals. However, in theory they were wrong. The difference is a matter of precision.

Numerical Representations of $\mathbb{I}, \mathbb{R}, \mathbb{C}$: When doing numerical work, one must consider how we may compute within the set of reals (i.e., which contain irrationals). There can be no irrational number representation on a computer. The international standard of computation, IEEE floating point numbers,[26] are actually rational approximations. The mantissa and the exponent are both integers, having sign and magnitude. The size of each integer depends on the precision of the number being represented. An IEEE floating-point number is rational because it has a binary (integer) mantissa, multiplied by 2 raised to the power of a binary (integer) exponent. For example, $\pi \approx a2^b$ with $a, b \in \mathbb{Z}$. In summary, IEEE floating-point numbers are not, and cannot, be irrational, because numerical representations would imply an infinite number of bits.

True floating point numbers contain irrational numbers, which must be approximated by rational numbers. This leads to the concept of *fractional representation*, which requires the definition of the *mantissa*, *base* and *exponent*, where both the mantissa and the exponent are signed. Numerical results must not depend on the base. One could dramatically improve the resolution of the numerical representation by the use of the *fundamental theorem of arithmetic* (Section 1.2.2, page 37). For example, one could factor the exponent into its primes and then represent the number as $a2^b3^c5^d7^e$ $(a, b, c, d, e \in \mathbb{Z})$, etc. Such a representation would improve the resolution of the representation. But even so, the irrational numbers would be approximate. For example, base ten[27] is

[25] One can define $\xi = x + 1j\ ct$ to be complex $(x, t \in \mathbb{R})$, with x in meters [m], t is in seconds [s], and the speed of light c_o [m/s].

[26] IEEE 754: `http://www.h-schmidt.net/FloatConverter/IEEE754.html`.

[27] Base 10 is the natural world-wide standard simply because we have 10 fingers.

natural using this representation since $10^n = 2^n 5^n$. Thus

$$\pi \cdot 10^5 \approx 314\,159.27\ldots = 3 \cdot 2^5 5^5 + 1 \cdot 2^4 5^4 + 4 \cdot 2^3 5^3 + \cdots + 9 \cdot 2^0 5^0 + \cancel{2 \cdot 2^{-1} 5^{-1}} \cdots .$$

If we approximate π by 22/7, then according to the Matlab/Octave DEC2BIN() routine, the binary representation is ck 4 errors

$$\pi \cdot 2^{17} \approx 131072_{10} \cdot 22/7 = 110, 0100, 1001, 0010, 0101_2,$$

where 1 and 0 are multipliers of powers of 2, which are then added together as follows:

$$2^{18} + 2^{17} + 2^{14} + 2^{11} + 2^8 + 2^5 + 2^2 + 2^0.$$

In base 16 (i.e., hexadecimal) $2^{17} \cdot 22/7 = 2^{18} \cdot 8_{16}/7_{16}$.

Computers keep track of the decimal point using the exponent, which in this case is the factor $2^{17} = 131072_{10}$. The concept of the decimal point is replaced by an integer, having the desired precision, and a scale factor of any base (radix). This scale factor may be thought of as moving the decimal point to the right (larger number) or left (smaller number). The mantissa "fine-tunes" the value about a scale factor (the exponent).

Example: $x = 2^{17} \times 22/7$, using IEEE 754 double precision,[28] $x = 411,940.5625_{10} = 2^{54} \times 1198372 = 0, 10010, 00, 110010, 010010, 010010, 010010_2 = 0x48c92492_{16}$. The exponent is 2^{18} and the mantissa is $4,793,490_{10}$. The commas in the binary (0,1) string are to help visualize the quasi-periodic nature of the bit-stream. The numbers are stored in a 32 bit format, with 1 bit for sign, 8 bits for the exponent and 23 bits for the mantissa. Perhaps a more instructive number is $x = 4,793,490.0 = 0, 100, 1010, 100, 100, 100, 100, 100, 100, 100, 100_2 = 0x4a924924_{16}$, which has a repeating binary bit pattern of $((100))_3$, broken by the scale factor $0x4a$. Even more symmetrical is $x = 0x24,924,924_{16} = 00, 100, 100, 100, 100, 100, 100, 100, 100, 100, 100_2 = 6.344,131,191,146,9 \times 10^{-17}$. In this example the repeating pattern is clear in the hex representation as a repeating $((942))_3$, as represented by the double brackets, with the subscript indicating the period, in this case, three digits. As before, the commas are to help with readability and have no other meaning.

The representation of numbers is not unique. For example, irrational complex numbers have approximate rational representations (i.e., $\pi \approx 22/7$). A better example is complex numbers $z \in \mathbb{C}$, which have many representations, as a pair of reals (i.e., $z = (x, y)$), or by Euler's formula, and matrices ($\theta \in \mathbb{R}$)

$$e^{j\theta} = \cos\theta + j\sin\theta \leftrightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

At a higher level, differentiable functions, aka, analytic functions, may be represented by a single-valued Taylor series expansion (Sect. 1.3.3, p. 67), limited by the its *region of convergence* (RoC).

Pythagoreans, Integers The integer is the cornerstone of the Pythagorean doctrine, so much so that it caused a fracture within the Pythagoreans when it was discovered that not all numbers are rational. The famous proof of irrational numbers is known as the spiral of Theodorus: the short side is 1 and the long side is recursively incremented by one, using a simple compass-ruler construction to maintain the right triangle.

To form triangles with perfect 90° angles, the lengths need to satisfy Eq. 1.1. Such triangles are useful in constructing buildings or roads made from bricks of uniform size.

[28] http://www.h-schmidt.net/FloatConverter/IEEE754.html

Public-key Security: An important application of prime numbers is public-key encryption, essential for internet security applications (e.g., online banking). Most people assume encryption is done by a personal login and passwords. Passwords are fundamentally insecure, for many reasons. Decryption depends on factoring large integers, formed from products of primes having thousands of bits.[29]The security is based on the relative ease of multiplying large primes, along with the virtual impossibility of factoring their products.

When a computation is easy in one direction, but its inverse is impossible, it is called a *trapdoor function*. We shall explore trapdoor functions in Chapter G. If everyone were to switch from passwords to public-key encryption, the internet would be much more secure.

Prob → HW

Puzzles: Another application of integers is imaginative problems that use integers. An example is the classic Chinese *four stone problem:* "Find the weight of four stones that can be used with a scale to weigh anything (e.g., salt, gold) between 0, 1, 2, . . . , 40 [gm]." As with the other problems, the answer is not as interesting as the method, since the problem may be easily recast into a related one. This type of problem can be found in airline magazines as amusement on a long flight. This puzzle is best cast as a linear algebra problem, with integer solutions. Again, once you know the trick, it is "easy."[30]

1.2.2 Lec 3: The role of physics in mathematics

Bells, chimes and eigen-modes: Integers naturally arose in art, music and science. Examples include the relations between musical notes, the natural eigen-modes (tones) of strings and other musical instruments. These relations were so common and well studied, it appeared that to understand the physical world (aka, the Universe), one needed to understand integers. This was a seductive view, but not actually correct. As will be discussed in Sections 1.3.1 (p. 56) and H.1.1 (p. 239), it is best to view the relationship between acoustics, music and mathematics as historical, since these topics played such an important role in the development of mathematics. Also interesting is the role that integers play in quantum mechanics, also based on eigen-modes, but in this case, those of atoms. Eigen-modes follow from solutions of the wave equation, which has natural delay due to d'Alembert's solution, along with reflecting boundary conditions (Eq. 1.163, p. 150), resulting in roots of the characteristic equation.

Engineers are so accustomed to working with real (or complex) numbers, the distinction between real (i.e., irrational) and fractional numbers is rarely acknowledged. Integers on the other hand arise in many contexts. One cannot master programming computers without understanding integer, hexadecimal, octal, and binary representations, since all numbers in a computer are represented in numerical computations in terms of rationals ($\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$).

As discussed in Section 1.2.1 (p. 29), the primary reason integers are so important is their absolute precision. Every integer $n \in \mathbb{Z}$ is unique,[31] and has the *indexing property*, which is essential for making lists that are ordered, so that one can quickly look things up. The alphabet also has this property (e.g., a book's index).

[29]It would seem that public-key encryption could work by having two numbers with a common prime, and then using the Euclidean algorithm, the greatest common divisor (GCD) could be worked out. One of the integers could be the public-key and the second could be the private key.

[30]Whenever someone tells you something is "easy," you should immediately appreciate that it is very hard, but once you learn a concept, the difficulty evaporates.

[31]Check out the history of $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

Because of the integer's absolute precision, the digital computer overtook the analog computer, once it was practical to make logic circuits that were fast. From 1946 the first digital computer was thought to be the University of Pennsylvania's *Eniac*. We now know that the code-breaking effort in Bletchley Park, England, under the guidance of Alan Turing, created the first digital computer (The Colossus) used to break the WWII German "Enigma" code. Due to the high secrecy of this war effort, the credit was only acknowledged in the 1970s when the project was finally declassified.

There is zero possibility of analog computing displacing digital computing, due to the importance of precision (and speed). But even with binary representation, there is a non-zero probability of error, for example on a hard drive, due to physical noise. To deal with this, error correcting codes have been developed, reducing the error by many orders of magnitude. Today error correction is a science, and billions of dollars are invested to increase the density of bits per area, to increasingly larger factors. A few years ago the terabyte drive was unheard of; today it is the standard. In a few years petabyte drives will certainly become available. It is hard to comprehend how these will be used by individuals, but they are essential for on-line (cloud) computing.

The role of mathematics in physics

Modern mathematics is built on a hierarchical construct of *fundamental theorems*, as summarized in Table 1.1. The importance of such theorems cannot be overemphasized. Gauss's and Stokes's laws play a major role in understanding and manipulating Maxwell's equations. Every engineering student needs to fully appreciate the significance of these key theorems. If necessary, memorize them. But that will not do over the long run, as each and every theorem must be fully understood. Fortunately most students already know several of these theorems, but perhaps not by name. In such cases, it is a matter of mastering the vocabulary.

The theorems are naturally organized and may be thought of in terms of Stillwell's three streams. For Stream 1 there is the *fundamental theorem of arithmetic* and the *prime number theorem*. For Stream 2 there is the *fundamental theorem of algebra* while for Stream 3 there are a host of theorems on calculus, ordered by their dimensionality. Some of these theorems verge on trivial (e.g., the fundamental theorem of arithmetic). Others are more challenging, such as the *fundamental theorem of vector calculus* and *Green's theorem*.

Complexity should not be confused with importance. Each of these theorems is, as stated, fundamental. Taken as a whole, they are a powerful way of summarizing mathematics.

Stream 1: Prime number theorems

There are three fundamental theorems about primes,

1. *The fundamental theorem of arithmetic*: This states that every integer $n \in \mathbb{Z}$ may be uniquely factored into prime numbers. This raises the question of the meaning of *factor* (split into a product). The *product* of two integers $m, n \in \mathbb{Z}$ is $mn = \sum_m n = \sum_n m$. For example, $2 * 3 = 2 + 2 + 2 = 3 + 3$.

2. *The prime number theorem:* One would like to know how many primes there are. That is easy: $|\mathbb{P}| = \infty$ (the size of the set of primes is infinite). One way of asking this questions is *What is the average density of primes, in the limit as* $n \to \infty$*?* This question was answered, for all practical purposes, by Gauss, who in his free time computed the first three million primes by hand. He discovered that, to a good approximation, the primes are equally likely

Table 1.1: *The fundamental theorems of mathematics*

1. Fundamental theorems of:
   (a) **Number systems: Stream 1**
      - arithmetic
      - prime number
   (b) **Geometry: Stream 2**
      - algebra
   (c) **Calculus: Stream 3**[a]
      - Leibniz $\mathbb{R}^1$
      - complex $\mathbb{C} \subset \mathbb{R}^2$
      - vectors $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$
        - Gauss's law (divergence theorem)
        - Stokes's law (curl theorem, or Green's theorem)
        - Vector calculus (Helmholtz's theorem)
2. Other key concepts:
   - Complex analytic functions (complex roots are finally accepted!)
     - Complex Taylor series (complex analytic functions)
     - Region of convergence (RoC) of complex analytic series (p. 70)
     - Laplace transform, and its inverse
     - Causal time $\implies$ complex frequency s
     - Cauchy integral theorem
     - Residue integration (i.e., Green's thm in $\mathbb{R}^2$)
   - Riemann mapping theorem (Gray, 1994; Walsh, 1973)
   - Complex impedance (Ohm's law) (Kennelly, 1893)

[a] Flanders, Harley (June–July 1973). "Differentiation under the integral sign." *American Mathematical Monthly* 80 (6): 615-627. doi:10.2307/2319163. JSTOR 2319163.

on a log scale. This is nicely summarized by the jingle, attributed to the mathematician Pafnuty Chebyshev (Stillwell, 2010, p. 585)

> Chebyshev said, and I say it again: *There is always a prime between n and 2n*

which nicely summarizes theorem 3.

When the ratio of two frequencies (pitch) is 2, the relationship is called an *octave*. Thus we might say, with a slight stretch of terminology, there is at least one prime per octave. Thus one might wonder about the maximum number of primes per octave. In modern music the octave is further divided into 12 intervals called *semitones* (factors), equal to the $\sqrt[12]{2}$. The product of 12 semitones is an octave. Thus one might ask how many primes there are per semitone? In the end, it is a question of the density of primes on a log (i.e., ratio) scale.

Stream 2: Fundamental theorem of algebra

This theorem states that every polynomial in x of degree N

$$P_N(x) = \sum_{k=0}^{N} a_k x^k \tag{1.5}$$

has at least one root (Section H.2.1, p. 244). When a common root is factored out, the degree of the polynomial is reduced by 1. Applied recursively, a polynomial of degree N has N roots. Note there are $N + 1$ coefficients (i.e., $[a_N, a_{N-1}, \cdots, a_0]$).

Stream 3: Fundamental theorems of calculus

In Sections 1.5.13 and 1.5.14 we will deal with each of the theorems for Stream 3, where we consider the several fundamental theorems of integration, starting with Leibniz's formula for integration on the real line ($\mathbb{R}$), then progressing to complex integration in the complex plane ($\mathbb{C}$) (*Cauchy's theorem*), which is required for computing the inverse Laplace transform. Gauss's and Stokes's laws for $\mathbb{R}^2$ require closed and open surfaces, respectively. One cannot manipulate Maxwell's equations, fluid flow, or acoustics without understanding these theorems. Any problem that deals with the wave equation in more than one dimension requires an understanding of these theorems, thus they are the basis of the derivation of the Kirchhoff voltage and current laws. The ∇ symbol is pronounced as "del" (preferred) or "nabla."

Finally we define the four basic vector operations based on the ∇ "operator:" The *gradient* $\nabla()$, *divergence* $\boldsymbol{\nabla} \cdot ()$ *curl* $\boldsymbol{\nabla} \times ()$ and the *Laplacian* $\boldsymbol{\nabla} \cdot \nabla() = \nabla^2()$. The first three operations are defined in terms of integral operations on a surface in 1, 2 or 3 dimensions, by taking the limit as that surface, and the volume contained within, goes to zero. These three differential operators are essential to fully understand Maxwell's equations, the crown jewel of mathematical physics. Hence mathematics plays a key role in physics, as does physics in math.

Other key concepts

Besides the widely recognized fundamental theorems for the three streams, there are a number of equally important theorems that have not yet been labeled as "fundamental."[32]

[32] It is not clear what it takes to reach this more official sounding category.

The widely recognized *Cauchy integral theorem* is an excellent example, since it is a stepping stone to Green's theorem and the *fundamental theorem of complex calculus*. In Section 1.4.5 (p. 118) we clarify the contributions of each of these special theorems.

Once these fundamental theorems of integration (Stream 3) have been mastered, the student is ready for the *complex frequency domain*, which takes us back to Stream 2 and the *complex frequency plane* ($s = \sigma + \omega\jmath \in \mathbb{C}$). While the Fourier and Laplace transforms are taught in mathematics courses, the concept of *complex frequency* is rarely mentioned. The *complex frequency domain* (p. 112) and *causality* are fundamentally related (Sects. 1.4.6–1.4.8, p. 120–122), and are critical for the analysis of signals and systems, and especially for the concept of impedance (Sect. 1.4.3, p. 109).

Without the concept of time and frequency, one cannot develop an intuition for the Fourier and Laplace transforms, especially within the context of engineering and mathematical physics. The Fourier transform covers signals, while the Laplace transform describes systems. Separating these two concepts, based on their representations as Fourier and Laplace transforms, is an important starting place for understanding physics and the role of mathematics. However, these methods, by themselves, do not provide the insight into physical systems necessary to be productive, or better, creative with these tools. One needs to master the tools of differential equations, and then partial differential equations to fully appreciate the world that they describe. Electrical and mechanical networks, composed of inductors, capacitors and resistors, are isomorphic to mechanical systems composed of masses, springs and dashpots. Newton's laws are analogous to those of Kirchhoff, which are the rules needed to analyze simple physical systems composed of linear (and nonlinear) sub-components. When lumped-element systems are taken to the limit, in several dimensions, we obtain Maxwell's partial differential equations, or the laws of continuum mechanics, and beyond.

The ultimate goal of this book is to make you aware of and productive in using these tools. This material can be best absorbed by treating it chronologically through history, so you can see how this body of knowledge came into existence, through the minds and hands of Galileo, Newton, Maxwell and Einstein. Perhaps one day you too can stand on the shoulders of the giants who went before you.

1.2.3 Lec 4: Prime numbers

If someone asked you for a theory of counting numbers, I suspect you would laugh and start counting. It sounds like either a stupid question, or a bad joke. Yet integers are a rich topic, so the question is not even slightly dumb. It is somewhat amazing that even birds and bees can count. While I doubt birds and bees can recognize primes, cicadas and other insects only crawl out of the ground in prime number cycles, (e.g., 13 or 17 year cycles). If you have ever witnessed such an event (I have), you will never forget it. Somehow they know. Finally, there is an analytic function, first introduced by Euler, based on his analysis of the sieve, now known as the *Riemann zeta function* $\zeta(s)$, which is complex analytic, with its poles at the logs of the prime numbers. The exact relationship between the primes and the poles will be discussed in Section H.4.2 (p. 259). The properties of this function are truly amazing, even fun.[33] Many of the questions and answers about primes go back to at least the early Chinese (c1500 BCE) (Stillwell, 2010).

[33]The Riemann zeta function is known as the million dollar equation as there is a cash reward for a proof of the Riemann Hypothesis.

The importance of prime numbers: Likely the first insight into the counting numbers started with the *sieve*, shown in Fig. 1.6. A sieve answers the question "How can one identify the prime numbers?" The answer comes from looking for irregular patterns in the counting numbers, by playing the counting numbers against themselves.

A recursive sieve method for finding primes was first devised by the Greek Eratosthenes[34] (276-194, BCE), and summarized in Fig. 1.6.

For example, starting from $\pi_1 = 2$ one strikes out $2{\cdot}2, 2{\cdot}3, 2{\cdot}4, 2{\cdot}5, \cdots, 2 \cdot \sqrt{N}$. By definition the multiples are products of the target prime (2 in our example) and every another integer ($n \geq 2$). In this way all the even numbers are removed in this first iteration. The next remaining integer (3 in our example) is identified as the next (second) prime π_2. Then all the $(N-2)/2$ multiples of $\pi_2 = 3$ are removed. The next remaining number is $\pi_3 = 5$, so all multiples of $\pi_3 = 5$ are removed (i.e., ~~10~~, ~~15~~, ~~25~~ etc., $\cdots$). This process is repeated until all the numbers of the list have either been canceled or identified as prime.

1. Write N integers from 2 to $N-1$. Let $k = 1$. The first element $\pi_1 = 2$ is a prime. Cross out $n \cdot \pi_n$ $(4, 8, 16, 32, \cdots)$.

	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~

2. Let $k = 2$, $\pi_2 = 3$. Cross out $n\pi_k$ $(9, 15, 21, 27, 33, 39, \cdots)$.

	2	**3**	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	35	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~

3. Let $k = 3, \pi_3 = 5$. Cross out $n\pi_3$ $(25, 35)$.

	2	**3**	~~4~~	**5**	~~6~~	**7**	~~8~~	~~9~~	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	~~40~~
41	~~42~~	**43**	~~44~~	~~45~~	~~46~~	**47**	~~48~~	~~49~~	~~50~~

4. Finally let $k = 4, \pi_4 = 7$. Cross out $n\pi_4$: (49). Thus there are 15 primes less than $N = 50$: $\pi_k = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$ (highlighted in red).

Figure 1.6: *Sieve of Eratosthenes for* $N = 50$.

As the word *sieve* implies, this process takes a heavy toll on the integers, rapidly pruning the non-primes. In four iterations of the sieve algorithm, all the primes below $N = 50$ are identified in

[34] https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes\#Euler.27s_Sieve

red. The final set of primes is displayed in step 4 of Fig. 1.6.

Once a prime greater than $N/2$ has been identified, the recursion stops, since twice that prime is greater than N, the maximum number under consideration. Once $\sqrt{N}$ has been reached all the primes have been identified (this follows from the fact that the next prime π_n is multiplied by an integer $n = 1, \ldots N$).

When using a computer, memory efficiency and speed are the main considerations. There are various schemes for making the sieve more efficient. For example, the recursion $n\pi_k = (n-1)\pi_k + \pi_k$ will speed up the process by replacing the multiply with an addition of π_k.

Two fundamental theorems of primes: Early theories of numbers revealed two fundamental theorems (there are many more than two), as discussed in Section 1.2.2 and G.1.1 (p. 225). The first of these is the *fundamental theorem of arithmetic*, which says that every integer greater than 1 may be uniquely factored into a product of primes

$$n = \prod_{k=1}^{K} \pi_k^{\beta_k}, \tag{1.6}$$

where $k = 1, \ldots, K$ indexes the integer's K *prime factors* $\pi_k \in \mathbb{P}$. Typically prime factors appear more than once, for example $25 = 5^2$. To make the notation compact we define the *multiplicity* β_k of each prime factor π_k. For example $2312 = 2^3 \cdot 17^2 = \pi_1^3\ \pi_7^2$ (i.e., $\pi_1 = 2, \beta_1 = 3; \pi_7 = 17, \beta_7 = 2$) and $2313 = 3^2 \cdot 257 = \pi_3^2\ \pi_{55}$ (i.e., $\pi_2 = 3, \beta_3 = 2; \pi_{55} = 257, \beta_{55} = 1$). Our demonstration of this is empirical, using the Matlab/Octave `factor(N)` routine, which factors N.[35]

What seems amazing is the unique nature of this theorem. Each counting number is *uniquely* represented as a product of primes. No two integers can share the same factorization. Once you multiply the factors out, the result is unique (N). Note that it's easy to multiply integers (e.g., primes), but expensive to factor them. And factoring the product of three primes is significantly more difficult than factoring two.

Factoring is much more expensive than division. This is not due to the higher cost of division over multiplication, which is less than a factor of 2.[36] Dividing the product of two primes, given one, is trivial, slightly more expensive that multiplying. Factoring the product of two primes is nearly impossible, as one needs to know what to divide by. Factoring means dividing by some integer and obtaining another integer *with remainder zero*.

Does this make sense?

Thus one could factor a product of primes $N = \pi_k \pi_l$ by doing M divisions, where M is the number of primes less than N. This assumes the list of primes less than N are known.

But the utility has to do with the density of primes (the prime number theorem, i.e., Gauss's hypothesis). If we were simply looking up a few numbers from a short list of primes, it would be easy, but the density of primes among the integers, is logarithmic (>1 per octave, Section G.1.1, p. 227).

This brings us to the *prime number theorem* (PNT). The security problem is the reason why these two theorems are so important: 1) Every integer has a unique representation as a product of primes, and 2) the density of primes is large (see the discussions on p. 39 and Section G.1.1, p. 226). Thus security reduces to the "needle in the haystack problem" due to the cost of a search.

exercise to show H is rm?

The formal way to measure the density is known as *Shannon entropy*, couched in terms of the

[35] If you wish to be a mathematician, you need to learn how to prove theorems. If you're a physicist, you are happy that someone else has already proved them, so that you can use the result.

[36] https://streamcomputing.eu/blog/2012-07-16/how-expensive-is-an-operation-on-a-cpu/

expected value of the log-probability of events: "What is the probability of finding a prime between N and $2N$?"[37]

HW problem?

1.2.4 Lec 5: Greatest common divisor (Euclidean algorithm)

The *Euclidean algorithm* is a method to find the *greatest common divisor* (GCD) k between two integers n, m, denoted $k = gcd(n, m)$, where $n, m, k \in \mathbb{N}$. For example, $15 = \text{gcd}(30, 105)$ since when factored $(30, 105) = (2 \cdot 3 \cdot 5,\ 7 \cdot 3 \cdot 5) = 3 \cdot 5 \cdot (2, 7) = 15 \cdot (2, 7)$. The Euclidean algorithm was known to the Chinese (i.e., not discovered by Euclid) (Stillwell, 2010, p. 41).

Why is the GCD important? The utility of the GCD algorithm arises directly from the fundamental difficulty in factoring large integers. Computing the GCD, using the Euclidean algorithm, is low cost, compared to factoring, which is extremely expensive. This utility surfaces when the two numbers are composed of very large primes. When two integers have no common factors they are said to be *coprime*, thus their GCD is 1. The ratio of two integers which are coprime is automatically in *reduced form* (they have no common factors). For example, 4/2 $\in \mathbb{Q}$ is not reduced since $2 = gcd(4, 2)$ (with no remainder). Canceling out the common factor 2 gives the reduced form 2/1=2 $\in \mathbb{N}$. Thus if we wish to form the ratio of two integers, first compute the GCD, then remove it from the two numbers to form the ratio. This assures the rational number is in its reduced form ($\in \mathbb{F}$ rather than $\in \mathbb{Q}$). If the GCD were 10^3 digits it is obvious that the common factor must be removed to greatly simplify further computation. This can make the difference between factoring it, or not, when using IEEE-754.

Exercise: Divide 10 into 99, and discuss where the GCD is used. **Solution:** When we divide a smaller number into a larger one, we must find the GCD and a remainder. For example 99/10 = 9 + 9/10 has a GCD of 9 and a remainder of 9/10. Thus we all learned how to compute the GCD in grade school, when we learned long division.

An example: Take the two integers $[873, 582]$. In factored form these are $[\pi_{25} \cdot 3^2, \pi_{25} \cdot 3 \cdot 2]$. Given the factors, we see that the largest common factor is $\pi_{25} \cdot 3 = 291$ ($\pi_{25} = 97$). When we take the ratio of the two numbers this common factor cancels

$$\frac{873}{582} = \frac{\cancel{\pi_{25}} \cdot \cancel{3} \cdot 3}{\cancel{\pi_{25}} \cdot \cancel{3} \cdot 2} = \frac{3}{2} = 1.5.$$

Of course if we divide 582 into 873 we will numerically obtain the answer $1.5 \in \mathbb{F}$.

Exercise: Show that in Matlab/Octave `rat(873/582) = 1+1/(-2)` gives the wrong answer. **Solution:** Since

$$\texttt{factor(873)} = 3 \cdot 3 \cdot 97 \text{ and } \texttt{factor(582)} = 2 \cdot 3 \cdot 97,$$

`3/2 = 1 + 1/2` is the correct answer. But due to rounding methods, it is not 3/2. As an example, in Matlab/Octave `rat(3/2)=2+1/(-2)`. One would expect `rat(3/2)=1+1/2`. Matlab's `rat()` command uses rounding rather than the floor function, which explains the difference. When the `rat()` function produces negative numbers, rounding is employed.

[37] When I understand this better, I'll do a better job of explaining it.

Greatest common divisor: `k=gcd(m,n)`

- *Examples* $(m, n, k \in \mathbb{Z})$*:*
  - *gcd*$(13 \cdot 5, 11 \cdot 5)$ *= 5 (The common 5 is the gcd)*
  - *gcd*$(13 \cdot 10, 11 \cdot 10)$ *= 10 (The gcd(130,110) = 10 =* $2 \cdot 5$*, is not prime)*
  - *gcd(1234,1024) =* $2(1234 = 2 \cdot 617, 1024 = 2^{10})$
  - *gcd*$(\pi_k \pi_m, \pi_k \pi_n)$ *=* π_k
  - `k=gcd(m,n)` *is the part that cancels in the fraction* $m/n \in F$
  - $m/\gcd(m, n) \in \mathbb{Z}$
- Coprimes $(m \perp n)$ *are numbers with no distinct common factors: i.e., gcd(m,n)=1*
  - *The gcd of two primes is always 1: gcd(13,11) = 1, gcd*(π_m, π_n)*=1* $(m \neq n)$
  - $m = 7 \cdot 13, n = 5 \cdot 19 \Rightarrow (7 \cdot 13) \perp (5 \cdot 19)$
  - *If* $m \perp n$ *then* $gcd(m, n) = 1$
  - *If* $gcd(m, n) = 1$ *then* $m \perp n$
- *The GCD may be extended to polynomials: e.g.,* $gcd(ax^2 + bx + c,\ \alpha x^2 + \beta x + \gamma)$
  - *gcd*$((x-3)(x-4),\ (x-3)(x-5))$*=* $(x-3)$
  - *gcd*$(x^2 - 7x + 12,\ 3(x^2 - 8x + 15))$*=* $3(x-3)$
  - *gcd*$(x^2 - 7x + 12,\ (3x^2 - 24x + 45)$*=*$3(x-3)$
  - *gcd(* $(x - 2\pi)(x-4),\ (x-2\pi)(x-5)$ *)=*$(x - 2\pi)$ *(Needs long division)*

Figure 1.7: The Euclidean algorithm for finding the GCD of two numbers is one of the oldest algorithms in mathematics, and is highly relevant today. It is both powerful and simple. It was used by the Chinese during the Han dynasty (Stillwell, 2010, p. 70) for reducing fractions. It may be used to find pairs of integers that are coprime *(their GCD must be 1), and it may be used to identify factors of polynomials by long division. It has an important sister algorithm called the* continued fraction algorithm *(CFA), that generalizes the Euclidean algorithm.*

Euclidean algorithm: The algorithm is best explained by a trivial example: Let the two numbers be $6, 9$. At each step the smaller number (6) is subtracted from the larger (9) and the difference (the remainder) and the smaller numbers are saved. This process continues until the two resulting numbers are equal, at which point the GCD equals that final number. If we were to take one more step, the final numbers would be the GCD and zero. For our example step 1 gives 9-6=3, leaving 6 and 3. Step 2 gives 6-3=3 and 3. Since the two numbers are the same, the GCD=3. If we take one more difference we obtain (3,0). We can easily verify this result since this example is easily factored (e.g., $3 \cdot 3, 3 \cdot 2) = 3(3, 2)$. It may be numerically verified using the Matlab/Octave GCD command `gcd(6,9)`, which returns 3.

Matrix method: The GCD may be written as a matrix recursion given the starting vector $(m_0, n_0)^T$. The recursion is then

$$\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}.$$

This recursion continues until $m_{k+1} < n_{k+1}$, at which point m and n must be swapped.

The direct method is inefficient because it recursively subtracts n many times until the resulting m is less than n, as shown in Fig. G.1 (p. 230). It also must test for $m < n$ at each iteration, and then swap m and n once that condition is met. This recursion is repeated until $m_{k+1} = 0$. At that stage the GCD is n_{k+1}. Figure G.1 (p. 230), along with the above matrix relation, gives insight into the Euclidean algorithm.

Below is a simple Matlab/Octave code to find `k=gcd(m,n)` based on the strict definition of the EA as described by Stillwell (2010):

```
function k = gcd(m,n)
 while m ~=0
  A=m; B=n;
  m=max(A,B); n=min(A,B); %m>n
  m=m-n;
 end
```

This program loops until $m = 0$. A much more efficient method is described in Section G.1.3, p. 228, using the `floor()` function, which is called *division with rounding*.

1.2.5 Lec 6: Continued fraction algorithm

In its simplest form, the *continued fraction algorithm* (CFA) starts from a single real decimal number $x \in \mathbb{R}$, and recursively expands it as a fraction $x \in \mathbb{F}$. Thus the CFA is used in finding rational approximations to any real number. For example, $\pi \approx 22/7$, which was well known by the Chinese mathematicians. The GCD (the Euclidean algorithm) on the other hand operates on a pair of integers $m, n \in \mathbb{N}$ and finds their greatest common divisor $k \in \mathbb{N}$. Thus $m/k, n/k \in \mathbb{N}$, reducing the ratio to its irreducible form, since $m/k \perp n/k$.

Despite this seemingly large difference between the two algorithms, apparently the CFA is closely related to the Euclidean algorithm (the GCD), so closely in fact, that (see Fig. 1.7) Gauss referred to the Euclidean algorithm as the Continued fraction algorithm (Stillwell, 2010, P. 48). At first glance it is not clear why Gauss would call the Euclidean algorithm the CFA. One must assume that Gauss had some deeper insight into this relationship. If so, that insight would be valuable to understand. [38]

In the following we refine the description of the CFA and give examples that go beyond the simple cases of expanding numbers. The CFA of any positive number, say $x_o \in \mathbb{R}^+$, is defined as follows:

1. Start with $n = 0$ and a positive input target $x_o \in \mathbb{R}^+$.
2. Define $a_n = \texttt{round}(x_n)$, which rounds to the nearest integer.
3. Define $r_n = x_n - a_n$, thus $-0.5 \leq r_n \leq 0.5$. If $r_n = 0$, the recursion terminates.
4. Define $x_{n+1} = 1/r_n$ and return to step 2, with $n = n + 1$.

The recursion may continue to any desired accuracy, since convergence is guaranteed.

An example: Let $x_o \equiv \pi \approx 3.14159\ldots$. Thus $a_o = 3$, $r_o = 0.14159$, $x_1 = 7.065 \approx 1/r_o$, and $a_1 = 7$. If we were to stop here we would have

$$\widehat{\pi}_2 \approx 3 + \frac{1}{7 + 0.0625\ldots} \approx 3 + \frac{1}{7} = \frac{22}{7}. \tag{1.7}$$

This approximation of $\pi \approx 22/7$ has a relative error of 0.04%

$$\frac{22/7 - \pi}{\pi} = 4 \times 10^{-4}.$$

[38] The resolution of this interrelationship is still unresolved.

For the next approximation we continue by reciprocating the remainder $1/0.0625 \approx 15.9966$ which rounds to 16 giving a negative remainder of $\approx -1/300$, resulting in the second approximation

$$\hat{\pi}_3 \approx 3 + 1/(7 + 1/16) = 3 + 16/(7 \cdot 16 + 1) = 3 + 16/113 = 355/113.$$

Note that if we had truncated 15.9966 to 15, the remainder would have been positive, but much larger, resulting in a much less accurate rational approximation for the same number of terms. It follows that there can be a dramatic difference depending on the rounding scheme, which, for clarity, is best specified rather than inferred.

Rational approximation examples

$$\begin{aligned}
\hat{\pi}_2 &= \frac{22}{7} = [3., 7] && \approx \hat{\pi}_2 + O(1.3 \times 10^{-3}) \\
\hat{\pi}_3 &= \frac{355}{113} = [3., 7, 16] && \approx \hat{\pi}_3 - O(2.7 \times 10^{-7}) \\
\hat{\pi}_4 &= \frac{104348}{33215} = [3., 7, 16, -249] && \approx \hat{\pi}_4 + O(3.3 \times 10^{-10})
\end{aligned}$$

Figure 1.8: The expansion of π to various orders, using the CFA, along with the order of the error of each rational approximation, with rounding. For example, $\hat{\pi}_2$ =22/7 has an absolute error ($|22/7 - \pi|$) of about 0.13%.

Notation: Writing out all the fractions can become tedious. For example, expanding $e = 2.7183\cdots$ using the Matlab/Octave command `rat(exp(1))` gives the approximation

$$\begin{aligned}
\exp(1) &= 3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7))))) - O\left(1.75 \times 10^{-6}\right), \\
&= [3. - 4, 2, 5, -2, -7] - O(1.75 \times 10^{-6}.
\end{aligned}$$

Since many entries are negative, we may deduce that rounding arithmetic is being used by Matlab (but this is not documented). Note that the leading integer part may be noted by an optional decimal point or semicolon.[39] If the process is carried further, the values of $a_n \in \mathbb{N}$ give increasingly more accurate rational approximations.

Rounding schemes: In Matlab/Octave there are five different rounding schemes (i.e., mappings): `round(c)`, `fix(c)`, `floor(c)`, `ceil(c)`, `roundb(c)` with $c \in \mathbb{R}$. If the rounding-down (`floor()`) is used $\hat{\pi}_{12} = [3., 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1]$, whereas true rounding to the nearest integer (`round()`) gives $\hat{\pi}_8 = [3., 7, 16, -294, 3, -4, 5, -15]$. Thus `round()` introduces negative remainders when a number rounds up to the nearest integer.

Exercise: Based on several examples, which rounding scheme is the most accurate? Explain why. **Solution:** Rounding will give a smaller remainder at each iteration, resulting in a smaller net error and thus faster convergence.

[39] Unfortunately Matlab/Octave does not support the bracket notation.

When the CFA is applied and the expansion terminates ($r_n = 0$), the target is rational. When the expansion does not terminate (which is not always easy to determine, as the remainder may be ill-conditioned due to small numerical rounding errors), the number is irrational. Thus the CFA has important theoretical applications regarding irrational numbers. You may explore this using Matlab's `rats(pi)` command.

One of the useful things about the procedure, besides its being so simple, is its generalizations to complex variables, one of which will be discussed in Section G.1.3 (p. 228).

Besides these five basic rounding schemes, there are two other important $\mathbb{R} \to \mathbb{N}$ functions (i.e., mappings): `mod(x,y)`, `rem(x,y)` with $x, y \in \mathbb{R}$, which will be needed later. The base 10 numbers may be generated from the counting numbers using `y=mod(x,10)`.

Exercise:

1. Show how to generate a base-10 real number $y \in \mathbb{R}$ from the counting numbers $\mathbb{N}$ using the `m=mod(n,10)` $+k10$ with $n, k \in \mathbb{N}$. **Solution:** Every time n reaches 10, $m = 0$. Keep track of this by adding 1 to k.

2. How would you generate binary numbers (base 2) using the `mod(x,b)` function? **Solution:** Use the same method as in the first example above, but with $b = 2$.

3. How would you generate hexadecimal numbers (base 16) using the `mod(x,b)` function? **Solution:** Use the same method as in the first example above, but with $b = 16$.

4. Write out the first 19 numbers in hex notation, starting from zero. **Solution:** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10, 11, 12, ...

5. What is FF_{16} in decimal notation? **Solution:** Using the Matlab/octave function `hex2dec('ff')` is 255_{10}.

Symmetry: A *continued fraction expansion* can have a high degree of symmetry. For example, the CFA of

$$R_1 \equiv \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1+\frac{1}{1+\cdots}} = 1.618033988749895\cdots . \tag{1.8}$$

Here a_n in the CFA is *always* 1 ($R_1 \equiv [1., 1, 1, \cdots]$), thus the sequence will not terminate, *proving* that $\sqrt{5} \in \mathbb{I}$. A related example is $R_2 \equiv$ `rat(1+sqrt(2))`, which gives $R_2 = [2., 2, 2, 2, \cdots]$.

When expanding a target irrational number ($x_o \in \mathbb{I}$), and the CFA is truncated, the resulting rational fraction approximates the irrational target. For the example above, if we truncate at three coefficients ($[1; 1, 1]$) we obtain

$$1 + \frac{1}{1+\frac{1}{1+0}} = 1 + 1/2 = 3/2 = 1.5 = \frac{1+\sqrt{5}}{2} + 0.118\cdots .$$

Truncation after six steps gives

$$[1., 1, 1, 1, 1, 1, 1] = 13/8 \approx 1.6250 = \frac{1+\sqrt{5}}{2} + .0070\cdots .$$

Because all the coefficients are 1, this example converges very slowly, When the coefficients are large (i.e., remainder small), the convergence will be faster. The expansion of π is an example of faster convergence.

In summary: Every rational number $m/n \in \mathbb{F}$, with $m > n > 1$, may be uniquely expanded as a continued fraction, with coefficients a_k determined using the CFA. When the target number is irrational ($x_o \in \mathbb{Q}$), the CFA does not terminate; thus, each step produces a more accurate rational approximation, converging in the limit as $n \to \infty$.

Thus the CFA expansion is an algorithm that can, in theory, determine when the target is rational, but with an important caveat: one must determine if the expansion terminates. This may not be obvious. The fraction $1/3 = 0.33333\cdots$ is an example of such a target, where the CFA terminates yet the fraction repeats. It must be that

$$1/3 = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \cdots .$$

Here 3*3=9. As a second example[40]

$$1/7 = 0.142857142857142857142857\cdots = 142857 \times 10^{-6} + 142857 \times 10^{-12} + \cdots$$

There are several notations for repeating decimals such as $1/7 = 0.1\overline{142857}$ and $1/7 = 0.1((142857))$. Note that $142857 = 999999/7$. Related identities include $1/11 = 0.090909\cdots$ and $11 \times 0.090909 = 999999$. When the sequence of digits repeats, the sequence is predictable, and it must be rational. But it is impossible to be sure that it repeats, because the length of the repeat can be arbitrarily long.

1.2.6 Lec 7: Pythagorean triplets (Euclid's formula)

Euclid's formula is a method for finding three integer lengths $[a, b, c] \in \mathbb{N}$, that satisfy Eq. 1.1. It is important to ask "Which set are the lengths [a,b,c] drawn from?" There is a huge difference, both practical and theoretical, if they are from the real numbers $\mathbb{R}$, or the counting numbers $\mathbb{N}$. Given $p, q \in \mathbb{N}$ with $p > q$, the three lengths $[a, b, c] \in \mathbb{N}$ of Eq. 1.1 are given by

$$a = p^2 - q^2, \qquad b = 2pq, \qquad c = p^2 + q^2. \tag{1.9}$$

This result may be directly verified, since

$$[p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2$$

or

$$p^4 + q^4 + \cancel{2p^2q^2} = p^4 + q^4 - \cancel{2p^2q^2} + \cancel{4p^2q^2}.$$

Thus, Eq. 1.9 is easily proven, once given. Deriving Euclid's formula is obviously more difficult.

A well-known example is the right triangle depicted in Fig. 1.9, defined by the integers $[3, 4, 5] \in N$, having angles $[0.54, 0.65, \pi/2]$ [rad], which satisfies Eq. 1.1 (p. 18). As quantified by Euclid's formula (Eq. 1.9), there are an infinite number of *Pythagorean triplets* (PTs). Furthermore the seemingly simple triangle, having angles of $[30, 60, 90] \in N$ [deg] (i.e., $[\pi/6, \pi/3, \pi/2] \in \mathbb{I}$ [rad]), has one irrational ($\mathbb{I}$) length ($[1, \sqrt{3}, 2]$).

The technique for proving Euclid's formula for PTs $[a, b, c] \in \mathbb{Q}$, derived in Fig. G.3 (p. 233) of Section G.2.1, is much more interesting than the PTs themselves.

[40] Taking the Fourier transform of the target number, represented as a sequence, could help to identify an underlying periodic component. The number $1/7 \leftrightarrow [[1, 4, 2, 8, 5, 7]]_6$ has a 50 [dB] notch at 0.8π [rad] due to its 6 digit periodicity, carried to 15 digits (Matlab/Octave maximum precision), Hamming windowed, and zero padded to 1024 samples.

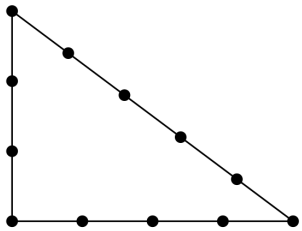

Figure 1.9: *Beads on a string form perfect right triangles when the number of unit lengths between beads for each side satisfies Eq. 1.1. For example, when $p = 2, q = 1$, the sides are $[3, 4, 5]$.*

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

Figure 1.10: *"Plimpton-322" is a stone tablet from 1800 [BCE], displaying a and c values of the Pythagorean triplets $[a, b, c]$, with the property $b = \sqrt{c^2 - a^2} \in \mathbb{N}$. Several of the c values are primes, but not the a values. The stone is item 322 (item 3 from 1922) from the collection of George A. Plimpton. –Stillwell (2010)*

The set from which the lengths $[a, b, c]$ are drawn was not missed by the Indians, Chinese, Egyptians, Mesopotamians, Greeks, etc. Any equation whose solution is based on integers is called a *Diophantine equation*, named after the Greek mathematician Diophantus of Alexandria (c250 CE) (Fig. 1.1, p. 19).

A stone tablet having the numbers engraved on it, as shown in Fig. 1.10, was discovered in Mesopotamia, from the 19^{th} century [BCE], and cataloged in 1922 by George Plimpton.[41] [42] These numbers are a and c pairs from PTs [a,b,c]. Given this discovery, it is clear that the Pythagoreans were following those who came long before them. Recently a second similar stone, dating between 350 and 50 [BCE] has been reported, that indicates early calculus on the orbit of Jupiter's moons, the very same moons that Rømer observed to show that the speed of light was finite (p. 24).[43]

[41] http://www.nytimes.com/2010/11/27/arts/design/27tablets.html

[42] https://en.wikipedia.org/wiki/Plimpton_322

[43] http://www.nytimes.com/2016/01/29/science/babylonians-clay-tablets-geometry-astronomy-jupiter.html

1.2.7 Lec 8: Pell's Equation

There is a venerable history for Pell's equation

$$x_n^2 - Ny_n^2 = (x_n - \sqrt{N}y_n)(x_n + \sqrt{N}y_n) = 1, \tag{1.10}$$

with non-square $N \in \mathbb{N}$ specified and $x, y \in \mathbb{N}$ unknown. It is suspected that Pell's equation is directly related to the Euclidean algorithm, as applied to polynomials having integer coefficients (Stillwell, 2010, 48). For example, with $N = 2$, one solution is $x = 17, y = 12$ (i.e., $17^2 - 2 \cdot 12^2 = 1$).

A 2x2 matrix recursion algorithm, likely due to the Chinese, was used by the Pythagoreans to investigate the $\sqrt{N}$, where we move the index outside the vector to save typing:

$$\begin{bmatrix} x \\ y \end{bmatrix}_n = \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_{n-1}. \tag{1.11}$$

As in the case of the CFA, $\boldsymbol{x}_n/y_n \to \sqrt{N} \in \mathbb{F}$. Starting with $[x_o, y_o]^T = [1, 0]^T$, results in solutions of Pell's equations (Stillwell, 2010, p. 44). Their approach was likely motivated by the Euclidean algorithm (GCD, p. 43), since $y_n/x_n \to \sqrt{2}$ (Stillwell, 2010, p. 37, 55).

Note that this is a composition method, of 2x2 matrices, since the output of one matrix multiply is the input to the next. They key question what is the relationship between Pell's equation and the linear recursion? Is it that Pell's equation may be trivially factored? There must be some simple way to prove that Eqs. 1.11 and 1.10 are equivalent, as demonstrated on Section G.2.3 (p. 235).

Asian solutions: The first solution of Pell's equation was published by Brahmagupta (c628), who independently discovered the equation (Stillwell, 2010, p. 46). Brahmagupta's novel solution introduced a different *composition method* (Stillwell, 2010, p. 69), and like the Greek result, this solution was incomplete.

Then in 1150CE, Bhâskara II obtained solutions using Eq. 1.11 (Stillwell, 2010, p.69). This is the solution method we shall explore here, as summarized in Fig. 1.11.

The best way to see how this recursion results in solutions to Pell's equation is by example. Initializing the recursion with the trivial solution $[x_o, y_o]^T = [1, 0]^T$ gives

$$\begin{aligned}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 1^2 - 2 \cdot 1^2 &= -1 \\
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & 3^2 - 2 \cdot 2^2 &= 1 \\
\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} & (7)^2 - 2 \cdot (5)^2 &= -1 \\
\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} &= \begin{bmatrix} 17 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} & 17^2 - 2 \cdot 12^2 &= 1 \\
\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} &= \begin{bmatrix} 41 \\ 29 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} & (41)^2 - 2 \cdot (29)^2 &= -1
\end{aligned}$$

Thus the recursion results in a modified version of Pell's equation

$$x_n^2 - 2y_n^2 = (-1)^n, \tag{1.12}$$

where only even values of n are solutions. This sign change had no effect on the Pythagoreans' goal, since they only cared about the ratio $y_n/x_n \to \pm\sqrt{2}$.

$$\begin{aligned}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_0 & \jmath^2 - 2 \cdot \jmath^2 &= 1 \\
\begin{bmatrix} x \\ y \end{bmatrix}_2 &= \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 & 3^2 - 2 \cdot 2^2 &= 1 \\
\begin{bmatrix} x \\ y \end{bmatrix}_3 &= \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix}_3 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 & (7\jmath)^2 - 2 \cdot (5\jmath)^2 &= 1 \\
\begin{bmatrix} x \\ y \end{bmatrix}_4 &= \begin{bmatrix} 17 \\ 12 \end{bmatrix}_4 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix}_3 & 17^2 - 2 \cdot 12^2 &= 1 \\
\begin{bmatrix} x \\ y \end{bmatrix}_5 &= \jmath \begin{bmatrix} 41 \\ 29 \end{bmatrix}_5 = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix}_4 & (41\jmath)^2 - 2 \cdot (29\jmath)^2 &= 1
\end{aligned}$$

Figure 1.11: *This summarizes the solution ($\in \mathbb{C}$) of Pell's equation for $N = 2$ using a slightly modified matrix recursion. Note that $x_n/y_n \to \sqrt{2}$ as $n \to \infty$, which was what the Pythagoreans were pursuing, and that the odd solutions are pure-imaginary. Note we have placed the recursion index n* outside *the brackets to reduce notional clutter.*

Modified recursion: As shown in Fig. 1.11, the recursion may be slightly modified to fix the $(-1)^n$ problem, by multiplying the 2x2 matrix by $\jmath = \sqrt{-1}$. This results in Pell's equation for every step of the recursion.

Solution to Pell's equation: By multiplying the matrix by $1\jmath$, all the solutions ($\boldsymbol{x}_k \in \mathbb{C}$) to Pell's equation are determined. From Fig. 1.11 we can see that every output of this slightly modified matrix recursion gives solutions to Pell's equation. The $1\jmath$ factor corrects the alternation in sign, so every iteration yields a solution. For $n = 0$ (the initial solution) $[x_0, y_0]$ is $[1, 0]_0$, $[x_1, y_1] = \jmath[1, 1]_1$, and $[x_2, y_2] = -[3, 2]_2$. These are easily checked using this recursion.

At each iteration, the ratio x_n/y_n approaches $\sqrt{2}$ with increasing accuracy, coupling it to the CFA. The value of $41/29 \approx \sqrt{2}$, with a relative error of <0.03%. The solution for $N = 3$ is given in Appendix E.1.1 (p. 204).

Relations to digital signal processing: Today we recognize Eq. 1.11 as a *difference equation,* which is a pre-limit (pre Stream 3) form of a differential equation. The Greek 2x2 form is an early precursor to 17^{th} and 18^{th} century developments in linear algebra. Thus the Greeks' recursive solution for the $\sqrt{2}$ and Bhâskara's (1030 CE) solution of Pell's equation are early precursors to discrete-time processing, as well as to calculus.

There are important similarities between Pell's equation and the Pythagorean theorem. As we shall see in Chapter G, Pell's equation is related to the geometry of a hyperbola, just as the Pythagorean equation is related to the geometry of a circle. One would assume there is a Euclidean formula for the case of Pell's equations, since these are all conic sections with closely related conic geometry. As we have seen, the solutions involve $\sqrt{-1}$.

Eigen-analysis: The key to the analysis of such equations is called the *eigen-analysis*, or *modal-analysis* method. These are also known as *resonant modes* in the physics literature. Eigen-modes describe the naturally occurring "ringing" found in physical wave-dominated boundary value problems. Each mode's "eigen-value" quantifies the mode's natural frequency. Complex eigen-values result in damped modes, which decay in time due to energy losses. Common examples include tuning forks, pendulums, bell, and strings of musical instruments, all of which have a characteristic

frequency.

Cauchy's residue theorem is used to find the time-domain response of each frequency-domain complex eigen-mode. Thus eigen-analysis and eigen-modes of physics are the same thing (see Sect. 1.4.3, p. 109), but are described using different (i.e., mutually unrecognizable) notional methods.[44] The eigen-method method is summarized in Appendix D, p. 199.

Taking a simple example of a 2x2 matrix $\boldsymbol{T} \in \mathbb{C}$, we start from the definition of the two eigen-equations

$$\boldsymbol{T}\boldsymbol{e}_{\pm} = \lambda_{\pm}\boldsymbol{e}_{\pm}, \tag{1.13}$$

corresponding to two eigen-values $\lambda_{\pm} \in \mathbb{C}$ and two 2x1 eigen-vectors $\boldsymbol{e}_{\pm} \in \mathbb{C}$. The eigen-values $\lambda_{\pm}$ may be merged into a 2x2 diagonal eigen-value matrix

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix},$$

while the two eigen-vectors $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ are merged into a 2x2 eigen-vector matrix

$$\boldsymbol{E} = [\boldsymbol{e}_+, \boldsymbol{e}_-] = \begin{bmatrix} e_1^+ & e_2^- \\ e_1^+ & e_2^- \end{bmatrix}, \tag{1.14}$$

corresponding to the two eigen-values. Using matrix notation, this may be compactly written as

$$\boldsymbol{T}\boldsymbol{E} = \boldsymbol{E}\Lambda. \tag{1.15}$$

Note that while $\lambda_{\pm}$ and $\boldsymbol{E}_{\pm}$ commute, $\boldsymbol{E}\Lambda \neq \Lambda\boldsymbol{E}$.

From Eq. 1.15 we may obtain two very important forms:

1. the diagonalization of $\boldsymbol{T}$

$$\Lambda = \boldsymbol{E}^{-1}\boldsymbol{T}\boldsymbol{E}, \tag{1.16}$$

   and

2. the eigen-expansion of $\boldsymbol{T}$

$$\boldsymbol{T} = \boldsymbol{E}^{-1}\Lambda\boldsymbol{E}^{-1}, \tag{1.17}$$

   which is useful for computing power of $\boldsymbol{T}$ (i.e., $\boldsymbol{T}^{100} = \boldsymbol{E}^{-1}\Lambda^{100}\boldsymbol{E}$).

Example: If we take

$$\boldsymbol{T} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

then the eigen-values are given by $(1 - \lambda_{\pm})(1 + \lambda_{\pm}) = -1$, thus $\lambda_{\pm} = \sqrt{2}$. This method of eigen-analysis is discussed in Section G.2.2 (p. 232) and Appendix E (p. 203).

The key idea of the 2x2 matrix solution, widely used in modern engineering, can be traced back to Brahmagupta's solution of Pell's equation, for arbitrary N. Brahmagupta's recursion, identical to that of the Pythagoreans' $N = 2$ case (Eq. 1.11), eventually led to the concept of linear algebra, defined by the simultaneous solutions of many linear equations. The recursion by the Pythagoreans

[44] During the discovery or creation of quantum mechanics, two alternatives were developed: Schrödinger's differential equation method and Heisenberg's matrix method. Eventually it was realized the two were equivalent.

(6^{th} BCE) predated the creation of algebra by al-Khwārizmī (9^{th} CE century) (Fig. 1.1) (Stillwell, 2010, p. 88).

WEEK 4

1.2.8 Lec 9: Fibonacci sequence

Another classic problem, also formulated by the Chinese, was the Fibonacci sequence, generated by the relation

$$f_{n+1} = f_n + f_{n-1}. \tag{1.18}$$

Here the next number $f_{n+1} \in \mathbb{N}$ is the sum of the previous two. If we start from $[0, 1]$, this difference equation leads to the Fibonacci sequence $f_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots]$. Alternatively, if we define $y_{n+1} = x_n$, then Eq. 1.18 may be compactly represented by a 2x2 matrix equation recursion

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n . \tag{1.19}$$

The correspondence of Eqs. 1.18 and 1.19 is easily verified. Starting with $[x, y]_0^T = [0, 1]^T$ we obtain for the first few steps

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_0 , \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 , \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 , \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 , \quad \cdots .$$

From the above $x_n = [0, 1, 1, 2, 3, 5, \ldots]$ is the Fibonacci sequence since the next x_n is the sum of the previous two and the next y_n is x_n.

Repeat eigen-analysis.

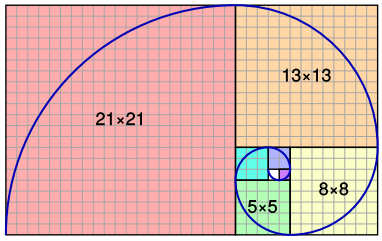

Figure 1.12: *This construction is called the Fibonacci spiral. Note how it is constructed out of squares having areas given by the square of the Fibonacci numbers. In this way, the spiral is smooth and the radius increases as the Fibonacci numbers (e.g., 8=3+5, 13 = 5+8, etc.). (Adapted from* `https://en.wikipedia.org/wiki/Golden_spiral`*.)*

The mean-Fibonacci sequence: Suppose that the Fibonacci sequence recursion is replaced by the mean of the last two values, namely let

$$f_{n+1} = \frac{f_n + f_{n-1}}{2}. \tag{1.20}$$

This seems like a small change. But how does the solution differ? To answer this question it is helpful to look at the corresponding 2x2 matrix.

Exercise: Find the 2x2 matrix corresponding to Eq. 1.20. **Solution:** The 2x2 matrix may be found using the *companion matrix* method (p. 65), giving

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}_n . \tag{1.21}$$

The eigen-values of this matrix are $[1, -1/2]$ (i.e., the roots of the binomial equation $\lambda^2 - \lambda/2 - 1/2 = 0$). Thus $[x_n, y_n] = [1, 1]^T$ and $[x_n, y_n] = [1, 1]^T$ are both solutions.

Exercise: Starting from $[x_n, y_n]^T = [1, 0]^T$ compute the first 5 values of $[x_n, y_n]^T$. **Solution:** Here is a Matlab/Octave code for computing x_n:

```
x(1:2,1)=[1;0];
A=[1 1;2 0]/2;
for k=1:10; x(k+1)=A*x(:,k); end
```

which gives the rational ($x_n \in \mathbb{Q}$) sequence: $1, 1/2, 3/4, 5/8, 11/2^4, 21/2^5, 43/2^6, 85/2^7, 171/2^8, 341/2^9, 683/2^{10}, \cdots$.

Exercise: Show that the solution to Eq. 1.20 is bounded, unlike that of the Fibonacci sequence, which diverges. Explain what is going on. **Solution:** Because the next value is the mean of the last two, the sequence is bounded. To see this one needs to compute the eigen-values of the matrix Eq. 1.21.

Exercise: Use the formula for the generalized diagonalization of a matrix to find the general solution of the mean-Fibonacci sequence. **Solution:** The eigen-values are given by the roots of

$$0 = -\lambda(1/2 - \lambda) - 1/2 = (\lambda - 1/4)^2 - 9/16$$

which are $[1, -1/2]$.

By studying the eigen-values of Eq. 1.21 one finds that the steady state solution approaches 1. Namely $f_n \to 1 = (f_{n-1} + f_{n-2})/2$ is the solution, as $n \to \infty$. Namely the average of the last two values must approach 1 for large n.

Exercise: Show that the geometric series formula holds for 2x2 matrices. Starting with the 2x2 identity matrix I_2 and $a \in \mathbb{C}$, with $|a| < 1$, show that

$$I_2(I_2 - aI_2)^{-1} = I_2 + aI_2 + a^2 I_2^2 + a^3 I_2^3 + \cdots .$$

Solution: Since $a^k I_2^k = a^k I_2$, we may multiply both sides by $I_2 - aI_2^k$ to obtain

$$\begin{aligned} I_2 &= I_2 + aI_2 + a^2 I_2^2 + a^3 I_2^3 + \cdots - aI_2(aI_2 + a^2 I_2^2 + a^3 I_2^3 + \cdots) \\ &= [1 + (a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + a^4 + \cdots)]I_2 \\ &= I_2 \end{aligned}$$

Summary: The GCD (Euclidean algorithm), Pell's equation and the Fibonacci sequence may all be written as compositions of 2x2 matrices. Thus Pell's equation and the Fibonacci sequence are special cases of 2x2 matrix composition

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n .$$

This is an important and common thread of these early mathematical findings. It will turn out that this 2x2 matrix recursion plays a special role in physics, mathematics and engineering, because such equations are solved using the eigen-analysis method. The first several thousands of years of mathematical trial and error set the stage for this breakthrough, but this took a long time to fully be appreciate.

Can CFA be written this w

Relate to PT, EA & CFA

Is the CFA the inverse of t GCD?

Discuss importance of eig analysis

Discuss the development of algebra (Fig 1.1 p. 19) and the discovery of Diophantus' book on "Arithmitic" by Bombelli, Fermat and Descarte (p. 23).

1.2.9 Lec 10: Exam I (In class)

1.3 Algebraic Equations: Stream 2

The era of 1640 to 1900 (Fig. 1.13) produced a continuous stream of fundamental theorems. A few of the individuals who played a notable role in this development, in chronological (birth) order, include: Galileo, Mersenne, Newton, d'Alembert, Fermat, Huygens, Descartes and Helmholtz. These individuals were some of the first to develop the basic ideas, in various forms, that were then later reworked into the proofs, that today we recognize as *the fundamental theorems of mathematics*.

Chronological history from the 17^{th} century

17^{th} *Newton 1642-1727, Bernoulli, Johann 1667–1748*

18^{th} *Bernoulli, Daniel, Cauchy 1789–1857, Euler 1707–1783, d'Alembert 1717–1783, Gauss 1777–1855*

19^{th} *Kirchhoff 1824–87, Helmholtz 1821–1894, Riemann 1826–1866, Maxwell 1831–1879, Rayleigh 1842–1919, Heaviside 1850–1925, Poincare 1854–1912,*

20^{th} *Sommerfeld 1686–1951, Einstein 1879–1955, Brillouin 1889–1969 . . .*

Time-Line

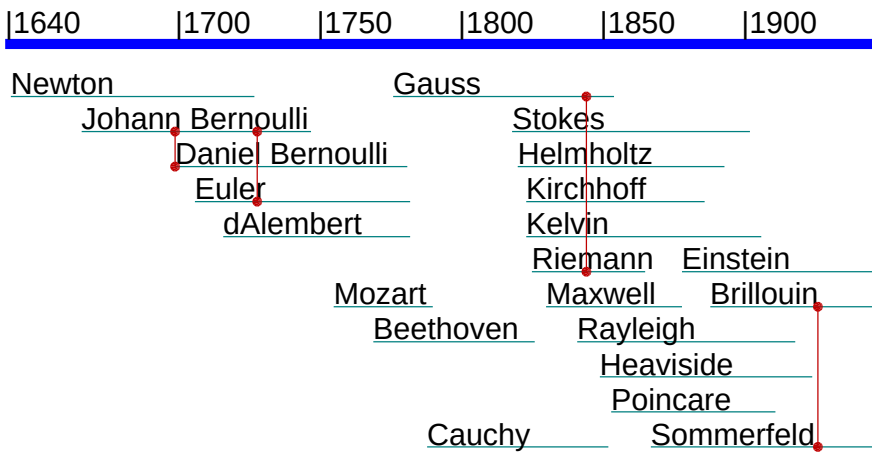

Figure 1.13: *Time-line of the three centuries from the mid 17^{th} to 20^{th} CE, one of the most productive times of all, because mathematicians were sharing information. Figure 1.2 (p. 23) (Bombelli-Gauss) provides a closer look at the 16–19 CE, and Fig. 1.24 (p.103) (Bombelli-Einstein) provides the full view 16–20 CE. [fig:TimeLine19CE]*

1.3.1 Lec 11 The physics behind nonlinear Algebra (Euclidean geometry)

Following Stillwell's history of mathematics, Stream 2 is geometry, which led to the merging of Euclid's geometrical methods and the 9^{th} century development of algebra by al-Khwarizmi (830 CE). This integration of ideas led Descartes and Fermat to develop *analytic geometry*. While not entirely a unique and novel idea, it was late in coming, given what was known at that time.

The mathematics up to the time of the Greeks, documented and formalized by Euclid, served students of mathematics for more than two thousand years. Algebra and geometry were, at first, independent lines of thought. When merged, the focus returned to the Pythagorean theorem, generalized as analytic conic sections rather than as geometry in Euclid's Elements. With the introduc-

tion of algebra, numbers, rather than lines, could be used to represent a geometrical length. Thus the appreciation for geometry grew, given the addition of the rigorous analysis using numbers.

Physics inspires algebraic mathematics: The Chinese used music, art, and navigation to drive mathematics. Unfortunately much of their knowledge has been handed down either as artifacts, such as musical bells and tools, or mathematical relationships documented, but not created, by scholars such as Euclid, Archimedes, Diophantus, and perhaps Brahmagupta. With the invention of algebra by al-Khwārizmī (830CE), mathematics became more powerful, and blossomed. During the 16th and 17th century, it had become clear that differential equations (DEs), such as the wave equation, can characterize a law of nature, at a single point in space and time. This principle was not obvious. A desire to understand motions of objects and planets precipitated many new discoveries. This period is illustrated in Fig. 1.2 (p. 23).

The law of gravity was first formulated by Galileo to explain the falling of two objects of different masses, and how they must obey conservation of energy. Kepler investigated the motion of the planets. While Kepler was the first to observe that the orbit of planets is described by ellipses, it seems he under-appreciated the significance of his finding, and continued working on his incorrect epicycle planetary model. Following up on Galileo's work, Newton (c1687) went on to show that there must be a gravitational potential between two masses (m_1, m_2) of the form

$$\phi_g(r(t)) \propto \frac{m_1 m_2}{r(t)}, \tag{1.22}$$

where $r = |x_1 - x_2|$ is the Euclidean distance between the two point masses at locations x_1 and x_2. Using algebra and his calculus, Newton formalized the equation of gravity, forces and motion (Newton's three laws) and showed that Kepler's discovery of planetary elliptical motion naturally follows from these laws. With the discovery of Uranus (1781) "Kepler's theory was ruined" (i.e., proven wrong) (Stillwell, 2010, p. 23).

Once Newton proposed the basic laws of gravity, he proceed to calculate, for the first time, the speed of sound. This required some form of the pressure *wave equation*

$$\frac{\partial^2}{\partial x^2}\varrho(x,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(x,t), \tag{1.23}$$

a key equation in mathematical physics. The speed of sound is

$$c_o = \sqrt{\frac{\eta_o P_o}{\rho_o}} = 343, \quad \text{[m/s]}$$

which is a function of the density $\rho_o = 1.12$ [kg/m^3] and the dynamic stiffness ηP_o of air.[45]

If we substitute for the pressure

$$\varrho(x,t) = e^{\jmath(\omega t \pm 2\pi k x)}, \tag{1.24}$$

where t is time and x is position, we find that $2\pi k = 2\pi/\lambda = 2\pi f/c_o$, because $f\lambda = c_o$.

[45] $\eta_o = C_p/C_v = 1.4$ is the ratio of two thermodynamic constants and $P_o = 10^5$ [Pa] is the barometric pressure of air.

Case of dispersive wave propagation: This classic relation $\lambda f = c$ is deceptively simple, thus confusing, because $k = 2\pi/\lambda$ becomes complex (has both real and imaginary parts) in dispersive media (e.g., acoustic waves in tubes) when losses are considered (Kirchhoff, 1868; Mason, 1927). A more important example is the case of electron waves in silicon crystals (Brillouin, 1953). In these more general cases, $k(f) = 2\pi f/c$ is replaced with the complex analytic function $\kappa(s)$ of s, i.e., and the wave becomes the eigen-function of the wave equation

$$p^{\pm}(x,t) = P_o(s)e^{st}e^{\pm\kappa(s)x}. \tag{1.25}$$

In these more general cases the wave number $\kappa(s)$ must be a *complex analytic function* of the *Laplace frequency* $s = \sigma + \omega\jmath$, as used with the Laplace transform. This is because electron "waves" in a dispersive semi-conductor, such as silicon, are "filtered" in magnitude and phase. Silicon is a highly dispersive "wave-filter," forcing the wavelength to be a complex function of frequency. This view is elegantly explained by Brillouin (1953, Chap. 1), in his historic context. Modern acoustics contains a rich source of related examples (Morse, 1948; Beranek, 1954; Beranek and Mellow, 2012).

Newton's *Principia* was finally published in 1687, and the general solution to Newton's wave equation [i.e., $p(x,t) = G(t \pm x/c)$], where G is *any* function, was first published 60 years later by d'Alembert (c1747), which showed that for sounds of a single frequency, the wavelength λ and frequency f were related by

$$f\lambda = c. \tag{1.26}$$

Today d'Alembert's analytic wave solution must be written as Eq. 1.25 having a complex wave number $\kappa(s) = 2\pi/\lambda(s)$ [m^{-1}]. This formulation led to the frequency domain concept of the Laplace analysis, based on the *linearity* (i.e., superposition) property of the wave equation (Postulate P2: Lec. 1.3.15, p. 100).

Newton's value for the speed of sound in air c_o was incorrect by the thermodynamic constant $\sqrt{\eta_o}$, a problem that would take more than two hundred years to resolve. What was needed was the *adiabatic process*, (the concept of constant-heat). For audio frequencies (0.02-20 [kHz]), the small temperature gradients cannot diffuse the distance of a wavelength in one cycle (Pierce, 1981; Boyer and Merzbach, 2011), "trapping" the heat energy in the wave.[46] The fix to Newton's formula for the sound speed was to define the *dynamic stiffness* of air $\eta_o P_o$, where P_o (1 [atm] or 10^5 [Pa]) is the static stiffness of air.

Newton's success was important because it quantified the physics behind the speed of sound, and demonstrated that momentum waves (mv), not mass m, was transported by the wave. His concept was correct, and his formulation using algebra and calculus represented a milestone in science. In air, assuming no visco-elastic losses, it is constant (i.e., $c_o = \sqrt{\eta_o P_o/\rho_o}$). When including losses the wave number becomes a complex function of frequency, leading to Eq. 1.25. In periodic structures, again the wave number becomes complex due to diffraction, as commonly observed in optics (e.g., diffraction gratings) and acoustics. Thus Eq. 1.26 only holds for the most simple cases, but in general it must be considered as a complex analytic function of s, as $\kappa(s)$ in Eq. 1.25.

The corresponding discovery for the formula for the speed of light was made 174 years after Principia, by Maxwell (c1861). Maxwell's formulation also required great ingenuity, as it was necessary to hypothesize an experimentally unmeasured term in his equations, to get the mathematics to correctly predict the speed of light.

[46] There were other physical enigmas, such as the observation that sound disappears in a vacuum or that a vacuum cannot draw water up a column by more than 34 feet.

The first Algebra:

Prior to the invention of algebra, people worked out problems as sentences using an obtuse description of the problem (Stillwell, 2010, p. 93). Algebra changed this approach, resulting in a compact language of mathematics, where numbers are represented as abstract symbols (e.g., x and α). The problem to be solved could be formulated in terms of sums of powers of smaller terms, the most common being powers of some independent variable (i.e., time or frequency). Since we set $a_n = 1$

$$P_N(z) \equiv z^n + a_{n-1}z^{n-1} + \cdots + a_o z^0 = \sum_{k=0}^{n} a_k z^k = \prod_{k=0}^{n}(z - z_k). \tag{1.27}$$

is called a monic rather than a polynomial. The coefficient a_n cannot be zero, or the polynomial would not be of degree n. The solution is to force $a_n = 1$, since this simplifies the expression, and does not change the roots.

The key question is: What values of $z = z_k$ result in $P_N(z_k) = 0$. In other words, what are the roots z_k of the polynomial? Answering this question consumed thousands of years, with intense efforts by many aspiring mathematicians. In the earliest attempts, it was a competition to evaluate mathematical acumen. Results were held as a secret to the death bed. It would be fair to view this effort as an obsession. Today the roots of any polynomial may be found, to high accuracy, by numerical methods. Finding roots is limited by the numerical limits of the representation, namely by IEEE-754 (p. 35). There are also a number of important theorems.

Of particular interest is composing a circle with a line, when the line does not touch the circle, and finding the roots. There was no solution to this problem using geometry. This question is addressed in the assignments.

Finding roots of polynomials

The problem of factoring polynomials has a history more than a millennium in the making. While the quadratic (degree $N = 2$) was solved by the time of the Babylonians (i.e., the earliest recorded history of mathematics), the cubic solution was finally published by Cardano in 1545. The same year, Cardano's student solved the quartic ($N = 4$). In 1826 (281 years later) it was proved that the quintic ($N = 5$) could not be factored by analytic methods.

As a concrete example we begin with trivial but important case of the quadratic

$$P_2(x) = ax^2 + bx + c. \tag{1.28}$$

First note that if $a = 0$, the quadratic reduces to the monomial $P_1(x) = bx + c$. Thus we have the necessary condition that $a \neq 0$. The best way to proceed is to divide a out and work directly with the normalized quadratic $\hat{P}_2(x) = \frac{1}{a}P_2(x)$. In this way we do not need to worry about the exception, $a = 0$.

The *roots* are those values of x such that $\hat{P}_2(x_k) = 0$. One of the first results (recorded by the Babylonians, c2000 BCE) was the factoring of this equation by *completing the square* (Stillwell, 2010, p. 93). One may isolate x by rewriting Eq. 1.28 as

$$\frac{1}{a}P_2(x) = (x + b/2a)^2 - (b/2a)^2 + c/a. \tag{1.29}$$

This is easily verified by expanding the squared term and canceling $(b/2a)^2$

$$\frac{1}{a}P_2(x) = [x^2 + (b/a)x + \cancel{(b/2a)^2}] - \cancel{(b/2a)^2} + c/a.$$

Setting Eq. 1.29 to zero and solving for the two roots $x_\pm$ gives the *quadratic formula*

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.30}$$

The roots of $\hat{P}_2(x)$, with $a = 1$, greatly simplify to

$$x_\pm = -\frac{1}{2}b \pm \sqrt{(b/2)^2 - c}. \tag{1.31}$$

This can be simplified even further. The term $b^2 - c > 0$ under the square root is called the discriminant, and in physics and engineering problems, 99.9% of the time it is negative. Finally $b/2 \ll \sqrt{c}$; thus, the most natural way (i.e., corresponding to the most common physical cases) of writing the solution is[47]

$$x_\pm = -\frac{1}{2}b \pm \jmath\sqrt{c - (b/2)^2} \approx -b/2 \pm \jmath\sqrt{c}. \tag{1.32}$$

This form separates the real and imaginary parts of the solution in a natural way. The term $b/2$ is called the damping, which accounts for losses in a resonant circuit, while the term $\sqrt{c}$, for mechanical, acoustical and electrical networks, is called the resonant frequency, typically written as ω_o. The last approximation ignores the (typically) minor correction to the resonant frequency, which in engineering practice is typically always ignored. Knowing that there is a correction is highlighted by this formula, making one aware it exists.

Summary: The quadratic equation and its solution are ubiquitous in physics and engineering. It seems obvious that instead of memorizing the meaningless Eq. 1.30, one should learn the physically meaningful solution, Eq. 1.32, obtained via Eq. 1.29, with $a = 1$. Arguably, the factored and normalized form (Eq. 1.29) is easier to remember, as a method (completing the square), rather than as a formula to be memorized.

Additionally, the real $(b/2)$ and imaginary $\sqrt{c}$ parts of the roots have physical significance as the damping and resonant frequency. Equation 1.30 has none.

No insight is gained by memorizing the quadratic formula. To the contrary, an important concept is gained by learning how to complete the square, which is typically easier than identifying a, b, c and blindly substituting them into Eq. 1.30. Thus it's worth learning the alternate solution (Eq. 1.32) since it is more common in practice and requires less algebra to interpret the final answer.

Exercise: By direct substitution demonstrate that Eq. 1.30 is the solution of Eq. 1.28. Hint: Work with $\hat{P}_2(x)$. **Solution:** Setting $a = 1$ the quadratic formula may be written

$$x_\pm = \frac{-b \pm 1\jmath\sqrt{4c - b^2}}{2}.$$

[47] This is the case for mechanical and electrical circuits having small damping. Physically $b > 0$ is the damping coefficient and $\sqrt{c} > 0$ is the resonant frequency. One may then simplify the form as $x^2 + 2bx + c^2 = (x + b + \jmath c)(x + b - \jmath c)$.

Substituting this into $\hat{P}_2(x)$ gives

$$\begin{aligned}\hat{P}_\pm(x_\pm) &= {x_\pm}^2 + bx_\pm + c \\ &= \left(\frac{-b \pm \sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b \pm \sqrt{b^2-4c}}{2}\right) + c \\ &= \frac{1}{4}\left(b^2 \mp 2b\sqrt{b^2-4c} + (b^2 - 4c)\right) + \frac{1}{4}\left(-2b^2 \pm 2b\sqrt{b^2-4c}\right) + \not{c} \\ &= 0.\end{aligned}$$

Exercise: By direct substitution, show that Eq. 1.32 satisfies Eq. 1.30. **Solution:** Putting Eq. 1.30 into Eq. 1.30 in factored form, gives

$$\begin{aligned}\hat{P}_2(x) &= (x - x_+)(x - x_-) \\ &= \left(x + \frac{1}{2}b + \jmath\sqrt{c - (b/2)^2}\right)\left(x + \frac{1}{2}b - \jmath\sqrt{c - (b/2)^2}\right) \\ &= \left(x + \frac{b}{2}\right)^2 + \left(c - b^2/4\right) \\ &= \left(x^2 + bx + \not{b^2/4}\right) + \left(c - \not{b^2/4}\right) \\ &= x^2 + bx + c.\end{aligned}$$

In third grade I learned the times-table trick for 9:[48]

$$9 \cdot n = (n-1) \cdot 10 + (10 - n). \tag{1.33}$$

With this simple rule I did not need to depend on my memory for the 9 times tables. By expanding the above, one can see why it works: $9n = n10 \not{-10} + \not{10} - n = n(10-1)$. Learning an algorithm is much more powerful than memorization of the 9 times tables. How one thinks about a problem can have great impact.

Newton's method for finding roots of $P_N(s)$ Newton is well known for an approximate but efficient method to find the roots of a polynomial.[49] Consider polynomial $P_N(s) \in \mathbb{C}$ for $s \in \mathbb{C}$

$$P_N(s) = c_N(s - s_0)^N + c_{N-1}(s - s_0)^{N-1} + \cdots + c_1(s - s_0) + c_0,$$

where we may use Taylor's formula (p. 67) to determine the coefficients

$$c_n = \frac{1}{n!}\frac{d^n}{ds^n}P_N(s)\bigg|_{s=s_0}.$$

[48] E.G.: $9 \cdot 7 = (7-1) \cdot 10 + (10-7) = 60 + 3$ and $9 \cdot 3 = (3-1) \cdot 10 + (9-3) = 20 + 7$. As a check, note that the two terms $(n-1)$ and $(10-n)$, add to 9.

[49] https://en.wikipedia.org/wiki/Newton's_method

If our initial guess for the root s_0 is close to a roots, then $|(s-s_0)|^n \ll |(s-s_0)|$ for $n \geq 2 \in \mathbb{N}$. Thus we may truncate $P_N(s-s_o)$ to its linear term

$$\begin{aligned} P_N(s) &\approx (s-s_0)\left.\frac{d}{ds}P_N(s)\right|_{s=s_0} + P_N(s_0) \\ &= (s-s_0)P'_N(s_0) + P_N(s_0), \end{aligned}$$

where $P'_N(s)$ is shorthand for $dP_N(s)/ds$.

This equation may be recursively iterated, defining a sequence s_n that converges to the root, such that $P_N(s_n) = 0$ as $n \to \infty$. Replacing s by s_n and s_0 with s_{n-1}, the formula becomes a recursion for the root s_n as $n \to \infty$

$$(s_n - s_{n+1})P'_N(s_{n+1}) + P_N(s_{n+1}) = P_N(s_n) \to 0.$$

With every step the expansion point moves closer to the root, converging to the root in the limit. As it comes closer, the linearity assumption becomes more accurate, ultimately resulting in the convergence to the root. Solving for s_n gives the key formula behind Newton's famous root-finding method

$$s_n = s_{n-1} - \frac{P_N(s_{n-1})}{P'_N(s_{n-1})}.$$

Here s_{n-1} is the old expansion point and s_n is the next approximation to the root. This expression is related to the log-derivative $d\log P(x)/dx = P'(x)/P(x)$. It follows that even for cases where fractional derivatives of roots are involved, Newton's method should converge, since the log-derivative linearizes them.[50]

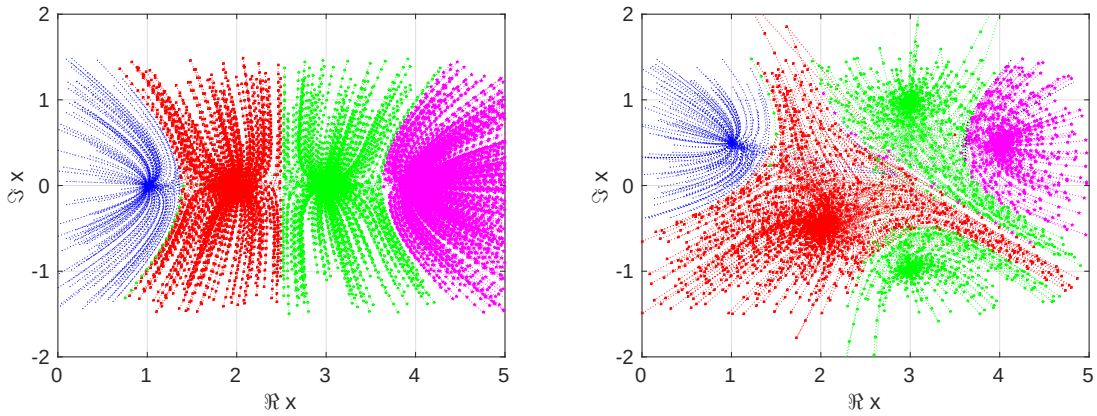

Figure 1.14: Newton's method applied to the polynomial having real roots [1, 2, 3, 4] (left) and 5 complex roots (right). A random starting point was chosen, and each curve shows the values of s_n as Newton's method converges to the root. Different random starting points converge to different roots. The method always results in convergence to a root. Claims to the contrary are a result of forcing the roots to be real (Stewart, 2012, p. 347). For convergence, one must work with $s_n \in \mathbb{C}$.

Newton's view: Newton believed that imaginary roots and numbers had no meaning (p. 105) and only sought real roots. In this case Newton's relation may be explored as a graph, which puts Newton's method in the realm of analytic geometry. The function $P'_N(x)$ is the slope of the

[50] https://en.wikipedia.org/wiki/Argument_principle

polynomial $P_N(x)$ at x_n. The value of $x_n \in \mathbb{R}$ is the estimate of the root after n iterations, with x_0 being the initial guess.

Example: When the polynomial is $P_2 = 1-x^2$ and $P_2'(x) = -2x$, Newton's iteration becomes

$$x_{n+1} = x_n + \frac{1-x_n^2}{2x_n}.$$

For the case of $N = 2$ the root of $P_2'(x)$ is the average of the the roots of $P_2(x)$.

To start the iteration ($n = 0$) we need an initial guess for x_0, which is a "best guess" of where the root will be. If we let $x_0 = 1/2$, then

$$x_1 = x_0 - \frac{1-x_0^2}{2x_0} = x_0 + \frac{1}{2}\left(x_0 - 1/x_0\right).$$

1. Let $P_2(x) = 1 - x^2$, and $x_0 = 1/2$. Draw a graph describing the first step of the iteration. **Solution:** Start with an (x, y) coordinate system and put points at $x_0 = (1/2, 0)$ and the vertex of $P_2(x)$, i.e.: $(0, 1)$ ($P_2(0) = 1$). Then draw $1 - x^2$, along with a line from x_0 to x_1.

2. Calculate x_1 and x_2. What number is the algorithm approaching? Is it a root of P_2? **Solution:** First we must find $P_2'(x) = -2x$. Thus the equation we will iterate is

$$x_{n+1} = x_n + \frac{1-x_n^2}{2x_n} = \frac{x_n^2+1}{2x_n}.$$

   By hand

$$\begin{aligned} x_0 &= 1/2 \\ x_1 &= \frac{(1/2)^2+1}{2(1/2)} = \frac{1}{4} + 1 = 5/4 = 1.25 \\ x_2 &= \frac{(5/4)^2+1}{2(5/4)} = \frac{(25/16)+1}{10/4} = \frac{41}{40} = 1.025. \end{aligned}$$

   These estimates are rapidly approaching $x = 1$, the positive (real) root of $P_2(x)$. Note that if one starts at the root of $P'(x) = 0$ (i.e., $x_0 = 0$), the first step is in-determinant.

3. Write a Matlab script to check your answer for part (a). **Solution:**

```
x=1/2;
for n = 1:3
x = x+(1-x*x)/(2*x);
end
```

   (a) For $n = 4$, what is the absolute difference between the root and the estimate, $|x_r - x_4|$? **Solution:** 4.6E-8 (very small!)

   (b) What happens if $x_0 = -1/2$? **Solution:** You converge on the negative root, $x = -1$.

4. Does Newton's method work for $P_2(x) = 1 + x^2$? Why?[51] Hint: What are the roots in this case? **Solution:** In this case $P_2'(x) = +2x$ thus the iteration gives

$$x_{n+1} = x_n - \frac{1+x_n^2}{2x_n}.$$

[51] https://en.wikipedia.org/wiki/Newton's_method#Complex_functions

In this case the roots are $x_\pm = \pm 1\jmath$, namely purely imaginary. Obviously *Newton's method fails*, because there is no way for the answer to become complex. If like Newton, you didn't believe in complex numbers, your method would fail to converge to the complex roots. Real in, real out.

5. What if you let $x_0 = (1+\jmath)/2$ for the case of $P_2(x) = 1 + x^2$? **Solution:** By starting with a complex initial value, we fix the Real in = Real out problem.

   Fractal diagrams: When the initial guess is real, but the nearest root is complex, Newton's iteration fails, as discussed in the exercises above. This failure to converge gives rise to fractal patterns, called the *Mandelbrot set*.[52]

Example: Assume that polynomial $P_3(s) = (s-a)^2(s-b)^\pi$. Then

$$\ln P_3(s) = 2\ln s - a + \pi \ln s - b$$

and

$$\frac{d}{ds}\ln P_3(s) = \frac{2}{s-a} + \frac{\pi}{s-b}.$$

Reduction by logarithmic derivative to simple poles: As shown by the above trivial example, any polynomial, having zeros of arbitrary degree (i.e., π in the example), may be reduced to the ratio of two polynomials, by taking the logarithmic derivative, since

$$Y_N(s) = \frac{N(s)}{D(s)} = \frac{d}{ds}\ln P_N(s) = \frac{P'_N(s)}{P_N(s)}. \tag{1.34}$$

Here the starting polynomial is the denominator $D(s) = P_n(s)$ while the numerator $N(s) = P'_N(s)$ is the derivative of $D(s)$. Thus the logarithmic derivative can play a key role in analysis of complex-analytic functions, as *it reduces higher order poles, even those of irrational degree, to simple poles*.

The logarithmic derivative $Y_N(s)$ has a number of special properties:

1. $Y_N(s)$ has simple poles s_p and zeros s_z.
2. The poles of $Y_N(s)$ are the zeros of $P_N(s)$.
3. The zeros of $Y_N(s)$ (i.e., $P'_N(s_z) = 0$) are the zeros of $P'_N(s)$.
4. $Y_N(s)$ is analytic everywhere other than its poles.
5. Since the zeros of $P_N(s)$ are simple (no second-order poles), it is obvious that the zeros of $Y_N(s)$ always lie close to the line connecting the two poles. One may easily demonstrate the truth of the statement numerically, and has been quantified by the *Gauss-Lucas theorem* which specifies the relationship between the roots of a polynomial and those of its derivative. Specifically, the roots of P'_{N-1} lie inside the convex hull of the roots of P_N.
6. Newton's method may be expressed in terms of the logarithmic derivative, since

   $$s_{k+1} = s_k + \epsilon_o / Y_N(s),$$

   where ϵ_o is called the *step size*, which is used to control the rate of convergence of the algorithm to the zeros of $P_n(s)$. If the step size is too large, the root finding path may jump to a different domain of convergence, thus a different root of $P_n(s)$.

Further analysis on Newton's method may be found in Section H.1.1, p. 241.

[52] https://en.wikipedia.org/wiki/Mandelbrot_set

1.3.2 Matrix formulation of the polynomial

There is a one-to-one relationship between polynomials and matrix analysis. These are best describe in terms of the *companion matrix*.

Companion Matrix

The companion matrix

$$C_N = \begin{bmatrix} -c_{N-1} & c_{N-2} & \cdots & \cdots & -c_0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \ldots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{1.35}$$

is derived from the monic polynomial of degree N

$$\begin{aligned} P_N(s) &= s^N + c_{N-1}s^{N-1} \cdots + c_2 s^2 + c_1 s + c_o \\ &= s^N + \sum_{n=0}^{N-1} c_n s^n, \end{aligned}$$

having coefficient vector having coefficients

$$\mathbf{c}_{N-1}^T = [c_{N-1}, c_{N-2}, \cdots c_o]^T.$$

An alternate form is (Horn and Johnson, 1988, p. 146)

$$C'_N = \begin{bmatrix} 0 & & & & & & -c_0 \\ 1 & 0 & & & \mathbf{0} & & -c_1 \\ 0 & 1 & 0 & & & & -c_2 \\ \vdots & 0 & 1 & 0 & \cdots & & \vdots \\ & & \cdots & \ddots & 0 & & \vdots \\ & \mathbf{0} & & & 1 & 0 & -c_{N-2} \\ & & & & 0 & 1 & -c_{N-1} \end{bmatrix}. \tag{1.36}$$

The *Companion matrix* has the same eigen-values as the roots of the monic polynomial $P_N(s)$. That is, the roots of monic polynomial of degree N are the eigen-values of the companion matrix C_N.

Exercise: Show that the eigen-values of the 3x3 companion matrix are the same as the roots of $P_3(s)$. **Solution:** Expanding the determinant of $C_3 - sI_3$ along the right-most column:

$$-\begin{vmatrix} -s & 0 & -c_0 \\ 1 & -s & -c_1 \\ 0 & 1 & -(c_2+s) \end{vmatrix} = c_0 + c_1 s + (c_2+s)s^2 = s^3 + c_2 s^2 + c_1 s + c_0.$$

This is the characteristic polynomial, which is equal to $-P_3(s)$.

is there a sign error?

Exercise: Find the companion matrix for the Fibonacci sequence, defined by the difference equation

$$f_{n+2} = f_{n+1} + f_n$$

with $f_0 = 1$. **Solution:** Taking the Z transform gives the polynomial $z^2 - z^1 - z^0 = 0$, having the coefficient vector $C = [1, -1, -1]$, resulting in the Fibonacci companion matrix

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

as discussed on page 53.

Example: Matlab/Octave: A polynomial is represented in Matlab/Octave in terms of its coefficient vector. When the polynomial vector for the poles of a differential equation is

$$\boldsymbol{c}_N = [1, c_{N-1}, c_{N-2}, \cdots c_0]^T,$$

the coefficient $c_N = 1$. This normalization guarantees that the leading term is not zero, and the number of roots (N) is equal to the degree of the monic polynomial.

Working with polynomials in Matlab/Octave:

In Matlab/Octave there are eight functions you need to become familiar with, that work together:

1. `R=root(A)` Vector `A` $= [a_N, a_{N-1}, \ldots, a_o] \in \mathbb{C}$ are the complex coefficients of polynomial $P_N(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}$, where $N \in \mathbb{N}$ is the degree of the polynomial. It is convenient to force $a_N = 1$, corresponding to dividing the polynomial by this value, when it is not 1, guaranteeing it cannot be zero, as mentioned above. Further `R` is the vector of roots, $[z_1, z_2, \cdots, z_n] \in \mathbb{C}$ such that `polyval(A,`z_k`)=0`.

2. `y=polyval(A,x)`: This evaluates the polynomial defined by vector `A` $\in \mathbb{C}^N$ evaluated at `x` $\in \mathbb{C}$, returning vector `y(x)` $\in \mathbb{C}$.

3. `P=poly(R)`: This is the inverse of `root()`, returning a vector of polynomial coefficients `P` $\in \mathbb{C}^N$ of the corresponding *characteristic polynomial*, starting from either a vector of roots R, or a matrix A, for example, defined with the roots on the diagonal. The characteristic polynomial is defined as the determinant of $|A - \lambda I| = 0$ having roots R.

   Due to IEEE-754 scaling issues, this can give strange results that are numerically correct, but only within the limits of IEEE-754 accuracy.

4. `R=polyder(N)` This routine takes the coefficients of a polynomial N and returns the coefficients of the derivative of N. This is useful when working with Newton's method, since each step is proportional to $P_N(x)/P'_{N-1}(x)$.

5. `[K,R]=residue(N,D)`: Given the ratio of two polynomials `N,D`, `residue(N,D)` returns vectors `K,R` such that

$$\frac{N(s)}{D(s)} = \sum_k \frac{K_k}{s - s_k}, \tag{1.37}$$

   where $s_k \in \mathbb{C}$ are the roots of the denominator `D` polynomial and `K` $\in \mathbb{C}$ is a vector of *residues*, which characterize the roots of the numerator polynomial $N(s)$. The use of `residue()` will be discussed in Sect. 1.3.5 (p. 76), and in greater detail in Sect. 1.5.6 (p. 142), and in Appendix F.1, p. 211.

6. `C=conv(A,B)`: Vector `C`$\in \mathbb{C}^{N+M-1}$ contains the polynomial coefficients of the convolution of the two vector of coefficients of polynomials `A,B`$\in \mathbb{C}^N$ and `B`$\in \mathbb{C}^M$. For example `[1, 2, 1]=conv([1, 1], [1, 1])`.

7. `[C,R]=deconv(N,D)`: Vectors `C, N, D` $\in \mathbb{C}$. This operation uses long division of polynomials to find $C(s) = N(s)/D(s)$ with remainder $R(s)$, where `N = conv(D,C)+R`, namely

$$C = \frac{N}{D} \quad \text{remainder} \quad R \tag{1.38}$$

8. `A=compan(D)`: Vector $D = [1, d_{N-1}, d_{N-2}, \cdots, d_0]^T$ contains the coefficients of polynomial

$$D(s) = s^N + \sum_{k=1}^{N} d_{N-k} s^k,$$

and A is the companion matrix, of vector D (Eq. F.3, p. 212). The eigen-values of A are the roots of monic polynomial $D(s)$.

Exercise: Practice the use of Matlab's/Octave's related functions, which manipulate roots, polynomials and residues: `root()`, `conv()`, `deconv()`, `poly()`, `polyval()`, `polyder()`, `residue()`, `compan()`.

Solution: Try Newton's method for various polynomials. Use `N=poly(R)` to provide the coefficients of a polynomial given the roots R. Then use `root()` to factor the resulting polynomial. Then use Newton's method and show that the iteration converges to the nearest root.[53]

1.3.3 Taylor series

The definition of an analytic function is one that

1. may be expanded in a series (called a Taylor series expansion)

$$P(x) = \sum_{n=0}^{\infty} c_n (x - x_o)^n, \tag{1.39}$$

2. converges for $|x - x_o| < 1$, called the RoC, with coefficients c_n that are determined by the derivatives of $P(x)$, evaluated at the expansion point $x = x_o$.

3. The Taylor series representation of $P(x)$ has special applications for solving differential equations because 1) it is single valued, and

4. all its derivatives are uniquely defined.

5. it may be trivially continued by making $x \in \mathbb{C}$, making its $\mathcal{LT}^{-1}$ causal.

[53] A Matlab/Octave program that does this may be downloaded from `http://jontalle.web.engr.illinois.edu/uploads/493/M/NewtonJPD.m`.

A limitation of the Taylor series expansion is that it is not valid outside of the RoC. One method for avoiding this limitation is to move the expansion point. But this analytic continuation is a non-trivial exercise because it requires working with the derivatives of $P(x)$ at the new expansion point, where $P(x)$ may not even have derivatives, due to a possible singularity.

The *Taylor series* coefficients c_n are defined by taking derivatives of $P(x)$ and evaluating them at the expansion point x_0, namely

$$c_n = \frac{1}{n!}\frac{d^n}{dx^n}P(x-x_o)\bigg|_{x=x_o}. \tag{1.40}$$

The Taylor formula is a prescription for how to uniquely define the coefficients c_n. Without the Taylor series formula, we would have no way of determining c_n. The proof of the Taylor formula is transparent, simply by taking successive derivative of Eq. 1.39, and then evaluating the result at the expansion point. If $P(x)$ is analytic then this procedure will always work. If $P(x)$ fails to have a derivative of any order, then the function is not analytic and Eq. 1.39 is not valid for $P(x)$. For example, if $P(x)$ has a pole at x_o then it is not analytic at that point.

Example: The trivial (corner) case is the geometric series $P(x) = 1/(1-x)$ about the expansion point $x = 0$. The function $P(x)$ is defined everywhere, except at the singular point $x = 1$, whereas the geometric series is only valid for $|x| < 1$.

Exercise: Verify that c_o and c_1 of Eq. 1.39 follow from Eq. 1.40. **Solution:** To obtain c_0, for $n = 0$, there is no derivative (d^0/dx^0 indicates no derivative is taken), so we must simply evaluate $P(x-x_o) = c_0 + c_1(x-x_o) + \cdots$ at $x = x_o$, leaving c_o. To find c_1 we take one derivative which results in $P'(x) = c_1 + 2c_2(x-x_0))\cdots$. Evaluating this at $x = x_0$ leaves c_1. Each time we take a derivative we reduce the degree of the series by 1, leaving the next constant term.

The Taylor series plays an important role in mathematics, as the coefficients of the series uniquely determine the analytic series representation via its derivatives. The implications and limitations of the power series representation are very specific. *First*, if the series fails to converge (i.e., outside the RoC), it is essentially meaningless.

The Taylor series does not need to be infinite to converge to the function it represents, since it obviously works for any polynomial $P_N(x)$ of degree N. But in the finite case ($N < \infty$), the RoC is infinite, and the series *is* the function $P_N(x)$ exactly, everywhere. Of course $P_N(x)$ is called a polynomial, of degree N. When $N \to \infty$, the Taylor series is only valid within the RoC, and it is (typically) the representation of the reciprocal of a polynomial.

These properties are both the curse and the blessing of the analytic function. On the positive side, analytic functions are the ideal starting point for solving differential equations, which is exactly how they were used by Newton and others. Analytic functions are "smooth" since they are infinitely differentiable, with coefficients given by Eq. 1.40. They are single valued, so there can be no ambiguity in their interpretation.

Two well-known analytic functions are the *geometric series*

$$\frac{1}{1-x} = 1 + x + x^2 + x^2 + \ldots = \sum_{n=0}^{\infty} x^n \tag{1.41}$$

and exponential series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \ldots = \sum_{n=0}^{\infty}\frac{1}{n!}x^n. \tag{1.42}$$

Brune impedance: A third very special family of functions are formed from ratios of polynomials, typically used to define impedances. Impedance functions as a class of functions are special because they must have a positive real part, so as to obey conservation of energy. A physical impedance cannot have a negative resistance (the real part); otherwise it would act like a power source, violating conservation of energy. Most impedances are in the class of Brune impedances, defined by the ratio of two polynomials, of degrees M and N

$$Z_{\text{Brune}}(s) = \frac{P_N(s)}{P_M(s)} = \frac{s^N + a_1 S^{N-1} \cdots a_0}{s^M + b_1 S^{M-1} \cdots b_0}, \tag{1.43}$$

where $M = N \pm 1$ (i.e., $N = M \pm 1$). This fraction of polynomials is sometimes known as a "Padé approximation," but more specifically this ratio is a *Brune impedance*, with poles and zeros, defined as the complex roots of the two polynomials. The key propriety of the Brune impedance is that the real part of the impedance is non-negative (positive or zero) in the right s half-plane

$$\Re Z(s) = \Re\left[R(\sigma,\omega) + jX(\sigma,\omega)\right] = R(\sigma,\omega) \geq 0 \qquad \text{for} \quad \Re s = \sigma \geq 0. \tag{1.44}$$

Since $s = \sigma + \omega\jmath$, the complex frequency (s) right half-plane (RHP) corresponds to $\Re s = \sigma \geq 0$). This condition defines the class of *positive-real* functions, also known as the the *Brune condition*, which is frequently written in abbreviated form as $\Re Z(\Re s \geq 0) \geq 0$.

As a result of this *positive-real* constraint on impedance functions, the subset of Brune impedances (those given by Eq. 1.43 satisfying Eq. 1.44) must be complex analytic in the entire right s half-plane. This is a powerful constraint that places strict limitations on the locations of both the poles and the zeros of every Brune impedance.

Exercise: Find the RoC of the following by application of Eq. 1.40.

1. $w(x) = \frac{1}{1-x\jmath}$. **Solution:** From a straightforward expansion we know the coefficients are

$$\frac{1}{1-x\jmath} = 1 + x\jmath + (x\jmath)^2 + (x\jmath)^3 \cdots = 1 + x\jmath - x^2 + -\jmath x^3 \cdots .$$

Working this out using Eq. 1.40 is more work:
$c_0 = \frac{1}{0!}w\Big|_0 = 1; c_1 = \frac{1}{1!}\frac{dw}{dx}\Big|_0 = -\frac{-\jmath}{(1-x\jmath)^2}\Big|_{x=0} = \jmath; c_2 = \frac{1}{2!}\frac{d^2w}{dx^2}\Big|_0 = \frac{1}{2!}\frac{-2}{(1-x\jmath)^3}\Big|_0 = -1;$
$c_3 = \frac{1}{3!}\frac{d^3w}{dx^3}\Big|_0 = \frac{-\jmath}{(1-x\jmath)^4}\Big|_0 = -j.$
However, if we take derivatives of the series expansion it is much easier, and one can even figure out the term for c_n:
$c_0 = 1; c_1 = \frac{d}{dx}\sum(\jmath x)^n\Big|_0 = j; c_2 = \frac{1}{2!}\frac{d^2}{dx^2}\sum(jx)^n\Big|_0 = 2(\jmath)^2;$
$c_3 = \frac{1}{3!}\frac{d^3}{dx^3}\sum(jx)^n\Big|_0 = (\jmath)^3 = -\jmath;$
$\cdots,$
$c_n = \frac{1}{n!}\jmath^n n! = \jmath^n.$

2. $w(x) = e^{x\jmath}$ **Solution:** $c_n = \frac{1}{n!}\jmath^n$.

Determining the region of convergence (RoC): Determining the RoC for a given analytic function is quite important, and may not always be obvious. In general the RoC is a circle whose radius extends from the expansion point out to the nearest pole. Thus when the expansion point is moved, the RoC changes since the location of the pole is fixed.

Example: For the geometric series (Eq. 1.41), the expansion point is $x_o = 0$, and the RoC is $|x| < 1$, since $1/(1-x)$ has a pole at $x = 1$. We may move the expansion point by a linear transformation, for example, by replacing x with $z+3$. Then the series becomes $1/((z+3)-1) = 1/(z+2)$, so the RoC becomes 2, because in the z plane the pole has moved to -2.

Example: A second important example is the function $1/(x^2+1)$, which has the same RoC as the geometric series, since it may be expressed in terms of its *residue expansion* (aka, partial fraction expansion)

$$\frac{1}{x^2+1} = \frac{1}{(x+1\jmath)(x-1\jmath)} = \frac{1}{2\jmath}\left(\frac{1}{x-1\jmath} - \frac{1}{x+1\jmath}\right).$$

Each term has an RoC of $|x| < |1\jmath| = 1$. The amplitude of each pole is called the *residue*, defined in Section 1.4.5 Eq. 1.107, p. 119. The residue for the pole at $1\jmath$ is $1/2\jmath$.

In summary, the function $1/(x^2+1)$ is the sum of two geometric series, with poles at $\pm 1\jmath$ which are not as obvious because the roots are complex, and conjugate. Once factored, it becomes clear what is going on.

Exercise: Verify the above expression is correct, and show that the residues are $\pm 1/2\jmath$. **Solution:** Cross-multiply and cancel, x cancels out and we are left with 1, as required.

Exercise: Find the residue of $\frac{d}{dz}z^\pi$. **Solution:** Taking the derivative gives $z^{\pi-1}$ which has a pole at $z = 0$. Applying the formula for the residue (Eq. 1.107, p. 119) we find

$$c^{-1} = \lim_{z\to 0} z z^{\pi-1} = \lim z^\pi = 0.$$

Thus the residue is zero. The exponential series converges for every finite value of $x \in \mathbb{R}$ (the RoC is the entire real line), thus the exponential is called an *entire function*.

Analytic functions:

Any function that has a Taylor series expansion is called an *analytic function*. Within the RoC, the series expansion defines a single-valued function. Polynomials $1/(1-x)$ and e^x are examples of analytic functions that are real functions of their real argument x. This is not the entire story. Because analytic functions are easily manipulated term by term, they may be used to find solutions of differential equations since the derivatives of a series are uniquely determined within the RoC, due to Eq. 1.40.

Every analytic function has a corresponding differential equation, that is determined by the coefficients a_k of the analytic power series. An example is the exponential, which has the property that it is the eigen-function of the derivative operation

$$\frac{d}{dx}e^{ax} = ae^{ax},$$

which may be verified using Eq. 1.42. This relationship is a common definition of the exponential function, which is very special because it is the eigen-function of the derivative.

The complex analytic power series (i.e.: complex analytic functions) may also be integrated, term by term, since

$$\int^x f(x)dx = \sum \frac{a_k}{k+1}x^{k+1}. \tag{1.45}$$

Newton took full advantage of this property of the analytic function and used the analytic series (*Taylor series*) to solve analytic problems, especially for working out integrals, allowing him to solve differential equations. To fully understand the theory of differential equations, one must master single-valued analytic functions and their analytic power series.

Single- vs. multi-valued functions: Polynomials and their ∞-degree extensions (analytic functions) are single valued: for each x there is a single value for $P_N(x)$. The roles of the domain and codomain may be swapped to obtain an *inverse function*, with properties that can be quite different from those of the function. For example, $y(x) = x^2 + 1$ has the inverse $x = \pm\sqrt{y-1}$, which is double valued, and complex when $y < 1$. Periodic functions such as $y(x) = \sin(x)$ are even more "exotic" since $x(y) = \arcsin(x) = \sin^{-1}(x)$ has an infinite number of $x(y)$ values for each y. This problem was first addressed in Riemann's 1851 PhD thesis, written while he was working with Gauss.

Exercise: Let $y(x) = \sin(x)$. Then $dy/dx = \cos(x)$. Show that $dx/dy = -1/\sqrt{1-x^2}$. Hint: $x(y) = \cos^{-1}(y) = \arccos(y)$. **Solution:** See the *implicit function theorem* (D'Angelo, 2017, p. 104). Add solution.

Exercise: Let $y(x) = \sin(x)$. Then $dy/dx = \cos(x\jmath)$. Show that $dx/dy = -\jmath/\sqrt{1+x^2}$. **Solution:** Add solution.

Exercise: Find the Taylor series coefficients of $y = \sin(x)$ and $x = \sin^{-1}(y)$. **Solution:** Add solution.

Complex analytic functions: When the argument of an analytic function $F(x)$ is complex, that is, $x \in \mathbb{R}$ is replaced by $s = \sigma + \omega\jmath \in \mathbb{C}$ (recall that $\mathbb{R} \subset \mathbb{C}$)

$$F(s) = \sum_{n=0}^{\infty} c_n(s - s_o)^n, \tag{1.46}$$

with $c_n \in \mathbb{C}$, that function is said to be a *complex analytic*.

For example, when the argument of the exponential becomes complex, it is periodic on the ω axis, since

$$e^{st} = e^{(\sigma+\omega\jmath)t} = e^{\sigma t}e^{\jmath\omega t} = e^{\sigma t}\left[\cos(\omega t) + \jmath\sin(\omega t)\right]. \tag{1.47}$$

Taking the real part gives

$$\Re\{e^{st}\} = e^{\sigma t}\frac{e^{\omega\jmath t} + e^{-\omega\jmath t}}{2} = e^{\sigma t}\cos(\omega t),$$

and $\Im\{e^{st}\} = e^{\sigma t}sin(\omega t)$. Once the argument is allowed to be complex, it becomes obvious that the exponential and circular functions are fundamentally related. This exposes the family of *entire circular functions* [i.e., $e^s, \sin(s), \cos(s), \tan(s), \cosh(s), \sinh(s)$] and their inverses [$\ln(s), \arcsin(s), \arccos(s), \arctan(s), \cosh^{-1}(s), \sinh^{-1}(s)$], first fully elucidated by Euler (c1750)

(Stillwell, 2010, p. 315). Note that because a function, such as $\sin(\omega t)$, is periodic, its inverse must be multi-valued. What is needed is some systematic way to account for this multi-valued property.

Given a complex analytic function of a complex variable, one must resort to the *extended complex plane*, *Riemann sheets* and *branch cuts*, as discussed in Section 1.3.11 (p. 92). The extended complex plane is a tool that extends the domain of complex analytic to include the point at infinity. This topic is critically important in engineering mathematics and will be discussed at length in Sections 1.3.11-1.3.14 (pp. 92-98).

Definition of the Taylor series of a complex analytic function: However there is a fundamental problem: we cannot formally define the Taylor series for the coefficients c_k until we have defined the derivative with respect to the complex variable $dF(s)/ds$, with $s \in \mathbb{C}$. Thus simply substituting s for x in an analytic function leaves a major hole in one's understanding of the complex analytic function.

To gain a feeling of the nature of the problem, we make take derivatives of a function with respect to various variables. For example,

$$\frac{d}{dt}e^{st} = se^{st},$$

$$e^{\omega \jmath t}\frac{d}{d\sigma}e^{\sigma t} = \sigma e^{st},$$

and

$$e^{\sigma t}\frac{d}{d\omega \jmath}e^{\omega \jmath} = \omega \jmath e^{st}.$$

It was Cauchy (1814) (Fig. 1.13) who uncovered much deeper relationships within complex analytic functions (Sect. 1.3.12, p. 95) by defining differentiation and integration in the complex plane, leading to several fundamental theorems of complex calculus, including the *fundamental theorem of complex integration*, and Cauchy's formula. We shall explore this and several fundamental theorems in Sect. 1.4.1 (p. 105).

There seems to be some disagreement as to the status of multi-valued functions: Are they functions, or is a function strictly single valued? If so, then we are missing out on a host of interesting possibilities, including all the inverses of nearly every complex analytic function. For example, the inverse of a complex analytic function is a complex analytic function (e.g., e^s and $\log(s)$).

Impact of complex-analytic mathematics on physics: It seems likely, if not obvious, that the success of Newton was his ability to describe physics by the use of mathematics. He was inventing new mathematics at the same time he was explaining new physics. The same might be said for Galileo. It seems likely that Newton was extending the successful techniques and results of Galileo (Galileo, 1638). Galileo died on Jan 8, 1642, and Newton was born Jan 4, 1643, just short of one year later. Certainly Newton was well aware of Galileo's great success, and naturally would have been influenced by it.

The application of complex analytic functions to physics was dramatic, as may be seen in the six volumes on physics by Arnold Sommerfeld (1868-1951), and from the productivity of his many (36) students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg[54] and Seebach, to

[54] https://www.aip.org/history-programs/niels-bohr-library/oral-histories/4661-1

name a few), notable coworkers (i.e., Leon Brillouin) and others (i.e., John Bardeen) upon whom Sommerfeld had a strong influence. Sommerfeld is famous for training many students who were awarded the Nobel Prize in Physics, yet he never won a Nobel (the prize is not awarded in mathematics). Sommerfeld brought mathematical physics (the merging of physical and experimental principles via mathematics) to a new level with the use of complex integration of analytic functions to solve otherwise difficult problems, thus following the lead of Newton who used real integration of Taylor series to solve differential equations (Brillouin, 1960, Ch. 3 by Sommerfeld, A.).

1.3.4 Lec 12: Polynomial root classification by convolution

Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem was being formulated by d'Alembert. The idea behind this theorem is that every polynomial of degree N (Eq. 1.27) has at least one root. This may be written as the product of the root and a second polynomial of degree of $N-1$. By the recursive application of this concept, it is clear that every polynomial of degree N has N roots. Today this result is known as the *fundamental theorem of algebra:*

> Every polynomial equation $P(z) = 0$ has a *solution* in the complex numbers. As Descartes observed, a solution $z = a$ implies that $P(z)$ has a factor $z - a$. The quotient
>
> $$Q(z) = \frac{P(z)}{z-a} = \frac{P(z)}{a}\left[1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots\right] \tag{1.48}$$
>
> is then a polynomial of one lower degree. ... We can go on to factorize $P(z)$ into n linear factors.
>
> —Stillwell (2010, p. 285).

The ultimate expression of this theorem is given by Eq. 1.27 (p. 59), which indirectly states that an n^{th} degree polynomial has n roots. We shall use the term *degree* when speaking of polynomials and the term *order* when speaking of differential equations. A general rule is order applies to the time domain and *degree* to the frequency domain, since the Laplace transform of a differential equation, having constant coefficients, of order N, is a polynomial of degree N in Laplace frequency s.

Exercise: Explore expressing Eq. 1.48 in terms of real 2x2 matrices, as described in Section 1.2.1, p. 32.

Today this theorem is so widely accepted we fail to appreciate it. Certainly about the time you learned the quadratic formula, you were prepared to understand the concept of polynomials having roots. The simple quadratic case may be extended a higher degree polynomial. The Matlab/Octave command `roots([1, a2, a1, ao])` provides the roots $[s_1, s_2, s_3]$ of the cubic equation, defined by the coefficient vector $[1, a_2, a_1, a_o]$. The command `poly([s1, s2, s3])` returns the coefficient vector. I don't know the largest degree that can be accurately factored by Matlab/Octave, but I'm sure its well over $N = 10^3$. Today, finding the roots numerically is a solved problem.

Factorization versus convolution: The best way to gain insight into the polynomial factorization problem is through the inverse operation, multiplication of monomials. Given the roots x_k, there is a simple algorithm for computing the coefficients a_k of $P_N(x)$ for any n, no matter how

large. This method is called *convolution*. Convolution is said to be a *trap-door* since it is easy, while the inverse, factoring (*deconvolution*), is hard, and analytically intractable for degree $N \geq 5$ (Stillwell, 2010, p. 102).

Convolution of monomials

As outlined by Eq. 1.27, a polynomial has two descriptions, first as a series with coefficients a_n and second in terms of its roots x_r. The question is "What is the relationship between the coefficients and the roots?" The simple answer is that they are related by *convolution*.

Let us start with the quadratic

$$(x+a)(x+b) = x^2 + (a+b)x + ab, \tag{1.49}$$

where in vector notation $[-a, -b]$ are the roots and $[1, a+b, ab]$ are the coefficients.

To see how the result generalizes, we may work out the coefficients for the cubic ($N = 3$). Multiplying the following three factors gives (p. 244)

$$(x-1)(x-2)(x-3) = (x^2-3x+2)(x-3) = x(x^2-3x+2) - 3(x^2-3x+2) = x^3 - 6x^2 + 11x - 6.$$

When the roots are $[1, 2, 3]$ the coefficients of the polynomial are $[1, -6, 11, -6]$. To verify, substitute the roots into the polynomial, and show that they give zero. For example, $r_1 = 1$ is a root since $P_3(1) = 1 - 6 + 11 - 6 = 0$.

As the degree increases, the algebra becomes more difficult. Imagine trying to work out the coefficients for $N = 100$. What is needed is a simple way of finding the coefficients from the roots. Fortunately, *convolution* keeps track of the book-keeping, by formalizing the procedure.

Convolution of two vectors: To get the coefficients by convolution, write the roots as two vectors $[1, a]$ and $[1, b]$. To find the coefficients we must convolve the root vectors, indicated by $[1, a] \star [1, b]$, where $\star$ denotes convolution. Convolution is a recursive operation. The convolution of $[1, a] \star [1, b]$ is done as follows: reverse one of the two monomials, padding unused elements with zeros. Next slide one monomial against the other, forming the local *dot product* (element-wise multiply and add):

$$\begin{array}{cccc} a & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ = & 0 & & \end{array} \qquad \begin{array}{ccc} a & 1 & 0 \\ 0 & 1 & b \\ = & x^2 & \end{array} \qquad \begin{array}{ccc} a & 1 & 0 \\ 1 & b & 0 \\ = & (a+b)x & \end{array} \qquad \begin{array}{ccc} 0 & a & 1 \\ 1 & b & 0 \\ = & abx^0 & \end{array} \qquad \begin{array}{cccc} 0 & 0 & a & 1 \\ 1 & b & 0 & 0 \\ = & 0 & & \end{array},$$

resulting in coefficients $[\cdots, 0, 0, 1, a+b, ab, 0, 0, \cdots]$.

By reversing one of the polynomials, and then taking successive dot products, all the terms in the sum of the dot product correspond to the same power of x. This explains why convolution of the coefficients gives the same answer as the product of the polynomials.

As seen by the above example, the positions of the first monomial coefficients are reversed, and then slid across the second set of coefficients, the dot-product is computed, and the result placed in the output vector. Outside the range shown, all the elements are zero. In summary,

$$[1, -1] \star [1, -2] = [1, -1-2, 2] = [1, -3, 2].$$

In general

$$[a, b] \star [c, d] = [ac, bc + ad, bd],$$

Convolving a third term $[1, -3]$ with $[1, -3, 2]$ gives (p. 244)

$$[1, -3] \star [1, -3, 2] = [1, -3 - 3, 9 + 2, -6] = [1, -6, 11, -6],$$

which is identical to the cubic example, found by the algebraic method.

By convolving one monomial factor at a time, the overlap is always two elements, thus it is never necessary to compute more than two multiplies and an add for each output coefficient. This greatly simplifies the operations (i.e., they are easily done in your head). Thus the final result is more likely to be correct. Comparing this to the algebraic method, convolution has the clear advantage.

Exercise:

1. What are the three nonlinear equations that one would need to solve to find the roots of a cubic? **Solution:** From our formula for the convolution of three monomials we may find the nonlinear "deconvolution" relations between the roots[55] $[-a, -b, -c]$ and the cubic's coefficients $[1, \alpha, \beta, \gamma]$

$$\begin{aligned}(x + a) \star (x + b) \star (x + c) &= (x + c) \star (x^2 + (a + b)x + ab) \\ &= x \cdot (x^2 + (a + b)x + ab) + c \cdot (x^2 + (a + b)x + ab) \\ &= x^3 + (a + b + c)x^2 + (ab + ac + cb)x + abc \\ &= [1, a + b + c, ab + ac + cb, abc].\end{aligned}$$

It follows that the nonlinear equations must be

$$\begin{aligned}\alpha &= a + b + c \\ \beta &= ab + ac + bc \\ \gamma &= abc.\end{aligned}$$

Clearly these are solve by the classic cubic solution which appears to be a *deconvolution* problem, also know as *long division of polynomials*. It follows that the following long-division of polynomials must be true:

$$\frac{x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc}{x + a} = x^2 + (b + c)x + bc$$

The product of monomial $P_1(x)$ with a polynomial $P_N(x)$ gives $P_{N+1}(x)$: This statement is another way of stating the *fundamental theorem of algebra*. Each time we convolve a monomial with a polynomial of degree N, we obtain a polynomial of degree $N + 1$. The convolution of two monomials results in a quadratic (degree 2 polynomial). The convolution of three monomials gives a cubic (degree 3). In general, the degree k of the product of two polynomials of degree n, m is the sum of the degrees ($k = n + m$). For example, if the degrees are each 5 ($n = m = 5$), then the resulting degree is 10.

[55] By working with the negative roots we may avoid an unnecessary and messy alternating sign problem.

In summary, the product of two polynomials of degree m, n having m and n roots, gives a polynomial of degree $m+n$. This is an analysis process, of merging polynomials, by coefficient convolution. Multiplying polynomials is a merging process, into a single polynomial.

seems in conflict with 1.2!

Note that the degree of a polynomial is one less than the length of the vector of coefficients. The coefficient on the leading term should always be set to 1 since it cannot be zero or the polynomial would not have degree N.

While you already know this theorem from high school algebra class, it is important to explicitly identify the *fundamental theorem of algebra*.

Composition of polynomials: Convolution is not the only important operation between two polynomials. Another is *composition*, which may be defined for two functions $f(z), g(z)$. Then the composition $c(z) = f(z) \circ g(z) = f(g(z))$. As a specific example, suppose $f(z) = 1 + z + z^2$ and $g(z) = e^{2z}$. With these definitions

$$f(z) \circ g(z) = 1 + e^{2z} + (e^{2z})^2 = 1 + e^{2z} + e^{4z}.$$

Note that $f(z) \circ g(z) \neq g(z) \circ f(z)$.

Exercise: Find $g(z) \circ f(z)$. **Solution:** $e^{2f(z)} = e^{2(1+z+z^2)} = e^2 e^{(1+z+z^2)} = e^3 e^z e^{z^2}$.

1.3.5 Lec 13 Residue expansions of rational functions

As discussed in Section 1.3.1, p. 66, there are 5 important Matlab/Octave routines that are closely related: `conv()`, `deconv()`, `poly()`, `polyder()`, `polyval()`, `residue()`, `root()`. Several of these are complements of each other, or do a similar operation in a slightly different way. Routines `conv`, `poly` build polynomials from the roots while `root` solves for the roots given the polynomial coefficients. The operation `residue()` converts the ratio of two polynomials and expands it in a partial fraction expansion, with poles and residues.

When lines and planes are defined, the equations are said to be *linear* in the independent variables. In keeping with this definition of *linear*, we say that the equations are *non-linear* when the equations have degree greater than 1 in the independent variables. The term *bilinear* has a special meaning, in that both the domain and codomain are linearly related by lines (or planes). As an example, impedance is defined in frequency as the ratio of the voltage over the current, but it frequently has a representation as the ratio of two polynomials $N(s)$ and $D(s)$

$$Z(s) = \frac{N(s)}{D(s)} = sL_o + R_o + \sum_{k=0}^{K} \frac{K_k}{s - s_k}. \tag{1.50}$$

Here $Z(s)$ is the impedance and V and I are the voltage and current at radian frequency ω.[56]

Such an impedance is typically specified as a *rational* or *bilinear* function, namely the ratio of two polynomials, $P_N(s) = N(s) = [a_N, a_{n-1}, \cdots, a_o]$ and $P_K(s) = D(s) = [b_K, b_{K-1}, \cdots, b_o]$ of degrees $N, K \in \mathbb{N}$, as functions of complex Laplace frequency $s = \sigma + \jmath\omega$, having simple roots. Most impedances are rational functions since they may be written as $D(s)V = N(s)I$. Since $D(s)$ and $N(s)$ are both polynomials in s, rational functions are also called *bilinear transformation* or in

[56] Note that the relationship between the impedance and the residues K_k is a linear one, best solved by setting up a linear system of equations in the unknown residues. Methods for doing this will be discussed in Appendix F.1 (p. 211).

the mathematical literature as *Möbius transformation*, which comes from a corresponding scalar differential equation, of the form

$$\sum_{k=0}^{K} b_k \frac{d^k}{dt^k} i(t) = \sum_{n=0}^{N} a_n \frac{d^n}{dt^n} v(t) \quad \leftrightarrow \quad I(\omega) \sum_{k=0}^{K} b_k s^k = V(\omega) \sum_{n=0}^{N} a_n s^n. \tag{1.51}$$

This equation, as well as 1.50, follow from the Laplace transform (See Section 1.3.14, p. 98) of the differential equation (on left), by forming the impedance $Z(s) = V/I = A(s)/B(s)$. This form of the differential equation follows from Kirchhoff's voltage and current laws (KCL, KVL) or from Newton's laws (for the case of mechanics).

The physical properties of an impedance: Based on d'Alembert's observation that the solution to the wave equation is the sum of forward and backward traveling waves, the impedance may be rewritten in terms of forward and backward traveling waves

$$Z(s) = \frac{V}{I} = \frac{V^+ + V^-}{I^+ - I^-} = r_o \frac{1 + \Gamma(s)}{1 - \Gamma(s)}, \tag{1.52}$$

where $r_o = P^+/I^+$ is called the *surge impedance* of the transmission line (e.g., wire) connected to the load impedance $Z(s)$, and $\Gamma(s) = P^-/P_+ = I^-/I^+$ is the reflection coefficient corresponding to $Z(s)$. Any impedance of this type is called a *Brune impedance* due to its special properties (discussed on p. 284) (Brune, 1931a). Like $Z(s)$, $\Gamma(s)$ is causal and complex analytic. Note that the impedance and the reflectance function must both be complex analytic, since the are connected by the bilinear transformation, which assures the mutual complex analytic properties.

Due to the bilinear transformation, the physical properties of $Z(s)$ and $\Gamma(s)$ are very different. Specifically, the real part of the load impedance must be non-negative ($\Re\{Z(\omega\jmath)\} \geq 0$), if and only if $|\Gamma(s)| \leq 1|$. In the time domain, the impedance $z(t) \leftrightarrow Z(s)$ must have a value of r_o at $t = 0$. Correspondingly, the time domain reflectance $\gamma(t) \leftrightarrow \Gamma(s)$ must be zero at $t = 0$.

This is the basis of conservation of energy, which may be traced back to the properties of the reflectance $\Gamma(s)$.

Exercise: Show that if the $\Re\{Z(s)\} \geq 0$ then $|\Gamma(s)| \leq 1$. **Solution:** Their two equivalent proofs, both of which start from the relation between $Z(s)$ and $\Gamma(s)$. Taking the real part of Eq. 1.52, which must be ≥ 0, we find

Lots to do here, with Log-derivative analysis.

$$\Re\{Z(s)\} = \frac{r_o}{2}\left[\frac{1+\Gamma(s)}{1-\Gamma(s)} + \frac{1+\Gamma^*(s)}{1-\Gamma^*(s)}\right] = r_o \frac{1-|\Gamma(s)|^2}{|1+\Gamma(s)|^2} \geq 0.$$

Thus $|\Gamma| \leq 1$.

1.3.6 Lec 14: Introduction to Analytic Geometry

Analytic geometry came about with the merging of Euclid's geometry with algebra. The combination of Euclid's (323 BCE) geometry and al-Khwarizmi's (830 CE) algebra resulted in a totally new powerful tool, *analytic geometry*, independently worked out by Descartes and Fermat (Stillwell, 2010). The addition of matrix algebra during the 18th century, allow an analysis in more than 3 dimensions, which today is one of the most powerful tools used in artificial intelligence, data

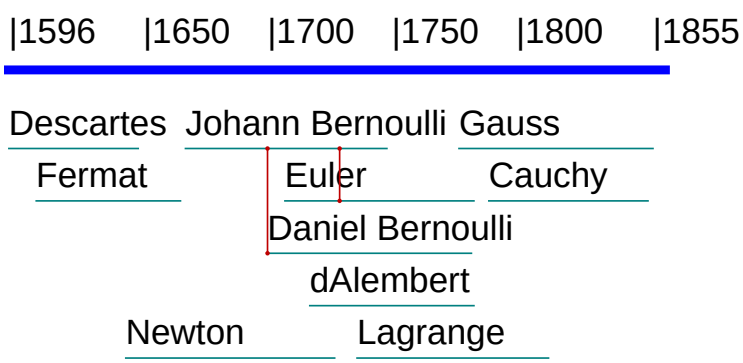

Figure 1.15: *Exploded time-line of the two centuries from 1596 to 1855CE during the development of the modern theories of analytic geometry, calculus, differential equations and linear algebra. The vertical red lines connect mentor-student relationships. Lagrange had a special role in the development of linear algebra. Note the overlap between Newton and Johann and his son Daniel Bernoulli, and Euler. Gauss had the advantage of input from Newton, Euler, d'Alembert and Lagrange. Likely Cauchy had an influence on Gauss as well.*

science and machine learning. The utility and importance of these new tools cannot be overstated. The time-line for this period is provided in Fig. 1.15

There are many important relationships between Euclidean geometry and 16th century algebra. An attempt at a detailed comparison is summarized in Table 1.2. Important similarities include vectors, their Pythagorean lengths $[a, b, c]$

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \tag{1.53}$$

$a = x_2 - x_1$ and $b = y_2 - y_1$, and the angles. Euclid's geometry had no concept of coordinates, thus of vectors. One of the main differences of analytic geometry is that one may compute with real, and soon after, complex numbers.

There are several new concepts that come with the development of analytic geometry:

1. Composition of functions: If $y = f(x)$ and $z = g(y)$ then the composition of functions f and g is denoted $z(x) = g \circ f(x) = g(f(x))$.

2. Elimination: Given two functions $f(x, y)$ and $g(x, y)$, elimination removes either x or y. This procedure, known to the Chinese, is called *Gaussian elimination*.

3. Intersection: While one may speak of the intersection of two lines to give a point, or two planes to give a line, the term *intersection* is also an important but very different concept in set theory. This is a special case of elimination when the functions $f(x, y), g(x, y)$ are linear in their arguments.

4. Vectors: Analytic geometry provides the new concept of a vector, as a line with length and orientation (i.e., direction). Analytic geometry defines vectors in any number of dimensions, as ordered sets of points.

5. Analytic geometry extends the ideas of Euclidean geometry with the introduction of the scalar (dot) product of two vectors $\boldsymbol{f} \cdot \boldsymbol{g}$, and the vector (cross) product $\boldsymbol{f} \times \boldsymbol{g}$ (see Fig. 1.16, p. 80).

What algebra also added to geometry was the ability to compute with complex numbers. For example, the length of a line (Eq. 1.53) was measured in Geometry with a compass: numbers

Table 1.2: An ad-hoc comparison between Euclidean geometry and analytic geometry. I am uncertain as to the classification of the items in the third column.

Euclidean geometry: $\mathbb{R}^3$	Analytic geometry: $\mathbb{R}^n$	Uncertain
1. Proof	1. Numbers	1. Cross product ($\mathbb{R}^3$)
2. Line length	2. Algebra	2. Recursion
3. Line intersection	3. Power series	3. Iteration $\in \mathbb{C}^2$ (e.g., Newton's method)
4. Point	4. Analytic functions	4. Iteration $\in \mathbb{R}^n$
5. Projection (e.g. scalar product)	5. Complex analytic functions: e.g., $\sin\theta, \cos\theta, e^{\theta_J}, \log z$	
6. Line direction	6. Composition	
7. Vector (sort of)	7. Elimination	
8. Conic section	8. Integration	
9. Square roots (e.g., spiral of Theodorus)	9. Derivatives	
	10. Calculus	
	11. Polynomial $\in \mathbb{C}$	
	12. Fund. thm. algebra	
	13. Normed vector spaces	
	14. ...	

played no role. Once algebra was available, the line's Euclidean length could be computed numerically, directly from the coordinates of the two ends, defined by the *3-vector*

$$\boldsymbol{e} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = [x, y, z]^T,$$

which represents a point at $(x, y, z) \in \mathbb{R}^3 \subset \mathbb{C}^3$ in three dimensions, having *direction*, from the origin $(0, 0, 0)$ to (x, y, z). An alternative *matrix notation* is $\boldsymbol{e} = [x, y, z]^T$, a column vector of three numbers. These two notations are different ways of representing exactly the same thing. I view them as equivalent notations.

By defining the vector, analytic geometry allows Euclidean geometry to become quantitative, beyond the physical drawing of an object (e.g., a sphere, triangle or line). With analytic geometry we have the Euclidean concept of a vector, a line having a magnitude (length) and direction, but analytic defined in terms of physical coordinates (i.e., numbers). The difference between two vectors defines a third vector, a concept already present in Euclidean geometry. For the first time, complex numbers were allowed into geometry (but rarely used until Cauchy and Riemann).

As shown in Fig. 1.16, there are two types of products, the 1) scalar $\boldsymbol{A} \cdot \boldsymbol{B}$ and 2) vector $\boldsymbol{A} \times \boldsymbol{B}$ products.

Scalar product of two vectors: When using algebra, many concepts, obvious with Euclid's geometry, may be made precise. There are many examples of how algebra extends Euclidean

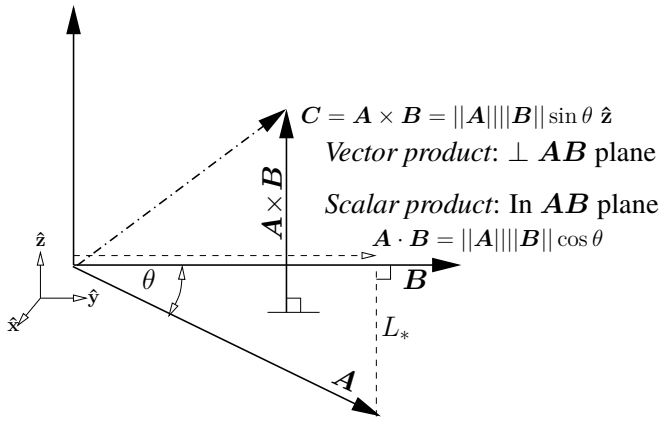

Figure 1.16: *Vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are used to define the* scalar product $\boldsymbol{A} \cdot \boldsymbol{B}$, vector (cross) product $\boldsymbol{A} \times \boldsymbol{B}$ *and* triple product $\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$. *The two vector products complement each other, one proportional to the* $\sin\theta$ *of the angle θ between them and the other to the* $\cos\theta$. *It follows that* $||\boldsymbol{A} \cdot \boldsymbol{B}||^2 + ||\boldsymbol{A} \times \boldsymbol{B}||^2 = ||\boldsymbol{A}||^2 ||\boldsymbol{B}||^2$ *The scalar product computes the projection of one vector on the other. The vector product $\boldsymbol{A} \times \boldsymbol{B}$ computes the area of the trapezoid formed by the two vectors, while the triple product $\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$ defines the volume of the formed parallelepiped (i.e., prism). When all the angles are* $90°$, *the volume becomes a* cuboid.

geometry, the most basic being the *scalar product* (aka dot product) between vectors

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{\kappa} &= (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \cdot (\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}}), \quad \in \mathbb{C} \\ &= \alpha x + \beta y + \gamma z. \end{aligned}$$

In matrix notation the scalar product is written as

$$\boldsymbol{x} \cdot \boldsymbol{\kappa} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} x, y, z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha x + \beta y + \gamma z. \tag{1.54}$$

The dot product takes the character of κ. For example, if $\kappa(s) \in \mathbb{C}$ is a function of complex frequency s, then the dot product is complex. If $\zeta \in \mathbb{R}$ is real, then the dot product is real.

Norm (length) of a vector: The *norm* of a vector

$$||\boldsymbol{e}|| \equiv +\sqrt{\boldsymbol{e} \cdot \boldsymbol{e}} \geq 0.$$

is *defined* as the positive square root of the scalar product of the vector with itself. This is a generalization of the length, in any number of dimensions, forcing the sign of the square-root to be non-negative. The length is a concept of Euclidean geometry, and it must always be positive and real. A complex (or negative) length is not physically meaningful. More generally, the Euclidean length of a line is given as the *norm* of the difference between two real vectors $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}$

$$\begin{aligned} ||\boldsymbol{e_1} - \boldsymbol{e_2}||^2 &= (\boldsymbol{e}_1 - \boldsymbol{e}_2) \cdot (\boldsymbol{e}_1 - \boldsymbol{e}_2) \\ &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \geq 0. \end{aligned}$$

From this formula we see that the norm of the difference of two vectors is simply a compact expression for the Euclidean length. A zero-length vector, such as is a point, is the result of the fact that

$$||x - x||^2 = (x - x) \cdot (x - x),$$

is zero.

Integral definition of a scalar product: Up to this point, following Euclid, we have only considered a vector to be a set of elements $\{x_n\} \in \mathbb{R}$, index over $n \in \mathbb{N}$, as defining a linear vector space with scalar product $x \cdot y$, with the scalar product defining the norm or length of the vector $||x|| = \sqrt{x \cdot x}$. Given the scalar product, the norm naturally follows.

At this point an obvious question presents itself: Can we extend our definition of vectors to differentiable functions (i.e., $f(t)$ and $g(t)$), indexed over $t \in \mathbb{R}$, with coefficients labeled by $t \in \mathbb{R}$, rather than by $n \in \mathbb{N}$? Clearly, if the functions are analytic, there is no obvious reason why this should be a problem, since analytic functions may be represented by a convergent series having Taylor coefficients, thus integrable term by term.

Specifically, under certain conditions, the function $f(t)$ may be thought of as a vector, defining a normed vector space. This intuitive and somewhat obvious idea is powerful. In this case the scalar product must be defined in terms of the integral

$$f(t) \cdot g(t) = \int_t f(t)g(t)dt \tag{1.55}$$

summed over $t \in \mathbb{R}$, rather than a sum over $n \in \mathbb{N}$.

This definition of the vector scalar product allows for a significant but straightforward generalization of our vector space, which will turn out to be both useful and an important extension of the concept of a normed vector space. In this space we can define the derivative of a norm with respect to t, which is not possible for the case of the discrete case, indexed over n. The distinction introduces the concept of continuity in the index t, which does not exist for the discrete index $n \in \mathbb{N}$.

Pythagorean theorem and the Schwarz inequality: Regarding Fig. 1.16, suppose we compute the difference between vector $\boldsymbol{A} \in \mathbb{R}$ and $\alpha\boldsymbol{B} \in \mathbb{R}$ as $L = ||\boldsymbol{A} - \alpha\boldsymbol{B}|| \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a scalar that modifies the length of $\boldsymbol{B}$. We seek the value of α, which we denote as α^*, that minimizes the length of L. From simple geometrical considerations, $L(\alpha)$ will be minimum when the difference vector is perpendicular to $\boldsymbol{B}$, as shown in the figure by the dashed line from the tip of $\boldsymbol{A} \perp \boldsymbol{B}$.

To show this algebraically we write out the expression for $L(\alpha)$ and take the derivative with respect to α, and set it to zero, which gives the formula for α^*. The argument does not change, but the algebra greatly simplifies, if we normalize $\boldsymbol{A}, \boldsymbol{B}$ to be unit vectors $\boldsymbol{a} = \boldsymbol{A}/||\boldsymbol{A}||$ and $\boldsymbol{b} = \boldsymbol{B}/||\boldsymbol{B}||$, which have norm = 1.

$$L^2 = (\boldsymbol{a} - \alpha\boldsymbol{b}) \cdot (\boldsymbol{a} - \alpha\boldsymbol{b}) = 1 - 2\alpha\boldsymbol{a} \cdot \boldsymbol{b} + \alpha^2. \tag{1.56}$$

Thus the length is shortest ($L = L_*$, as shown in Fig. 1.16) when

$$\frac{d}{d\alpha}L_*^2 = -2\boldsymbol{a} \cdot \boldsymbol{b} + 2\alpha^* = 0.$$

Solving for α^* we find $\alpha^* = \boldsymbol{a} \cdot \boldsymbol{b}$. Since $L_* > 0$ ($\boldsymbol{a} \neq \boldsymbol{b}$), Eq. 1.56 becomes

$$1 - 2|\boldsymbol{a} \cdot \boldsymbol{b}|^2 + |\boldsymbol{a} \cdot \boldsymbol{b}|^2 = 1 - |\boldsymbol{a} \cdot \boldsymbol{b}|^2 > 0.$$

In conclusion $\cos\theta \equiv |\boldsymbol{a} \cdot \boldsymbol{b}| < 1$. In terms of $\boldsymbol{A}, \boldsymbol{B}$ this is $|\boldsymbol{A} \cdot \boldsymbol{B}| < ||\boldsymbol{A}||\,||\boldsymbol{B}||\cos\theta$, as shown next to $\boldsymbol{B}$ in Fig. 1.16. Thus the scalar product between two vectors is their *direction cosine*. Furthermore since this forms a right triangle, the Pythagorean theorem must hold. The *triangle inequality* says that the lengths of the two sides must be greater than the hypotenuse. Note that $\Theta \in \mathbb{R} \notin \mathbb{C}$.

This derivation is an abbreviated version of a related discussion in Section H.2.1 (p. 85).

Vector ($\times$) product of two vectors: As shown in Fig. 1.16, the vector product (aka, cross-product) $\boldsymbol{a} \times \boldsymbol{b}$ is the second type of product between two vectors. The vector product defines a vector, perpendicular ($\perp$) to the plane of the two vectors being multiplied. The formula for computing the cross product is

$$\boldsymbol{a} \times \boldsymbol{b} = (a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}) \times (b_1\hat{\mathbf{x}} + b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

For example, if the two vectors are in $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, then the cross-product is $\hat{\mathbf{z}}$. It is strictly in $\hat{\mathbf{z}}$ if the two vectors are perpendicular to each other (i.e., $\hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}}$). The vector product of a vector with itself (or the difference between two vectors) is zero. For example, $\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$. Typically $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}$. If they are complex, the definition must be modified to be consistent with the physics.

The scalar product of a third vector $\boldsymbol{c}$ with the vector product $\boldsymbol{a} \times \boldsymbol{b}$, with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}$,

$$\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (\in \mathbb{R}^3),$$

is called the *triple product*, which represents the volume of a parallelepiped.

Impact of Analytic Geometry: The most obvious impact of analytic geometry, was a detailed analysis of the conic sections, using algebra, rather than drawings via a compass and ruler. An important example is the composition of the line and circle, a venerable construction, presumably going back to before Diophantus (250CE). Once algebra was invented, the composition could be done using formulas. With this analysis came complex numbers.

The first two mathematicians to appreciate this mixture of Euclid's geometry and the new algebra were Fermat and Descartes (Stillwell, 2010, p. 111-115); soon Newton contributed to this effort, by the addition of physics (calculations in acoustics, orbits of the planets, and the theory of gravity and light Stillwell (2010, p. 115-117)), significant concepts for 1687.

Given these new methods, many new solutions to problems emerged. The complex roots of polynomials continued to appear, without any obvious physical meaning. Complex numbers seem to have been viewed as more of an inconvenience than a problem. Newton's solution to this dilemma was to simply ignore the imaginary cases (Stillwell, 2010, p. 119).

Development of Analytic Geometry

rding needed

Intersection and Gaussian elimination: The first "algebra" (al-jabr) is credited to al-Khwarizmi (830 CE). Its invention advanced the theory of polynomial equations in one variable, Taylor series, and composition versus intersections of curves. The solution of the quadratic equation had been worked out thousands of year earlier, but with algebra a general solution could be defined. The Chinese had found the way to solve several equations in several unknowns, for example, finding the values of the intersection of two circles. With the invention of algebra by al-Khwarizmi, a powerful tool became available to solve the difficult problems.

Composition and Elimination In algebra there are two contrasting operations on functions: *composition* and *elimination*.

Composition: Composition is the merging of functions, by feeding one into the other. If the two functions are f, g then their composition is indicated by $f \circ g$, meaning the function $y = f(x)$ is substituted into the function $z = g(y)$, giving $z = g(f(x))$.

Composition is not limited to linear equations, even though that is where it is most frequently applied. To compose two functions, one must substitute one equation into the other. That requires solving for that substitution variable, which is not always possible in the case of nonlinear equations. However many tricks are available that may work around this restrictions. For example, if one equation is in x^2 and the other in x^3 or $\sqrt{x}$, it may be possible to multiply the first by x or square the second. The point is that one of the variables must be isolated so that when it is substituted into the other equations, the variable is removed from the mix.

Examples: Let $y = f(x) = x^2 - 2$ and $z = g(y) = y + 1$. Then

$$g \circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1. \tag{1.57}$$

In general composition does not commute (i.e., $f \circ g \neq g \circ f$), as is easily demonstrated. Swapping the order of composition for our example gives

$$f \circ g = f(g(y)) = z^2 - 2 = (y + 1)^2 - 2 = y^2 + 2y - 1. \tag{1.58}$$

Intersection: Complimentary to composition is *intersection* (i.e., decomposition) (Stillwell, 2010, pp. 119,149). For example, the intersection of two lines is defined as the point where they meet. Not sure ref is right. This is not to be confused with finding roots. A polynomial of degree N has N roots, but the points where two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a function (equation) of lower degree, implemented with Gaussian elimination.

Intersection of two lines Unless they are parallel, two lines meet at a point. In terms of linear algebra this may be written as 2 linear equations[57] (on the left), along with the intersection point $[x_1, x_2]^T$, given by the inverse of the 2x2 set of equations (on the right)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{1.59}$$

By substituting the expression for the intersection point $[\boldsymbol{x}_1, \boldsymbol{x}_2]^T$ into the original equation, we see that it satisfies the equations. Thus the equation on the right is the solution to the equation on the left.

Note the structure of the inverse: 1) The diagonal values (a, d) are swapped, 2) the off-diagonal values (b, c) are negated and 3) the 2x2 matrix is divided by the determinant $\Delta = ad - bc$. If $\Delta = 0$, there is no solution. When the determinant is zero ($\Delta = 0$), the slopes of the two lines

$$\textit{slope} = \frac{dx_2}{dx_1} = \frac{b}{a} = \frac{d}{c}$$

are equal, thus the lines are parallel. Only if the slopes differ can there be a unique solution.

[57] When writing the equation $Ax = y$ in matrix format, the two equations are $ax_1 + bx_2 = y_1$ and $dx_1 + ex_2 = y_2$ with unknowns (x_1, x_2), whereas in the original equations $ay + bx = c$ and $dy + ex = f$, they were y, x. Thus in matrix format, the names are changed. The first time you see this scrambling of variables, it can be confusing.

Exercise: Show that the equation on the right is the solution of the equation on the left. **Solution:** By a direct substitution of the right equation into the left equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, = \frac{1}{\Delta} \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -cb+ad \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix},$$

which gives the identity matrix.

Algebra will give the solution when geometry cannot. When the two curves fail to intersect on the real plane, the solution still exists, but is complex valued. In such cases, geometry, which only considers the real solutions, fails. For example, when the coefficients $[a, b, c, d]$ are complex, the solution exists, but the determinant can be complex. Thus algebra is much more general than geometry. Geometry fails when the solution has a complex intersection.

A system of linear $Ax = y$ equations has many interpretations, and one should not be biased by the notation. As engineers we are trained to view x as the input and y as the output, in which case then $y = Ax$ seems natural, much like the functional relation $y = f(x)$. But what does the linear relation $x = Ay$ mean, when x is the input? The obvious answer is that $y = A^{-1}x$. But when working with systems of equations, there are many uses of equations, and we need to become more flexible in our interpretation. For example $y = A^2x$ has a useful meaning, and in fact we saw this type of relationship when working with Pell's equation (p. 50) and the Fibonacci sequence (p. 53). As another example consider

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_{1x} & a_{1y} \\ a_{2x} & a_{2y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which is reminiscent of a three-dimensional surface $\mathbf{z} = \mathbf{f}(x, y)$. We shall find that such generalizations are much more than a curiosity.

1.3.7 Scalar products and its applications

ps move after Gaussian

4a

Another important example of algebraic expressions in mathematics is Hilbert's generalization of Eq. 1.1, known as the Schwarz inequality, shown in Fig. 1.17. What is special about this generalization is that it proves that when the vertex is 90°, the length of the leg is minimum.

Vectors may be generalize to have ∞ dimensions: $\vec{U}, \vec{V} = [v_1, v_2, \cdots, v_\infty]$). The inner product (i.e., dot product) between two such vectors generalizes the finite dimensional case

$$\vec{U} \cdot \vec{V} = \sum_{k=1}^{\infty} u_k v_k.$$

As with the finite case, the *norm* $||\vec{U}|| = \sqrt{\vec{U} \cdot \vec{U}} = \sqrt{\sum u_k^2}$ is the dot product of the vector with itself, defining the *length* of the infinite component vector. Obviously there is an issue of convergence, if the norm for the vector is to have a finite length.

It is a somewhat arbitrary requirement that $a, b, c \in \mathbb{R}$ for the Pythagorean theorem (Eq. 1.1). This seems natural enough since the sides are lengths. But, what if they are taken from the complex numbers, as for the lossy vector wave equation, or the lengths of vectors in $\mathbb{C}^n$? Then the equation generalizes to

$$\mathbf{c} \cdot \mathbf{c} = ||\mathbf{c}||^2 = \sum_{k=1}^{n} |c_k|^2,$$

where $||\mathbf{c}||^2 = (\mathbf{c}, \mathbf{c})$ is the inner (dot) product of a vector $\mathbf{c}$ with itself, where $|c_k|$ is the magnitude the complex c_k. As before, $||\mathbf{c}|| = \sqrt{||\mathbf{c}||^2}$ is the *norm* of vector $\mathbf{c}$, akin to a length.

Triangle inequality proof

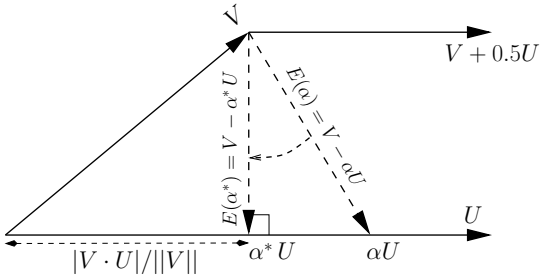

Figure 1.17: *The* Schwarz inequality *is related to the shortest distance (length of a line) between the ends of the two vectors.* $||U|| = \sqrt{U \cdot U}$ *as the dot product of that vector with itself.*

Schwarz inequality The *Schwarz inequality*[58] says that the magnitude of the inner product of two vectors is less than or equal to the product of their lengths

$$|U \cdot V| \leq ||U||\,||V||.$$

This may be simplified by normalizing the vectors to have unit length ($\widehat{U} = U/||U||, \widehat{V} = V/||V||$), in which case $-1 < \widehat{U} \cdot \widehat{V} \leq 1$. Another simplification is to define the scalar product in terms of the direction cosine

$$\cos\theta = |\widehat{U} \cdot \widehat{V}| \leq 1.$$

A proof of the Schwarz inequality is as follows: From these definitions we may define the minimum difference between the two vectors as the perpendicular from the end of one to the intersection of the second. As shown in Fig. 1.17, $U \perp V$ may be found by minimizing the length of the vector difference:

$$\begin{aligned}
\min_{\alpha} ||V - \alpha U||^2 = ||V||^2 + 2\alpha V \cdot U + \alpha^2 ||U||^2 > 0 \\
0 = \partial_\alpha\,(V - \alpha U) \cdot (V - \alpha U) \\
= V \cdot U - \alpha^* ||U||^2 \\
\therefore \quad \alpha^* = V \cdot U / ||U||^2.
\end{aligned}$$

The *Schwarz inequality* follows:

$$I_{\min} = ||V - \alpha^* U||^2 = ||V||^2 - \frac{|U \cdot V|^2}{||U||^2} > 0$$

$$0 \leq |U \cdot V| \leq ||U||\,||V||$$

An important example of such a vector space includes the definition of the *Fourier transform*, where we may set

$$U(\omega) = e^{-\omega_0 \jmath t} \qquad V(\omega) = e^{\omega \jmath t} \qquad U \cdot V = \int_\omega e^{\jmath \omega t} e^{-\jmath \omega_0 t} \frac{d\omega}{2\pi} = \delta(\omega - \omega_0).$$

It seems that the Fourier transform is a result that follows from a minimization, unlike the Laplace transform that follows from causal system parameters. This explains the important differences between the two, in terms of their properties (unlike the LT, the FT is not complex analytic). We also explored this topic in Lecture 1.3.13 (p. 97).

[58] A simplified derivation is provided in Sect. 1.3.6 (p. 77).

Power vs. power series, linear vs. nonlinear

Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage $v(t)$ and current $i(t)$ (or in mechanics as the force times the velocity). For example if we define $\mathcal{P} = v(t)i(t)$ to be the power $\mathcal{P}$ [watts], then the *total energy* [joules] at time t is (Van Valkenburg, 1964a, Chapter 14)

$$\mathcal{E}(t) = \int_0^t v(t)i(t)dt.$$

From this observe that the power is the rate of change of the total energy

$$\mathcal{P}(t) = \frac{d}{dt}\mathcal{E}(t),$$

reminiscent of the fundamental theorem of calculus [Eq. 1.86, (p. 106)].

Ohm's law and impedance: The ratio of voltage over the current is called the *impedance* which has units of [ohms]. For example given a resistor of $R = 10$ [ohms],

$$v(t) = R\, i(t).$$

Namely 1 amp flowing through the resistor would give 10 volts across it. Merging the linear relation due to Ohm's law with the definition of power, shows that the instantaneous power in a resistor is quadratic in voltage and current

$$\mathcal{P}(t) = v(t)^2/R = i(t)^2 R. \tag{1.60}$$

Note that Ohm's law is linear in its relation between voltage and current whereas the power and energy are *nonlinear*.

Ohm's law generalizes in a very important way, allowing the impedance (e.g., resistance) to be a linear complex analytic function of complex frequency $s = \sigma + \omega\jmath$ (Kennelly, 1893; Brune, 1931a). Impedance is a fundamental concept in many fields of engineering. For example:[59] Newton's second law $F = ma$ obeys Ohm's law, with mechanical impedance $Z(s) = sm$. Hooke's law $F = kx$ for a spring is described by a mechanical impedance $Z(s) = k/s$. In mechanics a "resistor" is called a *dashpot* and its impedance is a positive and real constant.[60]

s a vector, pressure and volt-
a potentials.

Kirchhoff's laws KCL, KVL: The laws of electricity and mechanics may be written down using Kirchhoff's laws current and voltage laws (KCL, KVL), which lead to linear systems of equations in the currents and voltages (velocities and forces) of the system under study, with complex coefficients having positive real parts.

Points of major confusion are a number of terms that are misused, and overused, in the fields of mathematics, physics and engineering. Some of the most obviously abused terms are *linear/nonlinear*, *energy*, *power*, *power series*. These have multiple meanings, which can, and are, fundamentally in conflict.

[59] In acoustics the pressure is a potential, like voltage. The force per unit area is given by $f = -\nabla p$ thus $F = -\int \nabla p \; dS$. Velocity is analogous to a current. In terms of the velocity potential, the velocity per unit area is $v = -\nabla\phi$.

[60] `https://en.wikipedia.org/wiki/Impedance_analogy`

Transfer functions (transfer matrix): The only method that seems to work, to sort this out, is to cite the relevant physical application, in specific contexts. The most common standard reference is a physical system that has an input $x(t)$ and an output $y(t)$. If the system is linear, then it may be represented by its *impulse response* $h(t)$. In such cases the system equation is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s)|_{\sigma=0} X(\omega);$$

namely, the convolution of the input with the impulse response gives the output. From Fourier analysis this relation may be written in the real frequency domain as a product of the Laplace transform of the impulse response, evaluated on the $\omega\jmath$ axis and the Fourier transform of the input $X(\omega) \leftrightarrow x(t)$ and output $Y(\omega) \leftrightarrow y(t)$.

If the system is nonlinear, then the output is not given by a convolution, and the Fourier and Laplace transforms have no obvious meaning.

The question that must be addressed is why is the power said to be nonlinear whereas a power series of $H(s)$ is said to be linear: Both have powers of the underlying variables. This is massively confusing, and must be addressed. The question will be further addressed in Section H.5.1 in terms of the system postulates of physical systems.

Whats going on? The domain variables must be separated from the codomain variables. In our example, the voltage and current are multiplied together, resulting in a nonlinear output, the power. If the frequency is squared, this is describing the degree of a polynomial. This is not nonlinear because it does not impact the signal output; it characterizes the Laplace transform of the system response.

Perhaps move Lec 36 (p. here, or before Lec 16?

1.3.8 Lec 15 Gaussian Elimination

Perhaps move 2 p.87?

The method for finding the intersection of equations is based on the recursive elimination of all the variables but one. This method, known as *Gaussian elimination*, works across a broad range of cases, but may be defined as a systematic algorithm when the equations are linear in the variables.[61] Rarely do we even attempt to solve problems in several variables of degree greater than 1. But Gaussian eliminations may still work in such cases (Stillwell, 2010, p. 90).

In Appendix B.3 (p. 194) the inverse of a 2x2 linear system of equations is derived. Even for a 2x2 case, the general solution requires a great deal of algebra. Working out a numeric example of Gaussian elimination is more instructive. For example, suppose we wish to find the intersection of the two equations

$$\begin{aligned} x - y &= 3 \\ 2x + y &= 2. \end{aligned}$$

This 2x2 system of equations is so simple that you may immediately visualize the solution: By adding the two equations, y is eliminated, leaving $3x = 5$. But doing it this way takes advantage of the specific example, and we need a method for larger systems of equations. We need a generalized (algorithmic) approach. This general approach is called *Gaussian elimination*.

Start by writing the equations in *matrix* format (note this is not in the standard form $Ax = y$)

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \tag{1.61}$$

[61] https://en.wikipedia.org/wiki/System_of_linear_equations.

Next, eliminate the lower left term ($2x$) using a scaled version of the upper left term (x). Specifically, multiply the first equation by -2, add it to the second equation, replacing the second equation with the result. This gives

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 - 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}. \tag{1.62}$$

Note that the top equation did not change. Once the matrix is "upper triangular" (zero below the diagonal) you have the solution. Starting from the bottom equation, $y = -4/3$. Then the upper equation gives $x - (-4/3) = 3$, or $x = 3 - 4/3 = 5/3$.

In principle, Gaussian elimination is easy, but if you make a calculation mistake along the way, it is very difficult to find the error. The method requires a lot of mental labor, with a high probability of making a mistake. Thus you do not want to apply this method every time. For example, suppose the elements are complex numbers, or polynomials in some other variable such as frequency. Once the coefficients become more complicated, the seemingly trivial problem becomes highly error prone. There is a much better way, that is easily verified, which puts all the numerics at the end, in a single step.

The above operations may be automated by finding a carefully chosen upper-diagonalization matrix U that does the same operation. For example, let

$$U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{1.63}$$

Multiplying Eq. 1.61 by U, we find

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \tag{1.64}$$

we obtain Eq. 1.62. At this point we can either back-substitute and obtain the solution, as we did above, or find a matrix L that finishes the job, by removing elements above the diagonal.

Exercise: Show that $det(U) = det(UA) = 1$. **Solution:** Since

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc,$$

and $det(UA) = detU \cdot det(A)$, and $det(U) = 1 - 0 = 1$, the result follows naturally. Common notation is to let $det(A) = |A|$.

In Appendix B.3 the inverse of a general 2x2 matrix is summarized in terms of three steps: 1) swap the diagonal elements, 2) reverse the signs of the off-diagonal elements and 3) divide by the determinant $\Delta = ab - cd$. Specifically

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.65}$$

There are very few things that you must memorize, but the inverse of a 2x2 is one of them. It needs to be in your tool-bag, like the quadratic formula.

While it is difficult to compute the inverse matrix from scratch (Appendix **??**), it takes only a few seconds to verify it (steps 1 and 2)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}. \tag{1.66}$$

Thus dividing by the determinant gives the 2x2 identity matrix. A good strategy, when you don't trust your memory, is to write down the inverse as best you can, and then verify.

Using the 2x2 matrix inverse on our example (Eq. 1.61), we find

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1+2}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ -6+2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -4/3 \end{bmatrix}. \tag{1.67}$$

If you use this method, you will rarely (never) make a mistake, and the solution is easily verified. Either you can check the numbers in the inverse, as was done in Eq. 1.66, or you can substitute the solution back into the original equation.

1.3.9 Lec 16: Transmission (ABCD) matrix composition method

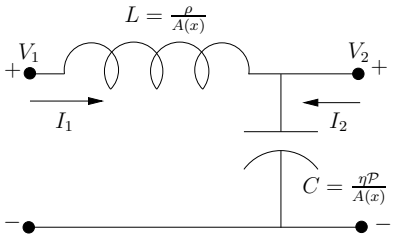

Figure 1.18: *A a single LC cell of the LC transmission line (see Fig. G.2 (p. 231). Every cell of any transmission line may be modeled by the ABCD method, as the product of two matrices. For the example shown here, the inductance L of the coil and the capacitance C of capacitor are in units of [henry/m] and [farad/m], thus they depend on length Δ_x [m] that the cell represents. Note the flows are always defined as into the + node.*

Matrix composition: Matrix multiplication represents a composition of 2x2 matrices, because the input to the second matrix is the output of the first (this follows from the definition of composition: $f(x) \circ g(x) = f(g(x))$). Thus the ABCD matrix is also known as the *transmission* matrix method, or occasionally the *chain matrix*. The general expression for an transmission matrix $\boldsymbol{T}(s)$ is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix}\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{1.68}$$

The four coefficients $\mathcal{A}(s), \mathcal{B}(s), \mathcal{C}(s), \mathcal{D}(s)$ are all complex functions of the Laplace frequency $s = \sigma + j\omega$ (p. 1.4.3). The derivation is repeated with more detail in Section H.3.2 (p. 252).

It is a standard convention to always define the current into the node, but since the input current (on the left) is the same as the output current on the right (I_2), hence the negative sign on I_2, to meet the convention of current into every node. When transmission matrices are cascaded, all the signs then match.

We have already used 2x2 matrix composition in representing complex numbers (p. 32), and for computing the $gcd(m, n)$ of $m, n \in \mathbb{N}$ (p. 43), Pell's equation (p. 50) and the Fibonacci sequence (p. 53).

Definitions of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$: The definitions of the four functions of Eq. 1.68 are easily read off of the equation, as

$$\mathcal{A}(s) = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \quad \mathcal{B}(s) = -\left.\frac{V_1}{I_2}\right|_{V_2=0}, \quad \mathcal{C}(s) = \left.\frac{I_1}{V_2}\right|_{I_2=0}, \quad \mathcal{D}(s) = -\left.\frac{I_1}{I_2}\right|_{V_2=0}. \tag{1.69}$$

These definitions follow trivially from Eq. 1.68. These relations have general names. $\mathcal{A}, \mathcal{D}$ are called voltage and current *transfer functions*, since they are the ratio of an output over an input, whereas $\mathcal{B}, \mathcal{C}$ are known as the *transfer impedance* and *transfer conductance*. Each expresses an output (port 2) in terms of an input at port 1.

Exercise: Explain why $\mathcal{C}$ is given as above. **Solution:** Writing out the lower equation $I_1 = \mathcal{C}V_2 - \mathcal{D}I_2$. Setting $I_2 = 0$, we obtain the equation for $\mathcal{C}$.

Thévenin parameters of a source: A very important concept in circuit theory is that of the *Thévenin parameters* (p. 89), the *open-circuit voltage* and the *short-circuit current*, the ratio of which define the *Thévenin impedance*. The open circuit voltage is defined as the voltage when the current is zero. In mechanics this is called the *isometric force*, the maximum force given zero velocity.

It trivially follows that the Thévenin voltage (open circuit voltage) is

$$V_{\text{Thév}} = \frac{1}{C} \equiv \left.\frac{V_2}{I_1}\right|_{I_2=0}.$$

For example, the voltage measured across a battery having no load is its Thévenin voltage.

We may find the impedance $Z_1 = \left.\frac{V_1}{I_1}\right|_{I_2=0}$ that would be seen looking into port 1, with nothing connected to port 2 ($I_2 = 0$) by taking the ratio of the upper and lower equations

$$Z_1 = \left.\frac{\mathcal{A}V_2 - \mathcal{B}I_2}{\mathcal{C}V_2 - \mathcal{D}I_2}\right|_{I_2=0} = \frac{\mathcal{A}}{\mathcal{C}} = \mathcal{A}V_{\text{Thév}}.$$

Using this relation we may find a second expression for the Thévenin voltage in terms of the source voltage V_1 rather than in terms of the source current.

$$V_{\text{Thev}} = \frac{Z_1}{\mathcal{A}} \equiv Z_1 \left.\frac{V_2}{V_1}\right|_{I_2=0}.$$

The choice of the appropriate expression depends on the physics of the actual source.

Exercise: Show that the Thévenin source impedance is

$$\mathbb{Z}_{\text{Thév}} = \frac{b}{C}.$$

Hint: Use the fact that $V_1 = -\mathcal{B}I_2|_{V_2=0}$. **Solution:** MISSING SOL

Properties of the transmission matrix: The transmission matrix is always constructed from the product of elemental matrices of the form

$$\begin{bmatrix} 1 & Z(s) \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ Y(s) & 1 \end{bmatrix}.$$

Thus for the case of reciprocal systems (P6, p. 101)

$$\det \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix} = 1,$$

since the determinant of each of these matrices is 1, the determinant, the product of many elemental matrices being 1, must be one. An anti-reciprocal system may be synthesized by the use of a gyrator, and in these cases $\Delta_T = -1$.

1.3.10 The impedance matrix

With a bit of algebra, one may find the impedance matrix in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ (Van Valkenburg, 1964a, p. 310)

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_T \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{1.70}$$

For reciprocal systems (P6, p. 101) $z_{12} = z_{21}$ since $\Delta_T = 1$. For anti-reciprocal systems, such as dynamic loudspeakers and microphones (Kim and Allen, 2013), $z_{21} = -z_{12} = 1/\mathcal{C}$, since $\Delta_T = -1$.

Impedance is a very general concept, closely tied to the definition of power $\mathcal{P}(t)$ (and energy). Power is defined as the product of the effort (force) and the flow (current). As described in Fig. 1.3, these concepts are very general, applying to mechanics, electrical circuits, acoustics, thermal circuits, or any other case where conservation of energy applies. Two basic variables are defined, *generalized force* and *generalized flow*, also called *conjugate variables*. The product of the conjugate variables is the power, and the ratio is the impedance. For example, for the case of voltage and current,

$$\mathcal{P}(t) \equiv \int^t v(t)i(t)dt, \qquad Z(s) \equiv \frac{V(\omega)}{I(\omega)}.$$

Ohm's law In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage is the 'force' and the current is the 'flow.' Ohm's law states that the voltage across and the current through a circuit element are related by the *impedance* of that element (which is typically a function of the Laplace frequency $s = \sigma + \omega\jmath$). For resistors, the voltage over the current is called the *resistance*, and is a constant (e.g. the simplest case, $V/I = R$). For inductors and capacitors, the impedance depends on the Laplace frequency s [e.g. $V/I = Z(s)$]).

Table 1.3: *Impedance is defined as the ratio of a force over a flow, a concept that also holds in mechanics and acoustics. In mechanics, the 'force' is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the 'flow' is the velocity. In acoustics, the 'force' is pressure, and the 'flow' is the volume velocity or particle velocity of air molecules.*

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I)	$Z = V/I$	ohms [Ω]
Mechanics	force (F)	velocity (U)	$Z = F/U$	mechanical ohms [Ω]
Acoustics	pressure (P)	particle velocity (V)	$Z = P/V$	acoustic ohms [Ω]
Thermal	temperature (T)	heat-flux (J)	$Z = T/J$	thermal ohms [Ω]

As discussed in Fig. 1.3, the impedance concept also holds for mechanics and acoustics. In mechanics, the 'force' is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the 'flow' is the velocity. In acoustics, the 'force' is pressure, and the 'flow' is the volume velocity or particle velocity of air molecules.

In this section we shall derive the method of linear composition of systems, also known as the *ABCD transmission matrix method*, or in the mathematical literature as the Möbius (bilinear) transformation. Using the method of *matrix composition*, a linear system of 2x2 matrices can represent a large and important family of networks. By the application of Ohm's law to the circuit shown in Fig. 1.18, we can model a cascade of such cells.

Example of the use of the ABCD matrix composition: In Fig. 1.18 we see the network is composed of a series inductor (mass) having an impedance $Z_l = sL$, and a shunt capacitor (compliance) having an impedance $Z_c = 1/sC$, where $s \in \mathbb{C}$. As determined by Ohm's law, each impedance is describe by a linear relation between the current and the voltage. For the inductive impedance, applying Ohm's law, we find

$$V_1 - V_2 = Z_l I_1$$

where Z_l is the complex impedance of the inductor. For the capacitive impedance, applying Ohm's law we find

$$V_2 = (I_1 + I_2)Z_c,$$

where Z_c is the complex impedance of the capacitor.

Each of these linear impedance relations may be written in matrix form. The series inductor equation gives (note $I_1 = -I_2$)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \tag{1.71}$$

while the shunt capacitor equation yields (note $V_1 = V_2$)

$$\begin{bmatrix} V_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \tag{1.72}$$

where $Y_c = 1/Z_c$ is called the *admittance*.

When the second matrix equation for the shunt admittance (Eq. 1.72) is substituted into the series impedance equation (Eq. 1.71), we find the ABCD matrix ($\boldsymbol{T}_1 \circ \boldsymbol{T}_2$), for the cell is simply the product of two matrices

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 + Z_l Y_c & Z_l \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{1.73}$$

Note that the determinant of the matrix $\Delta = \mathcal{AD} - \mathcal{BC} = 1$.

Thus $\mathcal{A}(s) = 1 + Z_L u_c = 1 + s^2 LC$, $\mathcal{B}(s) = Z_l$, $\mathcal{C}(s) = Y_c$ and $\mathcal{D} = 1$. This equation characterizes every possible relation between the input and output voltage and current of the cell.

For example, the ratio of the output to input voltage with the output unloaded ($I_2 = 0$), known as the *voltage divider relation* may be found from the upper equation with $I_2 = 0$. Writing this out gives

$$\left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{1}{\mathcal{A}(s)} = \frac{1}{1 + Z_l Y_c} = \frac{Z_c}{Z_c + Z_l}.$$

To derive the formula for the current divider equation, use the lower equation and set $V_2 = 0$.

$$\left.\frac{-I_2}{I_1}\right|_{V_2=0} = \frac{1}{\mathcal{D}} = 1.$$

1.3.11 Lec 17: Riemann Sphere: 3^d extension of chord and tangent method

Once algebra was formulated by c830 CE, mathematicians were able to expand beyond the limits set by geometry on the real plane, and the verbose descriptions of each problem in prose (Stillwell, 2010, p. 93). The geometry of Euclid's Elements had paved the way, but after 2000 years, the

addition of the language of algebra changed everything. The analytic function was a key development, heavily used by both Newton and Euler. Also the investigations of Cauchy made important headway with his work on complex variables. Of special note was integration and differentiation in the complex plane of complex analytic functions, which is the topic of stream 3.

It was Riemann, working with Gauss in the final years of Gauss's life, who made the breakthrough, with the concept of the *extended complex plane*.[62] This concept was based on the composition of a line with the sphere, similar to the derivation of Euclid's formula for Pythagorean triplets (Fig. G.3, p. 233). While the importance of the extended complex plane was unforeseen, it changed analytic mathematics forever, along with the physics it supported. It unified and thus simplified many important integrals, to the extreme. This idea is captured by the *fundamental theorem of complex integral calculus* (Table 1.6 p. 159) and 1.4, p. 103.

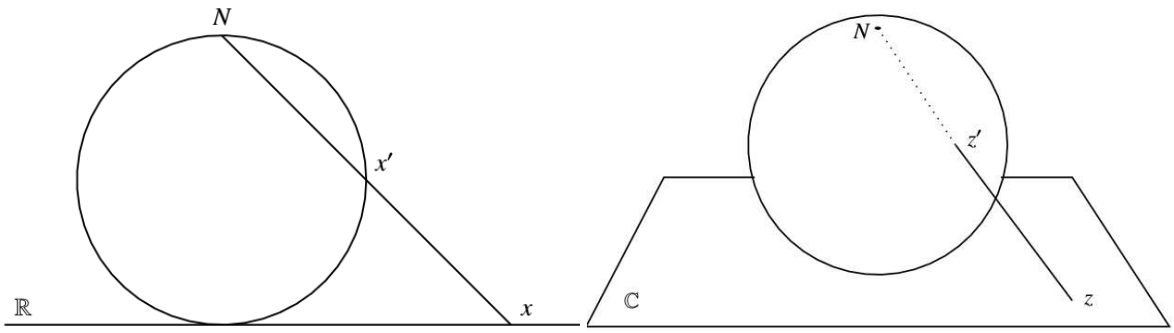

Figure 1.19: *The left panel shows how the real line may be composed with the circle. Each real x value maps to a corresponding point x' on the unit circle. The point $x \to \infty$ maps to the north pole N. This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane $z \to \infty$ maps onto the north pole N. This construction is important because while the plane is open (does not include $z \to \infty$), the sphere is analytic at the north pole. Thus the sphere defines the* closed extended plane. *Figure from Stillwell (2010, pp. 299-300).*

The idea is outlined in Fig. 1.19. On the left is a circle and a line. The difference between this case and the derivation of the Pythagorean triplets is that the line starts at the north pole, and ends on the real $x \in \mathbb{R}$ axis at point x. At point x', the line cuts through the circle. Thus the mapping from x to x' takes every point on $\mathbb{R}$ to a point on the circle. For example, the point $x = 0$ maps to the south pole (not indicated). To express x' in terms of x one must compose the line and the circle, similar to the composition used in Fig. G.3 (p. 233). The points on the circle, indicated here by x', require a traditional polar coordinate system, having a unit radius and an angle defined between the radius and a vertical line passing through the north pole. When $x \to \infty$ the point $x' \to N$, known as the *point at infinity*. But this idea goes much further, as shown on the right half of Fig. 1.19.

Here the real tangent line is replaced by a tangent complex plane $z \in \mathbb{C}$, and the complex puncture point $z' \in \mathbb{C}$, in this case on the complex sphere, called the *extended complex plane*. This is a natural extension of the chord/tangent method on the left, but with significant consequences. The main difference between the complex plane z and the extended complex plane, other than the coordinate system, is what happens at the north pole. The point at $|z| = \infty$ is not defined on the plane, whereas on the sphere, the point at the north pole is simply another point, like every other point on the sphere.

[62] "Gauss did lecture to Riemann but he was only giving elementary courses and there is no evidence that at this time he recognized Riemann's genius." `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Riemann.html` "In 1849 he [Riemann] returned to Göttingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851." `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Riemann.html`

Open vs. closed sets: Mathematically the plane is said to be an *open set*, since the limit $z \to \infty$ is not defined, whereas on the sphere, the point z' is a member of a *closed set*, since the north pole *is* defined. The distinction between an open and closed set is important, because the closed set allows the function to be complex-analytic at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

The z plane may be replaced with another plane, say the $w = F(z) \in \mathbb{C}$ plane, where w is some function F of $z \in \mathbb{C}$. For the moment we shall limit ourselves to *complex analytic functions* of z, namely $w = F(z) = u(x,y) + v(x,y)\jmath = \sum_{n=0}^{\infty} c_n z^n$.

In summary, given a point $z = x + y\jmath$ on the open complex plane, we map it to $w = F(z) \in \mathbb{C}$, the complex $w = u + v\jmath$ plane, and from there to the closed extended complex plane $w'(z)$. The point of doing this is that it allows us to allow the function $w'(z)$ to be analytic at the north pole, meaning it can have a convergent Taylor series at the point at infinity $z \to \infty$. Since we have not yet defined $dw(z)/dz$, the concept of a complex Taylor series remains undefined.

Bilinear transformation

In mathematics the *bilinear transformation* has special importance because it is linear in its action on both the input and output variables. Since we are engineers we shall stick with the engineering terminology. But if you wish to read about this on the internet, be sure to also search for the mathematical term, Möbius transformation.

When a point on the complex plane $z = x + y\jmath$ is composed with the bilinear transform ($a, b, c, d \in \mathbb{C}$), the result is $w(z) = u(x,y) + v(x,y)\jmath$

$$w = \frac{az + b}{cz + d}. \tag{1.74}$$

The transformation from $z \to w$ is a cascade of four independent compositions:

1. translation ($w = z + b$: $a = 1, b \in \mathbb{C}, c = 0, d = 1$),
2. scaling ($w = |a|z$: $a \in \mathbb{R}, b = 0, c = 0, d = 1$),
3. rotation ($w = \frac{a}{|a|}z$: $a \in \mathbb{C}, b = 0, c = 0, d = |a|$) and
4. inversion ($w = \frac{1}{z}$: $a = 0, b = 1, c = 1, d = 0$).

Each of these transformations is a special case of Eq. 1.74, with the inversion the most complicated. A wonderful video showing the effect of the bilinear (Möbius) transformation on the plane is available that I highly recommend: https://www.youtube.com/watch?v=0z1fIsUNhO4

The bilinear transformation is the most general way to move the expansion point in a complex analytic expansion. For example, starting from the harmonic series, the bilinear transform gives

$$\frac{1}{1-w} = \frac{1}{1 - \frac{az+b}{cz+d}} = \frac{cz + d}{(c-a)z + (d-b)}.$$

The RoC is transformed from $|w| < 1$ to $|(az-b)/(cz-d)| < 1$. An interesting application might be in moving the expansion point until it is on top of the nearest pole, so that the RoC goes to zero. This might be a useful way of finding a pole, for example.

not yet defined.

When the extended plane (Riemann sphere) is analytic at $z = \infty$, one may take the derivatives

there, defining the Taylor series with the expansion point at ∞. When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing the analytic nature at infinity. A second way to transform the point at infinity is by the bilinear transformation $\zeta = 1/z$, mapping a zero (or pole) at $z = \infty$ to a pole (or zero) at $\zeta = 0$. Thus this construction of the Riemann sphere and the Mbious (bilinear) transformation allows us to understand the point at infinity, and treat it like any other point. If you felt that you never understood the meaning of the point at ∞ (likely), this should help.

1.3.12 Lec 18: Complex analytic mappings (Domain-coloring)

Move AE3 #2 discussion h

One of the most difficult aspects of complex functions of a complex variable is understanding the mapping from $z = x + y\jmath$ to $w(z) = u + v\jmath$. For example, $w(z) = \sin(x)$ is trivial when $z = x + y\jmath$ is real (i.e., $y = 0$), because $\sin(x)$ is real. Likewise for the case where $x = 0$, where

$$\sin(y\jmath) = \frac{e^{-y} - e^{y}}{2\jmath} = \jmath \sinh(y)$$

is purely imaginary. But the general case, $w(z) = \sin(z) \in \mathbb{C}$

$$\sin(z\jmath) = \sin(x\jmath - y) = \jmath \sinh(z),$$

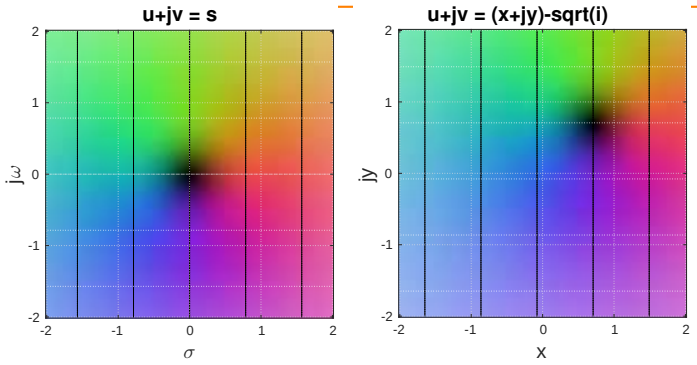

Figure 1.20: *Left: Domain–colorized map showing the complex mapping from the $s = \sigma + \omega\jmath$ plane to the $w(s) = u(\sigma, \omega) + v(\sigma, \omega)\jmath$ plane. This mapping may be visualized by the use of intensity (light/-dark) to indicate magnitude, and color (hue) to indicate angle (phase) of the mapping. Right: This shows the $w(z) = z - \sqrt{\jmath}$ plane, shift to the right and up by $\sqrt{2}/2 = 0.707$. The white and black lines are the iso-real and iso-imaginary contours of the mapping.*

is not easily visualized. Thus when $u(x, y)$ and $v(x, y)$ are not well known functions, $w(z)$ can be much more difficult to visualize.

Fortunately with computer software today, this problem can be solved by adding color to the chart. A Matlab/Octave script `zviz.m` has been used to make the make the charts shown here.[63] This tool is also known as *domain-coloring*.[64] Rather than plotting $u(x, y)$ and $v(x, y)$ separately, domain-coloring allows us to display the entire function on one chart. Note that for this visualization we see the complex polar form of $w(s) = |w|e^{\jmath \angle w}$, rather than the four-dimensional Cartesian graph $w(x + y\jmath) = u(x, y) + v(x, y)\jmath$.

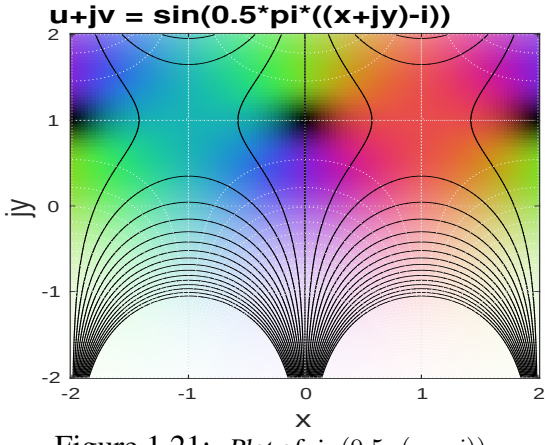

Figure 1.21: *Plot of* $\sin(0.5\pi(z - i))$.

Visualizing complex functions: The mapping from $z = x + iy$ to $w(z) = u(x, y) + iv(x, y)$ is a $2 \cdot 2 = 4$ dimensional graph. This is difficult to visualize, because for each point in the domain z, we would like to represent both the magnitude and phase (or real and imaginary parts) of $w(z)$. A good way to visualize these mappings is to use color (hue) to represent the phase and intensity

[63] http://jontalle.web.engr.illinois.edu/uploads/298/zviz.zip
[64] This is also called 'domain coloring': https://en.wikipedia.org/wiki/Domain_coloring

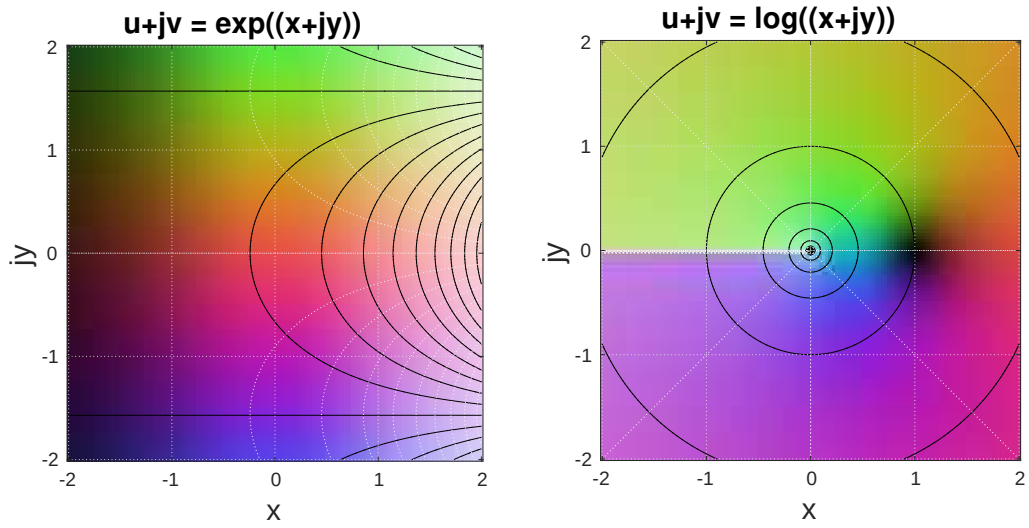

Figure 1.22: *This domain–color map allows one to visualize complex mappings by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase). The white and black lines are the iso-real and iso-imaginary contours of the mapping. Left: This figure shows the domain–color map for the complex mapping from the* $z = x + y\jmath$ *plane to the* $w(z) = u + v\jmath = e^{x+y\jmath}$ *plane, which goes to zero as* $x \to -\infty$*, causing the domain–color map to become dark for* $x < -2$*. The white and black lines are always perpendicular because* e^z *is complex analytic everywhere. Right: This shows the principal value of the inverse function* $u(x,y) + v((x,y)\jmath = \log(x + y\jmath)$*, which has a zero (dark) at* $z = 1$*, since there* $\log(1) = 0$ *(the imaginary part is zero). Note the branch cut from* $x = 0$ *to* $x = -\infty\jmath$*. On branches other than the one shown, there is are no zeros, since the phase (*$\angle z = 2\pi n$*) is not zero.*

(dark to light) to represent the magnitude. The Matlab program `zviz.m` has been provided to do this (see Lecture 17 on the class website).

To use the program in Matlab/Octave, use the syntax `zviz <function of z>` (for example, type `zviz z.^2`). Note the period between z and `^2`. This will render a 'domain coloring' (aka colorized) version of the function. Examples you can render with `zviz` are given in the comments at the top of the `zviz.m` program. A good example for testing is `zviz z-sqrt(j)`, which should show a dark spot (a zero) at $(1 + 1\jmath)/\sqrt{2} = 0.707(1 + 1\jmath)$.

Example: Figure 1.21 shows a colorized plot of $w(z) = \sin(\pi(z - i)/2)$ resulting from the Matlab/Octave command `zviz sin(pi*(z-i)/2)`. The abscissa (horizontal axis) is the real x axis and the ordinate (vertical axis) is the complex iy axis. The graph is offset along the ordinate axis by $1i$, since the argument $z-i$ causes a shift of the sine function by 1 in the positive imaginary direction. The visible zeros of $w(z)$ appear as dark regions at $(-2, 1), (0, 1), (2, 1)$. As a function of x, $w(x+1\jmath)$ oscillates between red (phase is zero degrees), meaning the function is positive and real, and sea-green (phase is 180°), meaning the function is negative and real.

Along the vertical axis, the function is either a $\cosh(y)$ or $\sinh(y)$, depending on x. The intensity becomes lighter as $|w|$ increases, and white as $w \to \infty$. The intensity becomes darker as $|w|$ decreases, and black as $w \to 0$.

Mathematicians typically use the more abstract (i.e., non–physical) notation $w(z)$, where $w = u + v\imath$ and $z = x + y\imath$. Engineers think in terms of a physical complex impedance $Z(s) = R(s) + X(s)\jmath$, having resistance $R(s)$ and reactance $X(s)$ [ohms], as function of the complex Laplace radian frequency $s = \sigma + \omega\jmath$ [rad], as used, for example, with the Laplace transform (Sect. 1.3.14, p. 98). In Fig. 1.20 we use both notations, with $Z(s) = s$ on the left and $w(z) = z - \sqrt{j}$ on the right, where we show this color code as a 2x2 dimensional domain-coloring graph. Intensity (dark to light) represents the magnitude of the function, while hue (color) represents the phase, where (see Fig. 1.20) red is 0°, sea-green is 90°, blue-green is 135°, blue is 180°, and violet is −90° (or

270°). The function $w = s$ has a dark spot (a zero) at $s = 0$, and becomes brighter away from the origin. On the right is $w(z) = z - \sqrt{j}$, which shifts the zero to $z = \sqrt{j}$. Thus domain–coloring gives the full picture of the complex analytic function mappings $w(x, y) = u(x, y) + v(x, y)\jmath$ in colorized polar coordinates.

Two additional examples are given in Fig. 1.22 to help you interpret the two complex mappings $w = e^s$ (left) and its inverse $s = \ln(w)$. The exponential is relatively easy to understand because $w(s) = |e^\sigma e^{\omega\jmath}| = e^\sigma$. The red region is where $\omega \approx 0$ in which case $w \approx e^\sigma$. As σ becomes large and negative, $w \to 0$, thus the entire field becomes dark on the left. The field is becoming light on the right where $w = e^\sigma \to \infty$. If we let $\sigma = 0$ and look along the ω axis, we see that the function is changing phase, sea-green (90°) at the top and violet (-90°) at the bottom.

In the right panel note the zero for $\ln(w) = \ln|w| + \omega\jmath$ at $w = 1$. The root of the log function is $\log(w_r) = 0$ $w_r = 1, \phi = 0$, since $\log(1) = 0$. More generally, the log of $w = |w|e^{\phi\jmath}$ is $s = \ln|w| + \phi\jmath$. Thus $s(w)$ can be zero only when the angle of w is zero.

The $\ln(w)$ function has a *branch cut* along the $\phi = 180°$ axis. As one crosses over the cut, the phase goes above 180°, and the plane changes to the next *sheet* of the log function. The only sheet with a zero is the principle value, as shown. For all others, the log function is either increasing or decreasing monotonically, and there is no zero, as seen for sheet 0 (the one showing in Fig. 1.22).

1.3.13 Lec 19: Signals: Fourier transforms

Two basic transformations in engineering mathematics are the Fourier and the Laplace transforms, which deal with time–frequency analysis.

Notation: The Fourier transform takes a time domain signal $f(t) \in \mathbb{R}$ and transforms it to the frequency domain $\omega \in \mathbb{R}$, where it is complex ($F(\omega) \in \mathbb{C}$). For the Fourier transform, both the time $-\infty < t < \infty$ and frequency $\infty < \omega < \infty$ are strictly real. The relationship between $f(t)$ and its transform $F(\omega)$ is indicated by the double arrow symbol

$$f(t) \leftrightarrow F(\omega).$$

Since the FT obeys superposition, it is possible to define a FT of a complex time function $f(t) \in \mathbb{C}, t \in \mathbb{R}$. This is useful in certain engineering applications (i.e., Hilbert envelope, Hilbert transforms). It is accepted in the engineering and physics literature to use the case of the variable to indicate the type of argument. A time function is $f(t)$, where t has units of seconds [s] and is in lower case, whereas its Fourier transform, a function of frequency, having units of either hertz [Hz] or radians [2πHz] is written using upper case $F(\omega)$. This helps the reader parse the type of variable under consideration. This is a helpful piece of notation, but not entirely in agreement with notation used in mathematics.

Definition of the Fourier transform: The forward transform takes $f(t)$ to $F(\omega)$ while the inverse transform takes $F(\omega)$ to $\widetilde{f}(t)$. The tilde symbol indicates that in general the recovered inverse transform signal can be slightly different from $f(t)$. We give examples of this in Table F.2.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\jmath\omega t}dt \qquad \widetilde{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\jmath\omega t}d\omega \tag{1.75}$$

$$F(\omega) \leftrightarrow f(t) \qquad \widetilde{f}(t) \leftrightarrow F(\omega). \tag{1.76}$$

Periodic signals: Besides these two basic types of time–frequency transforms, there are several variants that depend on the symmetry in time and frequency. For example, when the time signal is sampled (discrete in time), the frequency response becomes periodic, leading to the *discrete-time Fourier transform* (DTFT). When a time response is periodic, the frequency response is sampled (discrete in frequency), leading to the *Fourier series*. These two symmetries may be simply characterized as *periodic in time* $\Rightarrow$ discrete in frequency, and *periodic in frequency* $\Rightarrow$ discrete in time. When a function is both discrete in time and frequency, it is necessarily periodic in time and frequency, leading to the *discrete Fourier transform* (DFT). The DFT is typically computed with an algorithm called the *fast Fourier transform* (FFT), which can dramatically speed up the calculation when the data is a power of 2 in length.

A very important symmetry is when functions that are causal (in time) *and* periodic (in frequency). The best known example is the class of signals that have *z transforms*, which are causal (one-sided in time) discrete-time signals. The harmonic series (Eq. 1.41, p. 68) is the z-transform of the discrete-time step function, thus, by symmetry, analytic within the RoC in the complex frequency (z) domain.

The double brackets on $f((t))_{T_o}$ indicate that $f(t)$ is periodic in t with period T_o, i.e., $f(t) = f(t + kT_o)$ for all $k \in \mathbb{N}$. Averaging over on period and dividing by the T_o gives the average value of $f(t)$.

Exercise: Consider the $\mathcal{FT}$ as a scalar (dot) product (Eq. 1.54, p. 80) between "vectors" $f(t)$ and $e^{-\jmath\omega_k t}$

$$f(t) \cdot e^{-\jmath\omega_k t} \equiv \frac{1}{T_o}\int_0^{T_o} f(t)e^{-\jmath\omega_k t}dt,$$

where $\omega_0 = 2\pi/T_o$ and $\omega_k = k\omega_o$. If $f(t) = e^{\jmath\omega_n t}$ what is the value of the scalar product?
Solution: Evaluating the scalar product we find

$$e^{\jmath\omega_n t} \cdot e^{-\jmath\omega_k t} = \frac{1}{T_o}\int_0^{T_o} e^{\jmath\omega_n t}e^{-\jmath\omega_k t}dt. = \frac{1}{T_o}\int_0^{T_o} e^{2\pi\jmath(n-k)t/T_o}dt = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}.$$

The two signals (vectors) are orthogonal.

1.3.14 Lec 20: Systems: Laplace transforms

The *Laplace transform* takes a real signals $f(t) \in \mathbb{R}$, as a function of real time $t \in \mathbb{R}$, that are *causal*, i.e., strictly zero for negative time ($f(t) = 0$ for $t < 0$), and transforms them into complex functions $F(s) \in \mathbb{C}$ of complex frequency $s = \sigma + \omega\jmath$. As for the Fourier transform, there is the notation $f(t) \leftrightarrow F(s)$.

When a signal is zero for negative time $f(t < 0) = 0$, it is said to be *causal*, and the resulting transform $F(s)$ is then complex analytic over significant regions of the s plane. For a function of time to be causal, time *must* be real ($t \in \mathbb{R}$), since if it were complex, it would lose the order property (thus it could not be causal). It is helpful to emphasize the causal nature of $f(t)u(t)$ by indicating the causal nature, using the Heaviside step function $u(t)$.

Restrictions on a function (e.g., real, causal, periodic, positive real part, etc.) are called a *symmetry property*. There are many forms of symmetry (Section 1.3.15, p. 100). The concept of symmetry is very general and widely used in both mathematics and physics, where it is more generally known as *group theory*. We shall restrict ourselves to only a few very basic symmetries (Section H.5.1, p. 268).

Definition of the Laplace transform: The forward and inverse Laplace transforms are

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad f(t) = \frac{1}{2\pi\jmath}\int_{\sigma_o-\infty\jmath}^{\sigma_o+\infty\jmath} F(s)e^{st}ds \tag{1.77}$$

$$F(s) \leftrightarrow f(t) \qquad f(t) \leftrightarrow F(s) \tag{1.78}$$

Here $s = \sigma + j\omega \in \mathbb{C}$ [2πHz] is the complex Laplace frequency in radians and $t \in \mathbb{R}$ [s] is the time in seconds.

When dealing with engineering problems it is convenient to separate the *signals* we use from the *systems* that process them. We do this by treating signals, such as a music signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has no mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that they remain physical). These physical restrictions are described in terms of the *network postulates*, which are first discussed in Sect. 1.3.15, and then in greater detail in Sect. H.5.1. There is a question as to why postulates are need, and which are the best postulates.[65] There may be no answer to this question, but having a set of postulates is a way of thinking about physics.[66]

As discussed in Section 1.4.8 (p. 122), we must use the Cauchy residue theorem (CRT), requiring closure of the contour $\mathcal{C}$ at $\omega\jmath \rightarrow \pm j\infty$

$$\oint_{\mathcal{C}} = \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} + \int_{\subset_\infty}$$

where the path represented by '$\subset_\infty$' is a semicircle of infinite radius. For a causal, 'stable' (e.g. doesn't "blow up" in time) signal, all of the poles of $F(s)$ *must* be inside of the Laplace contour, in the left half s-plane.

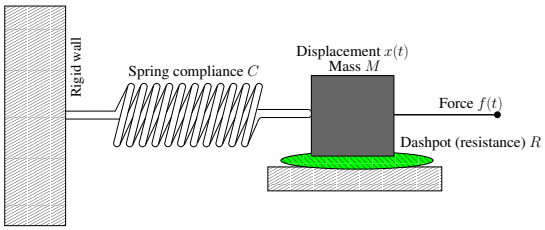

Figure 1.23: *This three-element mechanical resonant circuit consisting of a spring, mass and dashpot (e.g., viscous fluid).*

Hooke's law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dashpot is $f(t) = Rv(t)$, and Newton's famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for constant M is $f(t) = Mdv/dt$.

The equation of motion for the mechanical oscillator in Fig. 1.23 is given by Newton's second law; the sum of the forces must balance to zero

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t). \tag{1.79}$$

These three constants, the mass M, resistance R and stiffness K are all real and positive. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

[65] https://www.youtube.com/watch?v=JXAfEBbaz_4
[66] https://www.youtube.com/watch?v=YaUlqXRPMmY

Newton's second law (c1650) is the mechanical equivalent of Kirchhoff's (c1850) voltage law (KCL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$).

Equation 1.79 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

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Example: The divergent series

$$e^t u(t) = \sum_0^\infty \frac{1}{n!} t^n \leftrightarrow \frac{1}{s-1}$$

is a valid description of $e^t u(t)$, with an unstable pole at $s = 1$. For values of $|x - x_o| < 1$, the *analytic function* $P(x)$ has a *region of convergence* (RoC). For cases where the argument is complex ($x \in \mathbb{C}$), this is called the *radius of convergence* (RoC). We call the region $|x - x_o| > 1$ the *region of divergence* (RoD), and $|x - x_o| = 0$, the *singular circle*. Typically the underlying function $P(s)$, defined by the series, has a pole on the singular circle for $s \in \mathbb{C}$. One may isolate such poles by moving the expansion point s_o until the RoC approaches zero.

This is in conflict with the example of $e^{at} u(t)$, which has a divergent series, that agrees with the unstable function. I'm not sure how to justify this conflict, other than to point out that $t \in \mathbb{R}$, thus the series expansion of the diverging exponential is real-analytic, not complex analytic. It does have a Laplace transform, with a pole at $s = 1$, in agreement with its unstable nature. *Second*, the analytic function must be single valued. This follows from the fact that each term in Eq. 1.39 is single valued. *Third*, analytic functions are very "smooth," since they may be differentiated an ∞ number of times, and the series still converges. There can be no jumps or kinks in such functions.

The key idea that every impedance must be complex analytic and ≥ 0 for $\sigma > 0$ was first proposed by Otto Brune in his PhD at MIT, supervised by a student of Arnold Sommerfeld, Ernst Guillemin, an MIT ECE professor, who played a major role in the development of circuit theory. Brune's primary (non-MIT) advisor was Cauer, who was also well trained in 19th century German mathematics (Brune, 1931b).[67]

Summary: While the definitions of the FT ($\mathcal{FT}$) and LT ($\mathcal{LT}$) transforms appear superficially similar, they are not. The key difference is that the time response of the Laplace transform is causal, leading to a complex analytic frequency response. The frequency response of the Fourier transform is real, thus typically not analytic. These are not superficial differences. The concept of symmetry is helpful in understanding the many different types of time-frequency transforms. Two fundamental types of symmetry are causality and periodicity.

The Fourier transform $\mathcal{FT}$ characterizes the steady-state response while the Laplace transform $\mathcal{LT}$ characterizes both the transient and steady-state response. Given a system response $H(s) \leftrightarrow h(t)$ with input $x(t)$, the output is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s)\Big|_{s=\jmath\omega} X(\omega).$$

1.3.15 Lec 21: Ten network postulates

Solutions of differential equations, such as the wave equation, are conveniently described in terms of mathematical properties, which we present here in terms of 10 *network postulates*:

[67] It must be noted that Prof. 'Mac' Van Valkenburg from the University of Illinois, was arguably more influential in circuit theory, during the same period. Mac's book are certainly more accessible, but perhaps less widely cited.

(P1) *causality* (non-causal/acausal): Causal systems respond when acted upon. Virtually all physical systems obey causality. An example of a causal system is an integrator, which has a response of a step function. Filters are also examples of causal systems. Signals represent acausal responses. They do not have a clear beginning or end, such as the sound of the wind or traffic noise.

(P2) *linearity* (nonlinear): Linear systems obey superposition. Two signals $x(t)$ and $y(t)$ are the inputs to a linear system, producing outputs $x'(t)$ and $y'(t)$. When the inputs are presented together as $ax(t) + by(t)$ with weights $a, b \in \mathbb{C}$, the output will be $ax'(t) + by'(t)$. If either a or b is zero, the corresponding signal is removed from the output.

Nonlinear systems mix the two inputs, thereby producing signals not present in the input. For example, if the inputs to a nonlinear system are two sine waves, the output will contain distortion components, having frequencies not present at the input. An example of a nonlinear system is one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current flow only in one direction. Most physical systems have some degree of nonlinear response, but this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

(P3) *passive* (active): An active system has a power source, such as a battery, while a passive system has no power source. While you may consider a transistor amplifier to be active, it is only so when connected to a power source.

(P4) *real* (complex) time response : Typically systems are "real in, real out." They do not naturally have complex responses (real and imaginary parts). While a Fourier transform takes real inputs and produces complex outputs, this is not an example of a complex time response. P4 is a characterization of the input signal, not its Fourier transform.

(P5) *time-invariant* (time varying): For a system to be a time varying system the output must depend on when the input signal starts or stops. If the output, relative to the input, is independent of the starting time, then the system is said to be time-invariant (static).

(P6) *reciprocal* (non- or anti-reciprocal): In many ways this is the most difficult propriety to characterize and thus understand. It is characterized by the ABCD matrix. If $B = C$, the system is said to be reciprocal. If $B = -C$, it is said to be anti-reciprocal. The impedance matrix is reciprocal while a loudspeaker is anti-reciprocal and modeled by the gyrator rather than a transformer. All non-reciprocal systems are modeled by gyrators, which swap the force and flow variables.

Sect. H.5.1, p. 268.

(P7) *reversibility* (non-reversible): If swapping the input and output of a system leaves the system invariant, it is said to be reversible. When $A = D$ the system is reversible. Note the similarity and differences between reversible and reciprocal.

(P8) *space-invariant* (space-variant): If a system operates independently as a function of where it physically is in space, then it is space-invariant. When the parameters that characterize the system depend on position, it is space-variant.

(P9) *Deterministic* (random): Given the wave equation, along with the boundary conditions, the system's solution may be deterministic, or not, depending on its extent. Consider a radar or

sonar wave propagating out into uncharted territory. When the wave hits an object, the reflection can return waves that are not predicted, due to unknown objects. This is an example where the boundary condition is not known in advance.

(P10) *Quasi-static* ($ka < 1$) Quasi-statics follows from systems that have dimensions that are small compared to the local wavelength. This assumption fails when the frequency is raised (the wavelength becomes short). Thus this is also known as the *long-wavelength* approximation. Quasi-statics is typically stated as $ka < 1$, where $k = 2\pi/\lambda = \omega/c_o$ and a is the smallest dimension of the system. See p. 169 for a detailed discussion of the role of quasi-statics in acoustic horn wave propagation.

Postulate (P10) is closely related to the Feynman (1970c, Ch. 12-7) titled *The "underlying unity" of nature*, where Feynman asks Why do we need to treat the fields as smooth? The answer is related to the wavelength of the probing signal relative to the dimensions of the object being probed. This raises the fundamental question: Are Maxwell's equations a band-limited approximation to reality? I have no idea what the answer is.

Summary discussion of the 10 network postulates: Each postulate has two (or more) categories. For example, (P1) is either causal, non-causal or acausal while (P2) is either linear or non-linear. (P6) and (P9) only apply to 2-port *algebraic networks* (those having an input and an output). The others apply to both a 2– or 1–port networks (e.g., an impedance is a 1-port). An interesting example is the anti-reciprocal transmission matrix of a loudspeaker, shown in Fig. H.7 (p. 268).

Related forms of these postulates may be found in the network theory literature (Van Valkenburg, 1964a,b; Ramo *et al.*, 1965). Postulates (P1-P6) were introduced by Carlin and Giordano (1964) and (P7-P9) were added by Kim *et al.* (2016). As noted by Carlin and Giordano (1964, p. 5), while linearity (P2), passivity (P3), realness (P4) and time-invariance (P5) are independent, causality (P1) is a consequence of linearity (P2) and passivity (P3).

1.3.16 Lec 22: Exam II (Evening Exam)

1.4 Stream 3a: Scalar (i.e., Ordinary) Differential Equations

Stream 3 is ∞, a concept which typically means unbounded (immeasurably large), but in the case of calculus, ∞ means *infinitesimal* (immeasurably small), since taking a limit requires small numbers. Taking a limit means you may never reach the target, a concept that the Greeks called *Zeno's paradox* (Stillwell, 2010, p. 76).

When speaking of the class of *ordinary* (versus *vector*) differential equations, the term *scalar* is preferable, since the term "ordinary" is vague, if not a meaningless label. There are a special subset of fundamental theorems for scalar calculus, all of which are about integration, as summarized in Table 1.6 (p. 159), starting with Leibniz's theorem. These will be discussed below, and more extensively in Lec. 1.4.1 (p. 105) and Chapters I (p. 275) and J (p. 275).

Following the integral theorems on scalar calculus, are those on vector calculus, without which there could be no understanding of Maxwell's equations. Of these, the *fundamental theorem of complex calculus* (aka, Helmholtz decomposition), Gauss's law and Stokes's theorem, form the cornerstone of modern vector field analysis. These theorems allow one to connect the differential (point) and macroscopic (integral) relationships. For example, Maxwell's equations may be written as either vector differential equations, as shown by Heaviside (along with Gibbs and Hertz),[68] or in integral form. It is helpful to place these two forms side-by-side, to fully appreciate their significance. To understand the differential (microscopic) view, one must understand the integral (macroscopic) view. These are presented in Sections 1.5.13 (p. 154) and Fig. 1.36 (p. 159).

Chronological history post 16^{th} century

16^{th} *Bombelli 1526–1572*

17^{th} *Galileo 1564–1642, Kepler 1571–1630, Newton 1642–1727* Principia *1687; Mersenne; Huygen; Pascal; Fermat, Descartes (analytic geometry); Bernoullis Jakob, Johann & son Daniel*

18^{th} *Euler 1707–1783* Student of Johann Bernoulli*; d'Alembert 1717–1783; Kirchhoff; Lagrange; Laplace; Gauss 1777–1855*

19^{th} *Möbius, Riemann 1826–1866,*
Cauchy 1789–1857, Helmholtz 1821–1894, Maxwell 1831–1879, Heaviside 1850–1925, Rayleigh 1842–1919

20^{th} *Hilbert; Einstein; . . .*

Time-Line

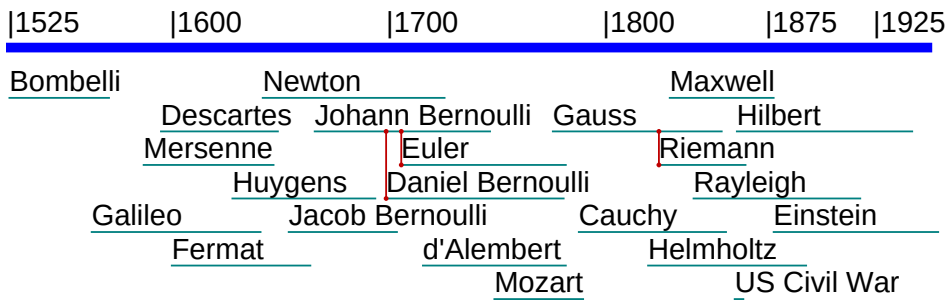


Figure 1.24: *Final overview of the time-line for the four centuries from the 16^{th} and 20^{th} CE covering Bombelli to Einstein. Mozart and the US Civil War are indicated along the bottom, for orientation. .*

[68] https://en.wikipedia.org/wiki/History_of_Maxwell\%27s_equations

The beginning of modern mathematics

As outlined in Fig. 1.24, mathematics as we know it today began in the 16th to 18th centuries, arguably starting with Galileo, Descartes, Fermat, Newton, the Bernoulli family, and Euler. Galileo was formidable, due to his fame, fortune, and his "successful" stance against the powerful Catholic establishment. His creativity in scientific circles was certainly well known due to his many skills and accomplishments. Descartes and Fermat were at the forefront of merging algebra and geometry. While Fermat kept meticulous notebooks, he did not publish, and tended to be secretive. Thus Descartes's contributions were more widely acknowledged, but not necessarily deeper.

Regarding the development of calculus, much was yet to be done by Newton and Leibniz, using term by term integration of functions based on Taylor series representation. This was a powerful technique, but as stated earlier, incomplete because the Taylor series can only represent single-valued functions, within the RoC. But more importantly, Newton (and others) failed to recognize the powerful generalization to complex analytic functions. The first major breakthrough was Newton's publication of Principia (1687), and a second was Riemann (1851), advised by Gauss.

Following Newton's lead, the secretive and introverted behavior of the typical mathematician dramatically changed with the Bernoulli family (Fig. 1.24). The oldest brother, Jacob, taught his much younger brother Johann, who then taught his son Daniel. But Johann's star pupil was Euler. Euler first mastered all the tools and then published, with a prolifically previously unknown.

Euler and the circular functions: The first major task was to understand the family of analytic circular functions, $e^x, \sin(x), \cos(x)$, and $\log(x)$, a task begun by the Bernoulli family, but mastered by Euler. Euler sought relations between these many functions, some of which may not be thought of as being related, such as the log and sin functions. The connection that may "easily" be made is through their complex Taylor series representation (Eq. 1.40, p. 68). By the manipulation of the analytic series representations, the relationship between e^x, and the $\sin(x)$ and $\cos(x)$, was precisely captured with the relation

$$e^{\jmath\omega} = \cos(\omega) + \jmath\sin(\omega), \tag{1.80}$$

and its analytic inverse (Greenberg, 1988, p. 1135)

$$\tan^{-1}(z) = \frac{1}{2\jmath}\ln\left(\frac{1\jmath - z}{1\jmath + z}\right). \tag{1.81}$$

Exercise: Starting from Eq. 1.80, derive Eq. 1.81. **Solution:** Solution: Let $z(\omega) = \tan\omega$, then

$$z(\omega) = \frac{\sin\omega}{\cos\omega} = \tan(\omega) = -\jmath\frac{e^{\omega\jmath} - e^{-\omega\jmath}}{e^{\omega\jmath} + e^{\omega\jmath}} = -\jmath\frac{e^{2\omega\jmath} - 1}{e^{2\omega\jmath} + 1}. \tag{1.82}$$

Solving this for $e^{2\omega\jmath}$ gives

$$e^{2\omega\jmath} = \frac{1 + z\jmath}{1 - z\jmath}. \tag{1.83}$$

Taking the square root results in the analytic relationship between $e^{\omega\jmath}$ and $\jmath\tan\omega$

$$e^{\omega\jmath} = \pm\sqrt{\frac{1 + z\jmath}{1 - z\jmath}}.$$

Taking the log, and using the definition of $\omega(z) = \tan^{-1}(z)$, we obtain Eq. 1.81.

These equations are the basis of transmission lines (TL) and the Smith chart. Here $z(\omega)$ is the TL's input impedance and Eq. 1.83 is the *reflectance*.

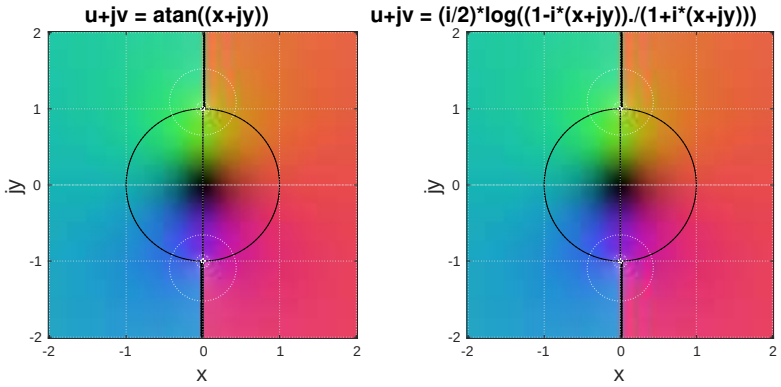


Figure 1.25: *Colorized plots of* $\omega(z) = tan^{-1}(z)$ *and* $\omega(z) = \frac{i}{2}\ln(1-iz)/(1+iz)$*, verifying they are the same complex analytic function.*

While many high school student memorize Euler's relation, it seems unlikely they appreciate the significance of *complex analytic functions* (Eq. 1.47, p. 71).

A brief history of complex analytic functions: Newton famously ignored imaginary numbers, and called them imaginary in a disparaging (pejorative) way. Given Newton's prominence, his view certainly must have attenuated interest in complex algebra, even though it had been previously quantified by Bombelli in 1525, likely based on his serendipitous finding of Diophantus's book *Arithmetic* in the Vatican library.

Euler derived his relationships using real power series (i.e., real analytic functions). While Euler was fluent with $\jmath = \sqrt{-1}$, he did not consider functions to be complex analytic. That concept was first explored by Cauchy almost a century later. The missing link to the concept of complex analytic is the definition of the derivative with respect to the complex argument

$$F'(s) = \frac{dF(s)}{ds} \tag{1.84}$$

where $s = \sigma + \omega\jmath$, without which the complex analytic Taylor coefficients may not be defined.

Euler did not appreciated the role of complex analytic functions, because these were first discovered well after his death (1783), by Augustin-Louis Cauchy (1789–1857).

1.4.1 Lec 23: Fundamental theorems of calculus

History of the fundamental theorem of calculus: It some sense, the story of calculus begins with the *fundamental theorem of calculus* (FTC), also known generically as *Leibniz's formula*. The simplest integral is the length of a line $L = \int_0^L dx$. If we label a point on a line as $x = 0$ and wish to measure the distance to any other point x, we form the line integral between the two points. If the line is straight, this integral is simply the Euclidean length given by the difference between the two ends (Eq. 1.3.6, p. 80).

If $F(\chi) \in \mathbb{R}$ describes a height above the line $\chi \in \mathbb{R}$, then $f(x)$

$$f(x) - f(0) = \int_{x=0}^{x} F(\chi) d\chi, \tag{1.85}$$

may be viewed as the *anti-derivative* of $F(\chi)$. Here χ is a dummy variable of integration. Thus the area under $F(\chi)$ only depends on the difference of the area evaluated at the end points. It makes intuitive sense to view $f(x)$ as the anti-derivative of $F(\chi)$.

This property of the area as an integral over an interval, only depending on the end points, has important consequences in physics in terms of conservation of energy, allowing for important generalizations. For example, as long as $\chi \in \mathbb{R}$, one may let $F(\chi) \in \mathbb{C}$ with no loss of generality, due to the linear propriety (P1, p. 101) of the integral.

If $f(x)$ is analytic (Eq. 1.39, p. 67), then

$$F(x) = \frac{d}{dx} f(x), \tag{1.86}$$

is an exact real differential. It follows that $F(x)$ is analytic. This is known as the *fundamental theorem of (real) calculus* (FTC). Thus Eq. 1.86 may be viewed as an *exact real differential*. This is easily shown by evaluating

$$\frac{d}{dx} f(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = F(x),$$

starting from the *anti-derivative* Eq. 1.85. If $f(x)$ is not analytic then the limit may not exist, so this is a necessary condition.

There are many important variations on this very basic theorem (i.e., Sect. 1.4, p. 103). For example, the limits could depend on time. Also when taking Fourier transforms, the integrand depends on both time $t \in \mathbb{R}$ and frequency $\omega \in \mathbb{R}$ via a complex exponential "kernel" function $e^{\pm \jmath \omega t} \in \mathbb{C}$, which is analytic in both t and ω.

The fundamental theorems of complex calculus:

The *fundamental theorem of complex calculus* (FTCC) states (Greenberg, 1988, p. 1197) that for any *complex analytic function* $F(s) \in \mathbb{C}$ with $s = \sigma + \omega \jmath \in \mathbb{C}$

$$f(s) - f(s_o) = \int_{s_o}^{s} F(\zeta) d\zeta. \tag{1.87}$$

Equations 1.85 and 1.87 differ because the path of the integral is complex. Thus the line integral is over $s \in \mathbb{C}$ rather than a real integral over $\chi \in \mathbb{R}$. The *fundamental theorem of complex calculus* (FTCC) states that the integral only depends on the end points, since

$$F(s) = \frac{d}{ds} f(s). \tag{1.88}$$

Comparing exact differentials Eq. 1.88 (FTCC) and Eq. 1.86 (FTC), we see that $f(s) \in \mathbb{C}$ must be *complex analytic*, and have a Taylor series in powers in $s \in \mathbb{C}$. It follows that $F(s)$ is also complex analytic.

Complex analytic functions: The definition of a *complex analytic function* $F(s)$ of $s \in \mathbb{C}$ is that the function may be expanded in a Taylor series (Eq. 1.46, p. 71) about an *expansion point* $s_o \in \mathbb{C}$. This definition follows the same logic as the FTC. Thus we need a definition for the coefficients $c_n \in \mathbb{C}$, which most naturally follow from Taylor's formula

$$c_n = \frac{1}{n!}\frac{d^n}{ds^n}F(s)\bigg|_{s=s_o}. \tag{1.89}$$

The requirement that $F(s)$ have a Taylor series naturally follows by taking derivatives with respect to s at s_o. *The problem is that both integration and differentiation of functions of complex Laplace frequency $s = \sigma + \omega\jmath$ have not yet been defined.*

Thus the question is: What does it mean to take the derivative of a function $F(s) \in \mathbb{C}$, $s = \sigma + \omega\jmath \in \mathbb{C}$, with respect to s, where s defines a plane rather than a real line? We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?

The answer is affirmative. The question may be resolved by applying the rules of the real derivative when defining the derivative in the complex plane. However for the complex case, there is an issue, regarding direction. Given any analytic function $F(s)$, is the partial derivative with respect to σ different from the partial derivative with respect to $\omega\jmath$? For complex analytic functions, the FTCC states that the integral is independent of the path in the s plane. *Based on the chain rule, the derivative must also be independent of direction at s_o.* This directly follows from the FTCC. *If the integral of a function of a complex variable is to be independent of the path, the derivative of a function with respect to a complex variable must be independent of the direction.* This follows from Taylor's formula, Eq. 1.89, for the coefficients of the complex analytic formula.

The Cauchy-Riemann conditions: The FTC defines the area as an integral over a real differential ($dx \in \mathbb{R}$), while the FTCC relates an integral over a complex function $F(s) \in \mathbb{C}$, along a complex interval (i.e., path) ($ds \in \mathbb{C}$). For the FTC the area under the curve only depends on the end points of the anti-derivative $f(x)$. But what is the meaning of an "area" along a complex path? The *Cauchy-Riemann conditions* provide the answer.

1.4.2 Lec 24: Cauchy-Riemann conditions

For the integral of $Z(s) = R(\sigma,\omega) + X(\sigma,\omega)\jmath$ to be independent of the path, the derivative of $Z(s)$ must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the *Cauchy-Riemann* conditions. This is an important generalization of Eq. 1.1, p. 18 which goes from real integration ($x \in \mathbb{R}$) to complex integration ($s \in \mathbb{C}$), based on lengths, thus on area.

To define

$$\frac{d}{ds}Z(s) = \frac{d}{ds}\left[R(\sigma,\omega) + \jmath X(\sigma,\omega)\right]$$

take partial derivatives of $Z(s)$ with respect to σ and $\jmath\omega$, and equate them:

$$\frac{\partial Z}{d\sigma} = \frac{\partial R}{\partial\sigma} + \jmath\frac{\partial X}{\partial\sigma} \qquad \equiv \qquad \frac{\partial Z}{\jmath d\omega} = \frac{\partial R}{\partial\jmath\omega} + \jmath\frac{\partial X}{\partial\jmath\omega}.$$

This says that a horizontal derivative, with respect to σ, is equivalent to a vertical derivative, with respect to $\jmath\omega$. Taking the real and imaginary parts gives the two equations

$$\textit{CR-1:}\ \frac{\partial R(\sigma,\omega)}{\partial\sigma} = \not{\jmath}\frac{\partial X(\sigma,\omega)}{\partial\omega\not{\jmath}} \quad \text{and} \quad \textit{CR-2:}\ \frac{\partial R(\sigma,\omega)}{\partial\omega\not{\jmath}} = -\not{\jmath}\frac{\partial X(\sigma,\omega)}{\partial\sigma}, \tag{1.90}$$

known as the *Cauchy-Riemann (CR) conditions*. The $\jmath$ cancels in CR-1, but introduces a $\jmath^2 = -1$ in CR-2. They may also be written in polar coordinates ($s = re^{\theta\jmath}$) as

$$\frac{\partial R}{\partial r} = \frac{1}{r}\frac{\partial X}{\partial \theta} \qquad \text{and} \qquad \frac{\partial X}{\partial r} = -\frac{1}{r}\frac{\partial R}{\partial \theta}.$$

If you are wondering what would happen if we took a derivative at 45 degrees, then we only need to multiply the function by $e^{\jmath\pi/4}$. But doing so will not change the derivative. Thus we may take the derivative in any direction by multiplying by $e^{\theta\jmath}$, and the CR conditions will not change.

The CR conditions are necessary conditions that the integral of $Z(s)$, and thus its derivative, be independent of the path, expressed in terms of conditions on the real and imaginary parts of Z. This is a very strong condition on $Z(s)$, which follows assuming that $Z(s)$ may be written as a Taylor series in s

$$Z(s) = Z_o + Z_1 s + \frac{1}{2}Z_2 s^2 + \cdots, \tag{1.91}$$

where $Z_n \in \mathbb{C}$ are complex constants given by the Taylor series formula (Eq. 1.89, p. 107). As with the real Taylor series, there is the convergence condition, that $|s| < 1$, called the *radius of convergence* (RoC). This is an important generalization of the region of convergence (R0C) for real $s = x$.

Every function that may be expressed as a *Taylor series* in $s - s_o$ about point $s_o \in \mathbb{C}$ is said to be *complex analytic* at s_o. This series, *which must be single valued*, is said to converge within a *radius of convergence* (RoC). This highly restrictive condition has significant physical consequences. For example, every impedance function $Z(s)$ obeys the CR conditions over large regions of the s plane, including the entire right half-plane (RHP) ($\sigma > 0$). This condition is summarized by the Brune condition $\Re\{Z(\sigma > 0)\} \geq 0$ (Section 1.4.3, Eq. 1.99, p. 112). When the CR condition is generalized to volume integrals, it is called *Green's theorem*, used heavily in the solution of boundary value problems in engineering and physics (Kusse and Westwig, 2010). Sections 1.4 (p. 103) and 1.5 (p. 126) depend heavily on these concepts.

We may merge these equations into a pair of second-order equations by taking a second round of partials. Specifically, eliminating the real part $R(\sigma, \omega)$ of Eq. 1.90 gives

$$\frac{\partial^2 R(\sigma,\omega)}{\partial\sigma\partial\omega} = \frac{\partial^2 X(\sigma,\omega)}{\partial^2\omega} = -\frac{\partial^2 X(\sigma,\omega)}{\partial^2\sigma}, \tag{1.92}$$

which may be compactly written as $\nabla^2 X(\sigma, \omega) = 0$. Eliminating the imaginary part gives

$$\frac{\partial^2 X(\sigma,\omega)}{\partial\omega\partial\sigma} = \frac{\partial^2 R(\sigma,\omega)}{\partial^2\sigma} = -\frac{\partial^2 R(\sigma,\omega)}{\partial^2\omega}, \tag{1.93}$$

which may be written as $\nabla^2 R(\sigma, \omega) = 0$.

In summary, for a function $Z(s)$ to be complex analytic, the derivative dZ/ds must be independent of direction (path), which requires that the real and imaginary parts of the function obey Laplace's equation, i.e.,

$$\text{CR-3: } \nabla^2 R(\sigma,\omega) = 0 \qquad \text{and} \qquad \text{CR-4: } \nabla^2 X(\sigma,\omega) = 0. \tag{1.94}$$

The CR equations are easier to work with because they are first-order, but the physical intuition is best understood by noting fact 1) the derivative of a complex analytic function is independent of

its direction, and fact 2) the real and imaginary parts of the function each obey Laplace's equation. Such relationships are known as *harmonic functions*.[69]

As we shall see in the next few lectures, complex analytic functions must be smooth since every analytic function may be differentiated an infinite number of times, within the RoC. The magnitude must attain its maximum *and* minimum on the boundary. For example, when you stretch a rubber sheet over a jagged frame, the height of the rubber sheet obeys Laplace's equation. Nowhere can the height of the sheet rise above or below its value at the boundary.

Harmonic functions define *conservative fields*, which means that energy (like a volume or area) is conserved. The work done in moving a mass from a to b in such a field is conserved. If you return the mass from b back to a, the energy is retrieved, and zero net work has been done.

1.4.3 Lec 25: Complex Analytic functions and Brune Impedance

It is rarely stated that the variable that we are integrating over, either x (space) or t (time), is real ($x, t \in R$), since that fact is implicit, due to the physical nature of the formulation of the integral. But this intuition must be refined once complex numbers are included with $s \in \mathbb{C}$, where $s = \sigma + \omega\jmath$.

That time and space are real variables is more than an assumption: it is a requirement, that follows from the real *order property*. Real numbers have order. For example, if $t = 0$ is now (the present), then $t < 0$ is the past and $t > 0$ is the future. The order property of time and space allows one to order these along a real axis. To have time travel, time and space would need to be complex (they are not), since if the space axis were complex, as in frequency s, the order property would be invalid. It follows that if we require order, time and space must be real ($t, x \in \mathbb{R}$). Interestingly, it was shown by d'Alembert (1747) that time and space are related by the pure delay due to the wave speed c_o. To obtain a solution to the governing wave equation, that d'Alembert first proposed for sound waves, $x, t \in \mathbb{R}^3$ may be combined as functions of

$$\zeta_\pm = t \pm x/c_o,$$

where c [m/s] is the phase velocity of the waves. The d'Alembert solution to the wave equation, describing waves on a string under tension, is

$$u(x,t) = F(t - x/c_o) + G(t + x/c_o), \tag{1.95}$$

which describes the transverse velocity (or displacement) of two independent waves $F(\zeta_-), G(\zeta_+)$ on the string, which represent forward and backward traveling waves.[70] For example, starting with a string at rest, if one displaces the left end, at $x = 0$, by a step function $u(t)$, then that step displacement will propagate to the right as $u(t - x/c_o)$, arriving at location x_o [m], at time x_o/c_o [s]. Before this time, the string will not move to the right of the wave-front, at x_o [m], and after t_o [s] it will have displacement 1. Since the wave equation obeys superposition (postulate P1, p. 100), it follows that the "plane-wave" eigen-functions of the wave equation for $\boldsymbol{x}, \boldsymbol{k} \in \mathbb{R}^3$ are given by

$$\psi_\pm(\boldsymbol{x}, t) = \delta(t \mp \boldsymbol{k} \cdot \boldsymbol{x}) \leftrightarrow e^{st \pm \jmath \boldsymbol{k} \cdot \boldsymbol{x}}, \tag{1.96}$$

where $|\boldsymbol{k}| = 2\pi/|\boldsymbol{\lambda}| = \omega/c_o$ is the *wave number*, $|\boldsymbol{\lambda}|$ is the wavelength, and $s = \sigma + \omega\jmath$.

[69] When the function is the ratio of two polynomials, as in the cases of the Brune impedance, they are also related to Möbius transformations, also know as *bi-harmonic* operators.

[70] D'Alembert's solution is valid for functions that are not differentiable, such as $\delta(t - c_o x)$.

When propagation losses are considered, we must replace $\jmath\boldsymbol{k}$ with a complex analytic wave number $\boldsymbol{\kappa}(s) = \boldsymbol{k}_r(s) + \jmath\boldsymbol{k}(\boldsymbol{s})$, which is denoted as either the *complex propagation function*, *propagation function*, or the *dispersion relation*. Forms of loss include viscosity and radiation. Important examples include acoustic plane waves, electromagnetic wave propagation, antenna theory, and the more complex case of 3D electron wave propagating in crystals (i.e., silicon), where electrons and EM waves are in a state of equilibrium.

Electrons and photons are simply different EM states, where $\boldsymbol{\kappa}(\boldsymbol{x}, s)$ describes the crystal's dispersion relations as functions of both frequency and direction, famously known as *Brillouin zones*. Dispersion is a property of the medium such that the wave velocity is a function of frequency and direction, as in silicon.[71] Informed discussions on the history of this topic may be found in Brillouin (1953).

Complex impedance functions

Conservation of energy (or power) is a cornerstone of modern physics. It may have first been under consideration by Galileo Galilei (1564-1642). Today the question is not if it is true, the questions is, why. Specifically, what is the physics behind conservation of energy? Surprisingly, the answer is straight forward, based on its definition, and the properties of impedance. Recall that the power is the product of the force and flow, and impedance is their ratio.

The power is given by the product of two variables, sometimes called *conjugate* variables, the force and the flow. In electrical terms, these are voltage (force) ($v(t) \leftrightarrow V(\omega)$) and current (flow) ($i(t) \leftrightarrow I(\omega)$), thus the electrical power at any instant of time is

$$\mathcal{P}(t) = v(t)i(t). \tag{1.97}$$

The total energy $w(t)$ is the integral of the power, since $\mathcal{P}(t) = dw/dt$. Thus if we start with all the elements at rest (no currents or voltages), then the energy as a function of time is always positive

$$w(t) = \int_0^t \mathcal{P}(t)dt \geq 0, \tag{1.98}$$

and is simply the total energy applied to the network (Van Valkenburg, 1964a, p. 376). Since the voltage and current are related by either an impedance or an admittance, conservation of energy depends on the property of impedance. From Ohm's law and P1 (every impedance is causal)

$$v(t) = z(t) \star i(t) = \int_{\tau=0}^{t} z(\tau)i(t-\tau)d\tau \leftrightarrow Z(s)I(s).$$

Verify!

From the expression for the energy, in terms of Ohm's law

[71] In case you missed it, I'm suggesting is that photons (propagating waves) and electrons (evanescent waves) are different wave "states." The difference is the medium, which determines the dispersion relation (Papasimakis *et al.*, 2018).

$$\begin{aligned} w(t) &= u(t) \star (v(t)i(t)) \\ &= u(t) \star i(t) \star z(t) \star i(t) \\ &= \int_{t=0}^{t} i(t) \Big(\int_{\tau=0}^{t} z(\tau) i(t-\tau) d\tau \Big) dt \\ &= \int_{\tau=0}^{t} z(\tau) \Big(\int_{t=0}^{t} i(t) i(t-\tau) dt \Big) d\tau \\ &\leq \int_{\tau=0}^{t} z(\tau) |w|^2(\tau)] d\tau \\ &\leftrightarrow \frac{1}{s} Z(s) |I(\omega)|^2 \geq 0. \end{aligned}$$

The step from time to frequency follows from the fact that

$$|w|^2(\tau) = i(t) \star i(t) = \int_{\tau=0}^{t} i(\tau) i(t-\tau) \rightarrow |I(\tau, \omega)|^2 > 0$$

always has a positive Fourier transform for every possible $i(t)$

Example: Let $i(t) = \delta(t)$. Then $|w|^2(\tau) = i(t) \star i(t) = \delta(\tau)$. Thus

$$w(t) = \int_{\tau=0}^{t} z(\tau) |w|^2(\tau) d\tau = \int_{\tau=0}^{t} z(\tau) \delta(\tau) d\tau = \int_{0}^{t} z(\tau) d\tau.$$

The Brune impedance always has the form $z(t) = r_o \delta(t) + \zeta(t)$. The surge impedance is defined as

$$r_o = \int_{0^-}^{\infty} z(t) dt.$$

The integral of the reactive part ($\zeta(t)$) is always zero, since the reactive part cannot not store energy.

Perhaps easier to visualize is when working in the frequency domain where the total energy, equal to the integral of the real part of the power, is

$$\frac{1}{s} \Re VI = \frac{1}{2s} (V^* I + V I^*) = \frac{1}{2s} (Z^* I^* I + Z I I^*) = \frac{1}{s} \Re Z(s) |I|^2 \geq 0.$$

Formally this is related to a positive definite operator where the positive resistance forces the definiteness, which is sandwiched between the current.

In conclusion conservation of energy is totally dependent on the properties of the impedance. Thus of the most important and obvious applications of complex functions of a complex variable is the impedance function. This seems to be the ultimate example of the FTC, applied to $z(t)$, in the name of conservation of energy.

Poles and zeros of PR functions must be first degree: We conjecture that this proof also requires that the poles and the zeros of the impedance function be simple (only first degree). Second degree poles would have a reactive "secular" response of the form $h(t) = t \sin(\omega_k t + \phi) u(t)$, and these terms would not average to zero, depending on the phase, as is required of an impedance. As a result, only single degree poles would be possible.[72] Furthermore, when the impedance is the ratio of two polynomials, where the lower degree polynomial is the derivative of the higher degree one, then the poles and zeros must alternate. This is a well-known property of the Brune

[72] Secular terms result from second degree poles since $u(t) \star u(t) = tu(t)$.

impedance, that has never been adequately explained except for very special cases (Van Valkenburg, 1964b, p. 104-107). I suspect that no one has ever reported an impedance having second degree poles and zeros as that would be rare impedance. Network analysis books never report 2nd degree poles and zeros in their impedance functions. Nor has there ever been any guidance as to where the poles and zeros might lie in the *left hand s plane*. Understanding the exact relationships between pairs of poles and zeros to assure that the real part of the impedance is real, would resolve a long-standing unsolved problem (Van Valkenburg, 1964b).

Every impedance must obey conservation of energy (P3): Every impedance is defined by a Laplace transform pair

$$z(t) \leftrightarrow Z(s) = R(\sigma, \omega) + \jmath X(\sigma, \omega)$$

with $R, X \in \mathbb{R}$. According to Postulate (P3) Sect. 1.3.15 (p. 100), a system is passive if it does not contain a power source. Drawing power from an impedance violates conservation of energy. This propriety is also called *positive real*, which is defined as (Brune, 1931b,a)

$$\Re\{Z(s \geq 0)\} \geq 0. \tag{1.99}$$

or $R(\omega, \omega) \geq 0$. In words: the real part of every impedance must be non-negative for $\sigma \geq 0$. When this condition holds, one cannot draw more power than is stored in the impedance.[73] A second condition requires that the impedance has simple poles. If there were a pole in the region $\sigma > 0$, then the first condition would not be met. Therefore, there can only be simple poles (degree of 1) in the region $\sigma \leq 0$. The region $\sigma \leq 0$ is called the *left half s plane* (LHP, and the complementary region, $\sigma > 0$, is called the *right half s plane* (RHP). The condition on the simple poles is sufficient, but not necessary, as $Z(s) = 1/\sqrt{s}$ is a physical impedance, but is not a first-degree pole. The impedance function $Z(s) = R(\sigma, \omega) + \jmath X(\sigma, \omega)$ has resistance R and reactance X as a function of complex frequency $s = \sigma + \jmath\omega$. From the causality postulate (P1) of Sections 1.3.15 and H.5.1 (p. 268), $z(t < 0) = 0$.

As an example, a series resistor R_o and capacitor C_o has an impedance given by (Table F.3, p. 210)

$$Z(s) = R_o + 1/sC_o \leftrightarrow R_o\delta(t) + \frac{1}{C_o}u(t) = z(t), \tag{1.100}$$

with constants $R_o, C_o \in \mathbb{R} > 0$. In mechanics an impedance composed of a dashpot (damper) and a spring has the same form. A resonant system has an inductor, resistor and a capacitor, with an impedance given by (Table F.4, p. 211)

$$Z(s) = \frac{sC_o}{1 + sC_oR_o + s^2C_oM_o} \leftrightarrow C_o\frac{d}{dt}\left(c_+e^{s_+t} + c_-e^{s^-t}\right) = z(t), \tag{1.101}$$

which is a second degree polynomial with two complex resonant frequencies $s_\pm$. When $R_o > 0$ these roots are in the left half s plane, with $z(t) \leftrightarrow Z(s)$.

Systems (networks) containing many elements, and transmission lines can be much more complicated, yet still have a simple frequency domain representation. This is the key to understanding how these physical systems work, as will be described below.

[73] Does this condition hold for the LHP $\sigma < 0$? It does for Eq. 1.101.

Complex analytic functions: To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However, he only used real functions of a real variable, due to the fundamental lack of appreciation of the complex analytic series. This same method is how one finds solutions to scalar differential equations today, but using an approach that makes the solution method less obvious. Rather than working directly with the Taylor series, today we use the complex exponential, since the complex exponential is an eigen-function of the derivative

$$\frac{d}{dt}e^{st} = se^{st}.$$

Since e^{st} may be expressed as a Taylor series, having coefficients $c_n = 1/n!$, in some real sense the modern approach is a compact way of doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency s, by looking for solutions of the form $A(s)e^{st}$, transforming the differential equation into a polynomial $A(s)$ in complex frequency. For example

$$\frac{d}{dt}f(t) + af(t) \leftrightarrow (s+a)F(s).$$

The root of $A(s_r) = s_r + a = 0$ is the eigen-value of the differential equation. A powerful tool for understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain. The Taylor series has been replaced by e^{st}, transforming Newton's real Taylor series into the complex exponential eigen-function. In some sense, these are the same method, since

$$e^{st} = \sum_{n=0}^{\infty} \frac{(st)^n}{n!}. \tag{1.102}$$

Taking the derivative with respect to time gives

$$\frac{d}{dt}e^{st} = se^{st} = s\sum_{n=0}^{\infty} \frac{(st)^n}{n!}, \tag{1.103}$$

which is also complex analytic. Thus if the series for $F(s)$ is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key here is "Where does the series fail to converge?" in which case, the entire representation fails. This is the main message behind the FTCC (Eq. 1.87).

The FTCC (Eq. 1.88) is formally the same as the FTC (Eq. 1.86) (Leibniz formula), the key (and significant) difference being that the argument of the integrand $s \in \mathbb{C}$. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depends on the path of integration.

But, according to FTCC, it does not. In fact they are clearly distinguishable from the FTC. And the reasoning is the same. If $F(s) = df(s)/ds$ is complex analytic (i.e., has a power series $f(s) = \sum_k c_k s^k$, with $f(s), c_k, s \in \mathbb{C}$), then it may be integrated, and *the integral does not depend on the path.* At first blush, this is sort of amazing. The key is that $F(s)$ and $f(s)$ must be complex analytic, which means they are differentiable. This all follows from the Taylor series formula Eq. 1.89 (p. 107) for the coefficients of the complex analytic series. For Eq. 1.87 to hold, the derivatives must be independent of the direction, as discussed in Section 1.4.2. The concept of a complex analytic function therefore has eminent consequences, in the form of several key theorems on complex integration discovered by Cauchy (c1820).

The use of the complex Taylor series generalize the functions it describes, with unpredictable consequences, as nicely shown by the domain coloring diagrams presented in Section 1.3.12 (p. 95). Cauchy's tools of complex integration were first exploited in physics by Sommerfeld (1952), to explain the onset transients in waves, as explained in detail in Brillouin (1960, Chap. 3).

Up to 1910, when Sommerfeld first published his results using complex analytic signals and saddle point integration in the complex plane, there was a poor understanding of the implications of the causal wave-front. It would be reasonable to say that his insights changed our understanding of wave propagation, for both light and sound. Sadly this insight has not been fully appreciated, even to this day. If you question my summary, please read Brillouin (1960, Chap. 1).

The full power of the complex analytic function was first appreciated by Bernard Riemann (1826-1866), in his University of Göttingen PhD Thesis of 1851, under the tutelage of Carl Friedrich Gauss (1777-1855), and drawing heavily on the work of Cauchy.

The key definition of a complex analytic function is that it has a Taylor series representation over a region of the complex frequency plane $s = \sigma + j\omega$, that converges in a *region of convergence* (RoC) about the expansion point, with a radius determined by the nearest pole of the function. A further surprising feature of all analytic functions is that within the RoC, the inverse of that function also has a complex analytic expansion. Thus given $w(s)$, one may also determine $s(w)$ to any desired accuracy, critically depending on the RoC.

1.4.4 Lec 26: Multi-valued functions

In the field of mathematics there seems to have been a tug-of-war regarding the basic definition of the concept of function. The accepted definition today seems to be a single-valued mapping from the domain to the codomain (or range). This makes the discussion of multi-valued functions somewhat tedious. In 1851 Riemann (working with Gauss) seems to have resolved this problem for the natural set of multi-valued functions by introducing the concept of the *branch-cut* and *sheets*.

Two simple examples of multi-valued functions are the circle $z^2 = x^2 + y^2$ and $w = \log(z)$. For example, assuming z is the radius of the circle, solving for $y(x)$ gives the multi-valued function

$$y(x) = \pm\sqrt{z^2 - x^2}.$$

If we accept the modern definition of a function as the mapping from one set to a second, then $y(x)$ is not a function, or even two functions. For example, what if $x > z$? Or worse, what if $z = 2\jmath$ with $|x| < 1$? Riemann's construction, using branch cuts for multivalued function, resolves all these difficulties (as best I know).

To proceed we need definitions and classifications of the various types of complex singularities:

1. Poles of degree 1 are called *simple poles*. Their amplitude called the *residue* (e.g. α/s has residue α). Simple poles are special (Eq. 1.106, p. 119)[74] and play a key role in mathematical physics. Consider the function $y(s) = \sqrt[\alpha]{1 - s^\alpha}$ with $\alpha \in \mathbb{Z}, \mathbb{F}, \mathbb{R}$ and $\mathbb{C}$.

2. When the numerator and denominator of a rational function have a common root (i.e., factor), that root is said to be *removable*.

3. A singularity that is not 1) removable, 2) a pole or 3) a branch point, is called *essential*.

[74] https://en.wikipedia.org/wiki/Pole_(complex_analysis)

4. When the first derivative of a function $Z(s)$ has a simple pole at s_o, then s_o is said to be a *branch point* of $Z(s)$ (e.g., $d\ln(s^\alpha)/ds = \alpha/s$). However, the converse does not necessarily hold.

5. A complex function which is analytic, except for isolated poles, is called *metamorphic*.[75] Metamorphic functions can have any number of poles, even an infinite number.

More complex typologies are being researched today, and progress is expected to accelerate due to modern computing technology.[76] It is helpful to identify the physical meaning of these more complex surfaces, to guide us in their interpretation and possible applications.[77]

Branch cuts: Up to this point we have only considered poles of degree k, of the form $1/s^k$, with $k \in \mathbb{Z}$. The concept of a *branch cut* allows one to manipulate (and visualize) multi-valued functions, for which $k \in \mathbb{F}$. This is done by breaking each region into a single-valued *sheets*. The concepts of the branch cut, the sheets, and the *extended plane*, were first devised by Riemann, working with Gauss (1777-1855), and first described in his thesis of 1851. Of course it was these three mathematical and geometrical constructions that provide the deep insight to complex analytic functions, supplementing the important earlier work of Cauchy (1789-1857) on the calculus of complex analytic functions. For an intuitive discussion of Riemann sheets and branch cuts, see Boas (1987, Section 29, pp. 221-225).

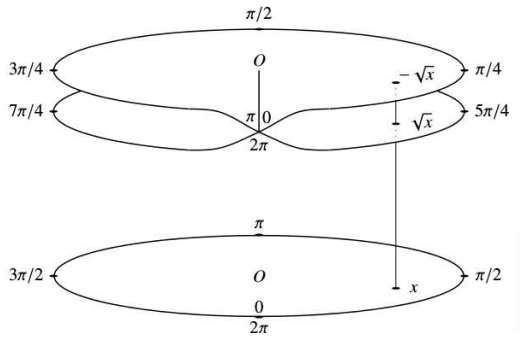


Figure 1.26: *FIX figure Here we see the mapping for the square root function $w(z) = \pm\sqrt{z}$ which has two single-valued sheets, corresponding to the two signs of the square root. The plot at the bottom represents the z plane while the two interleaved plots on top represent the two sheets of $\pm\sqrt{z}$. The lower sheet is $+\sqrt{z}$ while the upper sheet is $-\sqrt{z}$. The branch cut runs from $z = 0$ along the real axis out to ∞. The location of the branch cut may be moved by rotating the z coordinate system. For example, $w(z) = \pm\jmath\sqrt{z}$ and $w(z) = \pm\sqrt{-z}$ have a different branch cuts, as may be easily verified using the Matlab/Octave commands `j*zviz(z)` and `zviz(-z)`, as shown in Fig. 1.28. Every function is analytic on the branch cut, since the cut may be moved. If a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the complex analytic series (i.e., the Taylor formula, Eq. 1.89) does not depend on the location of a branch cut, as it only describes the function uniquely (as a single-valued function), valid in its local region of convergence. This figure has been taken from Stillwell (2010, p. 303). A more comprehensive view is presented in the text. The branch cut lies in the domain ($x \in \mathbb{C}$), not in the codomain $w(x) \in \mathbb{C}$ (Kusse and Westwig, 2010). This becomes clear by studying how `zviz.m` works, and understanding its output. A example that shows this is Fig. 1.28, where the axes are s and $\omega\jmath$, with the branch cut along the negative σ axis ($\theta = \pi$).*

Square root function: The branch cut is a line that separates the various single-valued parts of a multi-valued function. For example, in Fig. 1.26 we see the single-valued function $w(z) = s^2$ (left), and on the right, its inverse, the double-valued mapping of $s(w) = \pm\sqrt{w}$.

[75] `https://en.wikipedia.org/wiki/Meromorphic_function`

[76] `https://www.maths.ox.ac.uk/about-us/departmental-art/theory`

[77] `https://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201`

The multi-valued nature of the square root is best understood by working with the function in polar coordinates. Let

$$s_k = re^{\theta\jmath}e^{2\pi k\jmath},$$

where k is the sheet-index, and

$$w = \rho e^{\psi\jmath} = \sqrt{r}e^{\jmath\theta/2}e^{\jmath\pi k},$$

where $r, \rho \in \mathbb{R}$ are the magnitudes and $\theta, \psi \in \mathbb{R}$ are the angles. The domain-coloring program `zviz.m` assumes that the angles go from $-\pi < \theta < \pi$, with $\theta = 0$ being a light red and $\pm\pi$ a blue color. This angle to color map is shown in the left panel of Fig. 1.28. The first Riemann sheet for $k = 0$ $-\pi < \theta < \pi$ is shown in Fig. 1.28 (MIDDLE), which differs from the Principle cut shown in Fig. 1.27 (Right).

The second sheet ($k = 1$) (Fig. 1.28) picks up at $\theta = \pi$ and continues on to $\pi + 2\pi = 3\pi$. The first sheet maps the angle of w (i.e., $\phi = \angle w = \theta/2$) from $-\pi/2 < \phi < \pi/2$ ($w = \sqrt{r}e^{\jmath\theta/2}$). This corresponds to $u = \Re\{w(s)\} > 0$. The second sheet maps $\pi/2 < \psi < 3\pi/2$ (i.e., 90° to 270°), which is $\Re\{w\} = u < 0$. In summary, twice around the s plane is once around the $w(s)$ plane, because the angle is half due to the $\sqrt{s}$. This then describes the multi-valued nature of the square root function.

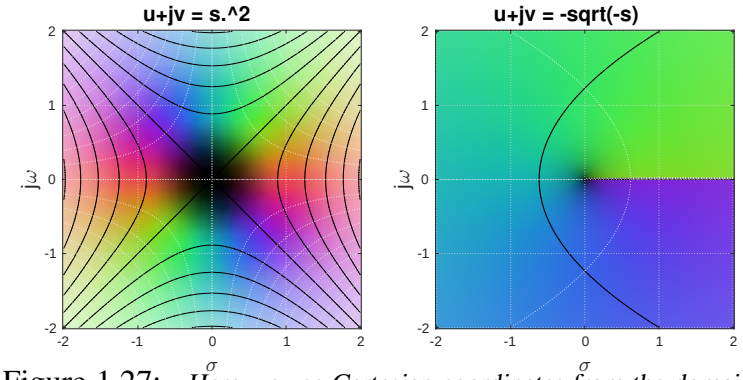


Figure 1.27: *Here we use Cartesian coordinates from the domain $s = \sigma + \omega\jmath$ to the range $w(\sigma, \omega) = u + v\jmath$. Left: Mapping: $w(s) = s^2$. Right: Mapping of the principal branch from $-s$ to $w(s) = -\sqrt{-s}$ (i.e., the rotated inverse of s^2). Here the branch cut is along the positive σ axis.*

This concept of analytic inverses becomes important only when the function is multi-valued. For example, if $w(s) = s^2$ then $s(w) = \pm\sqrt{w}$ is multivalued. Riemann dealt with such extensions with the concept of a *branch-cut* with multiple *sheets*, labeled by a sheet number. Each sheet describes an analytic function (Taylor series) that converges within some RoC, having a radius out to the nearest pole. Thus Riemann's branch cuts and sheets explicitly deal with the need to define unique single-valued inverses of multi-valued functions. Since the square root function has two overlapping regions, corresponding to the $\pm$ due to the radical, there must be two connected regions, sort of like mathematical Siamese-twins: distinct, yet the same.

Branch cuts emanate and terminate at *branch points*, defined as singularities (poles), that can even have fractional degree, as for example $1/\sqrt{s}$, and terminate at one of the matching roots, which includes the possibility of ∞.[78] For example, suppose that in the neighborhood of the pole, at s_o the function is

$$f(s) = \frac{w(s)}{(s - s_o)^k},$$

where $w, s, s_o \in \mathbb{C}$ and $k \in \mathbb{Q}$. When $k = 1$, $s_o = \sigma_o + \omega_o\jmath$ is a *first degree "simple pole"*, having degree 1 in the s plane, with *residue* $w(s_o)$.[79] Typically the order (and degree) are positive integers, but fractional degrees and orders are not uncommon in modern Engineering applications (Kirchhoff, 1868; Lighthill, 1978). Here we shall allow both the degree and order to be fractional ($\in \mathbb{F}$). When $k \in \mathbb{F} \subset \mathbb{R}$ (a real fraction), it defines the *degree* of a fractional pole. In such cases

[78] This presumes that poles appear in pairs, one of which may be at ∞.

[79] We shall refer to the *order* of a derivative, or differential equation, and the *degree* of a polynomial, as commonly used in Engineering applications (`https://en.wikipedia.org/wiki/Differential_equation#Equation_order`).

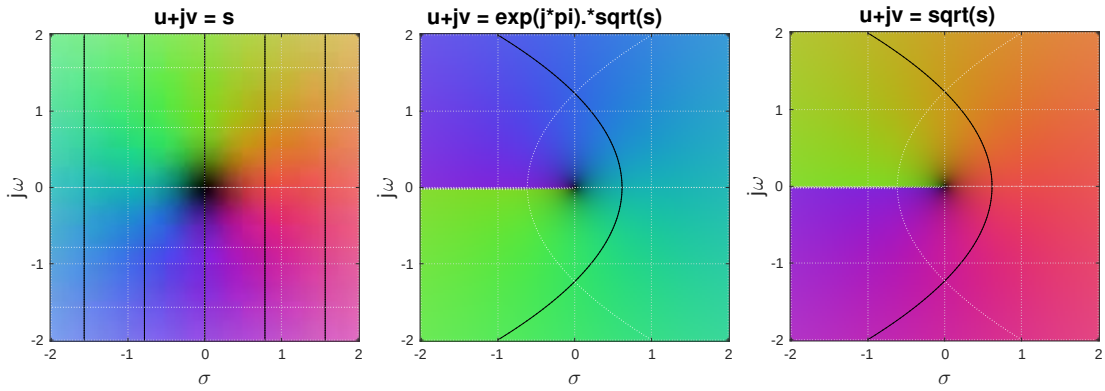


Figure 1.28: *Colorized plots of two sheets of $w(s) = \pm\sqrt{s}$ with the branch cut at $\theta = \pi$.* **Left:** *Color map reference plane ($z = s$).* **Center:** *$e^{\pi\jmath}\sqrt{s} = -\sqrt{s}$. This sheet was generated by rotating w by 180°. This should be compared to the right panel of Fig. 1.27 which shows $-\sqrt{-s}$, and to the upper sheet of Fig. 1.26. Note how the panel on the right of Fig. 1.27 matches the right half of s (purple = -90°, yellow/green = +90°) while the middle panel above comes from the left side of s (green to purple). The center panel is green at -180°, and purple at +180°, which matches the right panel at $\pm$180°respectively (i.e., $e^{\pi\jmath}\sqrt{s}$).* **Right:** *The lower sheet of Fig. 1.26.*

there must be a branch cut, of degree k. For example, if $k = 1/2$, the singularity (branch cut) is of degree 1/2, and there are two Riemann sheets, as shown in Figs. 1.26 and 1.27.

An important example is the Bessel function

$$\delta(t) + \frac{1}{t}J_1(t)u(t) \leftrightarrow \sqrt{s^2+1},$$

which is related to the solution to the wave equation in two-dimensional cylindrical coordinates (Table F.3, p. 210). Bessel functions are the solutions to guided acoustic waves in round pipes, or surface waves on the earth (seismic waves), or waves on the surface of a pond.

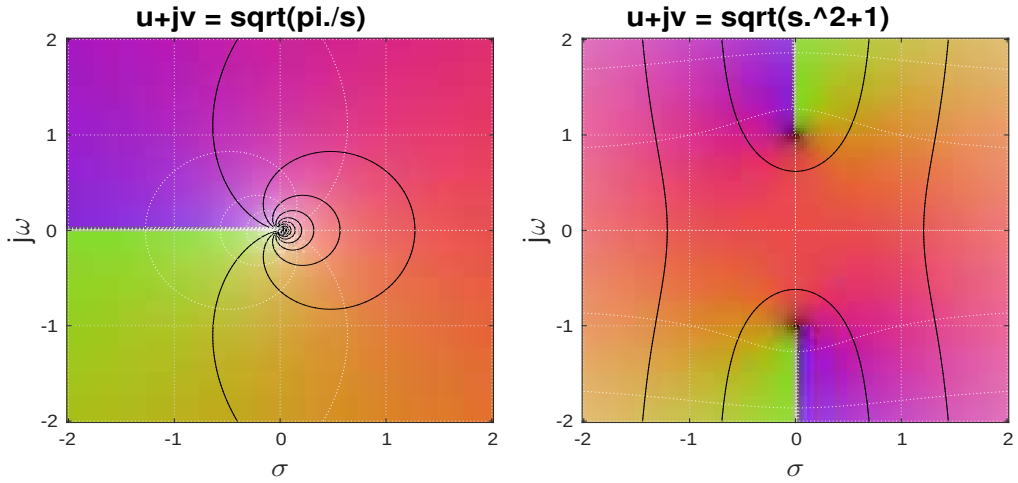


Figure 1.29: *Colorized plot of two $\mathcal{LT}$ pairs: Left: $\sqrt{\pi/s} \leftrightarrow u(t)/\sqrt{t}$. Right: $\sqrt{s^2+1} \leftrightarrow \delta(t) + \frac{1}{t}J_1(t)u(t)$.*

It is important to understand that the function is analytic on the branch cut, but not at the branch point. One is free to move the branch cut (at will). It does not need to be on a line: it could be cut in almost any connected manner, such as a spiral. The only rule is that it must start and stop at the matching branch points, or at ∞, which must have the same degree.

There are a limited number of possibilities for the degree, $k \in \mathbb{Z}$ or $\in \mathbb{F}$. If the degree is drawn from $\mathbb{R} \notin \mathbb{F}$, the pole cannot not have a residue. According to the definition of the residue, $k \in \mathbb{F}$

will not give a residue. But there remains open the possibility of generalizing the concept of the Riemann integral theorem, to include $k \in \mathbb{F}$. One way to do this is to use the logarithmic derivative which renders fractional poles to simple poles with fractional residues.

If the singularity had an irrational degree ($k \in \hat{\mathbf{x}}$), the branch cut has the same "irrational degree." Accordingly there would be an infinite number of Riemann sheets, as in the case of the log function. An example is $k = \pi$, for which

$$F(s) = \frac{1}{s^\pi} = e^{-\log(s^\pi)} = e^{-\pi \log(s)} = e^{-\pi \log(\rho)} e^{-\pi \theta \jmath},$$

where the domain is expressed in polar coordinates $s = \rho e^{\theta \jmath}$. When $k \in \mathbb{F}$ it may be maximally close (e.g., $\pi_{152} = 881$ and $\pi_{153} = 883$)[80] the branch cut could be very subtle (it could even be unnoticed), but it would have a significant impact on the nature of the function, and of course, on the inverse Laplace transform.

Multivalued functions: The two basic functions we review, to answer the questions about multi-valued functions and branch cuts, are $w(s) = s^2$ and $w(s) = e^s$, along with their inverse functions $w(s) = \sqrt{s}$ and $w(s) = \log(s)$. For uniformity we shall refer to the *complex abscissa* ($s = \sigma + \omega \jmath$) and the *complex ordinate* $w(s) = u + v\jmath$. When the complex abscissa and domain are swapped, by taking the inverse of a function, multi-valued functions are a common consequence. For example, $f(t) = \sin(t)$ is single valued, and analytic in t, thus has a Taylor series. The inverse function $t(f)$ is not so fortunate as it is multivalued.

The modern terminology is the *domain* and *range*, or alternatively the *co-domain*.[81]

Log function: Next we discuss the multi-valued nature of the log function. In this case there are an infinite number of Riemann sheets, not well captured by Fig. 1.22 (p. 96), which only displays the principal sheet. However if we look at the formula for the log function, the nature is easily discerned. The abscissa s may be defined as multi-valued since

$$s_k = re^{2\pi k \jmath} e^{\theta \jmath}.$$

Here we have extended the angle of s by $2\pi k$, where k is the sheet index $\in \mathbb{Z}$. Taking the log

$$\log(s) = \log(r) + (\theta + 2\pi k)\jmath.$$

When $k = 0$ we have the *principal value* sheet, which is zero when $s = 1$. For any other value of k $w(s) \neq 0$, even when $r = 1$, since the angle is not zero, except for the $k = 0$ sheet.

1.4.5 Lec 27: Three Cauchy Integral Theorems

Cauchy's theorems for integration in the complex plane

There are three basic definitions related to Cauchy's integral formula. They are closely related, and can greatly simplify integration in the complex plane.

[80] Since there are no even primes other than $\pi_1 = 2$, the minimum difference is 2. Out of 10^6 primes, 5 have a spacing of 80, with a uniform distribution on a log scale.

[81] The best way to create confusion is to rename something. The confusion grows geometrically with each renaming. I suspect that everyone who cares knows the terms abscissa and ordinate, and some fraction know the equivalent terms *domain* and *range*.

1. **Cauchy's (Integral) Theorem:**

$$\oint_{\mathcal{C}} F(s)ds = 0, \tag{1.104}$$

if and only if $F(s)$ is complex-analytic inside of a simple closed curve $\mathcal{C}$ (Boas, 1987, p. 45),(Stillwell, 2010, 319). The FTCC (Eq. 1.87) says that the integral only depends on the end points if $F(s)$ is complex-analytic. By closing the path (contour $\mathcal{C}$) the end points are the same, thus the integral must be zero, as long as $F(s)$ is complex analytic.

2. **Cauchy's Integral Formula:**

$$\frac{1}{2\pi j}\oint_{\mathcal{B}} \frac{F(s)}{s-s_o}ds = \begin{cases} F(s_o), s_o \in \mathcal{C} \text{ (inside)} \\ 0, s_o \notin \mathcal{C} \text{ (outside).} \end{cases} \tag{1.105}$$

Here $F(s)$ is required to be analytic everywhere within (and on) the contour $\mathcal{C}$ (Greenberg, 1988, p. 1200),(Boas, 1987, p. 51), (Stillwell, 2010, p. 220). The value $F(s_o) \in \mathbb{C}$ is the *residue* of the pole s_o of $F(s)/(s-s_o)$.

3. **The (Cauchy) Residue Theorem** (Greenberg, 1988, p. 1241), (Boas, 1987, p. 73)

$$\oint_{\mathcal{C}} F(s)ds = 2\pi j \sum_{k=1}^{K} c_k = \sum_{k=1}^{K} \oint \frac{F(s)}{s-s_k}ds, \tag{1.106}$$

where the *residues* $c_k \in \mathbb{C}$, corresponding to the kth poles of $f(s)$, enclosed by the contour $\mathcal{C}$. By the use of Cauchy's integral formula, the last form of the residue theorem is equivalent to the residue form.[82]

How to calculate the residue: The case of first degree poles, while special, is important, since the Brune impedance only allows simple poles and zeros, increasing the utility of this special case. The residues for simple poles are $F(s_k)$, which is complex analytic in the neighborhood of the pole, but not at the pole.

Consider the function $f(s) = F(s)/(s-s_k)$, where we have factored $f(s)$ to isolate the first-order pole at $s = s_k$, with $F(s)$ analytic at s_k. Then the residue of the poles at $c_k = F(s_k)$. This coefficient is computed by removing the singularity, by placing a zero at the pole frequency, and taking the limit as $s \to s_k$, namely

$$c_k = \lim_{s \to s_k} [(s-s_k)F(s)] \tag{1.107}$$

(Greenberg, 1988, p. 1242), (Boas, 1987, p. 72).

When the pole is an N^{th} degree, the procedure is much more complicated, and requires taking $N-1$ order derivatives of $f(s)$, followed by the limit process (Greenberg, 1988, p. 1242). Higher degree poles are rarely encountered: thus it is good to know that this formula exists, but perhaps it is not worth the effort to memorize it.

Summary and examples: These three theorems, all attributed to Cauchy, collectively are related to the fundamental theorems of calculus. Because the names of the three theorems are so similar, they are easily confused.

[82] This theorem is the same as a 2D version of Stokes's thm (citations).

1. In general it makes no sense to integrate *through* a pole, thus the poles (or other singularities) must not lie on $\mathcal{C}$.

2. The Cauchy integral theorem (Eq. 1.104), follows trivially from the fundamental theorem of complex calculus (Eq. 1.87, p. 106), since if the integral is independent of the path, and the path returns to the starting point, the closed integral must be zero. Thus Eq. 1.104 holds when $F(s)$ is complex analytic within C.

3. Since the real and imaginary parts of every complex analytic function obey Laplace's equation (Eq. 1.94, p. 108), it follows that every closed integral over a Laplace field, i.e., one defined by Laplace's equation, must be zero. *In fact this is the property of a conservative system,* corresponding to many physical systems. If a closed box has fixed potentials on the walls, with any distribution whatsoever, and a point charge (i.e., an electron) is placed in the box, then a force equal to $F = qE$ is required to move that charge, and thus work is done. However if the point is returned to its starting location, the net work done is zero.

4. Work is done in charging a capacitor, and energy is stored. However when the capacitor is discharged, all of the energy is returned to the load.

5. Soap bubbles and rubber sheets on a wire frame obey Laplace's equation.

6. These are all cases where the fields are Laplacian, thus closed line integrals must be zero. Laplacian fields are commonly observed because they are so basic.

7. We have presented the impedance as the primary example of a complex analytic function. Physically, every impedance has an associated stored energy, and every system having stored energy has an associated impedance. This impedance is usually defined in the frequency s domain, as a force *over* a flow (i.e., voltage over current). The power $\mathcal{P}(t)$ is defined as the force *times* the flow and the energy $\mathcal{E}(t)$ as the time integral of the power

$$\mathcal{E}(t) = \int_{-\infty}^{t} \mathcal{P}(t)dt, \tag{1.108}$$

which is similar to Eq. 1.85 (p. 106) [see Section H.2.1, Eq. 1.60 (p. 86)]. In summary, impedance and power and energy are all fundamentally related.

1.4.6 Lec 28: Cauchy Integral Formula & Residue Theorem

The Cauchy integral formula (Eq. 1.105) is an important extension of the Cauchy integral theorem (Eq. 1.104) in that a pole has been explicitly injected into the integrand at $s = s_o$. If the pole location is outside of the curve $\mathcal{C}$, the result of the integral is zero, in keeping with Eq. 1.104. When the pole is inside of $\mathcal{C}$, the integrand is no longer complex analytic at the enclosed pole. When this pole is simple, the residue theorem applies. By a manipulation of the contour in Eq. 1.105, the pole can be isolated with a circle around the pole, and then taking the limit, the radius may be taken to zero, in the limit, isolating the pole.

For the related *Cauchy residue theorem* (Eq. 1.106) the same result holds, except it is assumed that there are K simple poles in the function $F(s)$. This requires the repeated application of Eq. 1.105, K times, so it represents a minor extension of Eq. 1.105. The function $F(s)$ may be written as $f(s)/P_K(s)$, where $f(s)$ is analytic in $\mathcal{C}$ and $P_K(s)$ is a polynomial of degree K, with all of its roots $s_k \in \mathcal{C}$.

Non-integral degree singularities: The key point is that this theorem applies when $n \in \hat{\mathbf{x}}$, including fractionals $n \in \mathbb{F}$, but for these cases the residue is always zero, since by definition, the residue is the amplitude of the $1/s$ term (Boas, 1987, p. 73).

Examples:

1. When $n \in \mathbb{F}$ (e.g., $n = 2/3$), the residue is, by definition, zero.
2. The function $1/\sqrt{s}$ has a zero residue (apply the definition of the residue Eq. 1.107).
3. When $n \neq 1 \in \hat{\mathbf{x}}$, the residue is, by definition, zero.
4. When $n = 1$, the residue is given by Eq. 1.107).
5. This method is necessary when computing the inverse Laplace transform.

1.4.7 Lec 29: Inverse Laplace transform & Cauchy residue theorem

The inverse Laplace transform Eq. 1.78 transforms a function of complex frequency $F(s)$ and returns a causal function of time $f(t)$

$$f(t) \leftrightarrow F(s),$$

where $f(t) = 0$ for $t < 0$. Examples are provided in Table F.3 (p. 210). We next discuss the details of finding the inverse transform by use of the Cauchy residue theorem, and how the causal requirement $f(t < 0) = 0$ comes about.

The integrand of the inverse transform is $F(s)e^{st}$ and the limits of integration are $-\sigma_o \mp \omega\jmath$. To find the inverse we must close the curve, at infinity, and show that the integral at $\omega\jmath \to \infty$. There are two ways to close these limits – to the right $\sigma > 0$ (RHP), and to the left $\sigma < 0$ (LHP) – but there needs to be some logical reason for this choice. That logic is the sign of t. For the integral to converge the term e^{st} must go to zero as $\omega \to \infty$. In terms of the real and imaginary parts of $s = \sigma + \omega\jmath$, the exponential may be rewritten as $e^{\sigma t}e^{\jmath\omega t}$. Note that both t and ω go to ∞. So it is the interaction between these two limits that determines how we pick the closure, RHP vs. LHP.

Case for causality ($t < 0$): Let us first consider negative time, including $t \to -\infty$. If we were to close $\mathcal{C}$ in the left half-plane ($\sigma < 0$), then the product σt is positive ($\sigma < 0, t < 0$, thus $\sigma t > 0$). In this case as $\omega \to \infty$, the closure integral $|s| \to \infty$ will diverge. Thus we may not close in the LHP for negative time. If we close in the RHP $\sigma > 0$ then the product $\sigma t < 0$ and e^{st} will go to zero as $\omega \to \infty$. This then justifies closing the contour, allowing for the use the Cauchy theorems.

If $F(s)$ is analytic in the RHP, the FTCC applies, and the resulting $f(t)$ must be zero, and the inverse Laplace transform must be causal. This argument holds for any $F(s)$ that is analytic in the RHP ($\sigma > 0$).

Case of unstable poles: An important but subtle point arises: If $F(s)$ has a pole in the RHP, then the above argument still applies if we pick σ_o to be to the right of the RHP pole. This means that the inverse transform may still be applied to unstable poles (those in the RHP). This explains the need for the σ_o in the limits. If $F(s)$ has no RHP poles, then $\sigma_o = 0$ is adequate, and this factor may be ignored.

Case for zero time ($t = 0$): When time is zero, the integral does not, in general, converge, leaving $f(t)$ undefined. This is most clear in the case of the step function $u(t) \leftrightarrow 1/s$, where the integral may not be closed, because the convergence factor $e^{st} = 1$ is lost for $t = 0$.

The fact that $u(t)$ does not exist at $t = 0$ explains the Gibbs phenomenon in the inverse Fourier transform. At times where a jump occurs, the derivative of the function does not exist, and thus the time response function is not analytic. The Fourier expansion cannot converge at places where the function is not analytic. A low pass filter may be used to smooth the function, but at the cost of temporal resolution. Forcing the function to be analytic at the discontinuity, by smoothing the jumps, is an important computational method.

1.4.8 Lec 30: Inverse Laplace transform ($t > 0$)

Case of $t > 0$: Next we investigate the convergence of the integral for positive time $t > 0$. In this case we must close the integral in the LHP ($\sigma < 0$) for convergence, so that $st < 0$ ($\sigma \leq 0$ and $t > 0$). When there are poles on the $\omega\jmath = 0$ axis, $\sigma_o > 0$ assures convergence by keeping the on-axis poles inside the contour. At this point the Cauchy residue theorem (Eq. 1.106) is relevant. If we restrict ourselves to simple poles (as required for a Brune impedance), the residue theorem may be directly applied.

The most simple example is the step function, for which $F(s) = 1/s$ and thus

$$u(t) = \oint_{\text{LHP}} \frac{e^{st}}{s} \frac{ds}{2\pi\jmath} \leftrightarrow \frac{1}{s},$$

which is a direct application of the Cauchy Residue theorem, Eq. 1.106 (p. 119). The forward transform of $u(t)$ is straight forward, as discussed in Section 1.3.14 (p. 98). This is true of most if not all of the elementary forward Laplace transforms. In these cases, causality is built into the integral by the limits, so is not a result, as it must be in the inverse transform. An interesting problem is proving that $u(t)$ is not defined at $t = 0$.

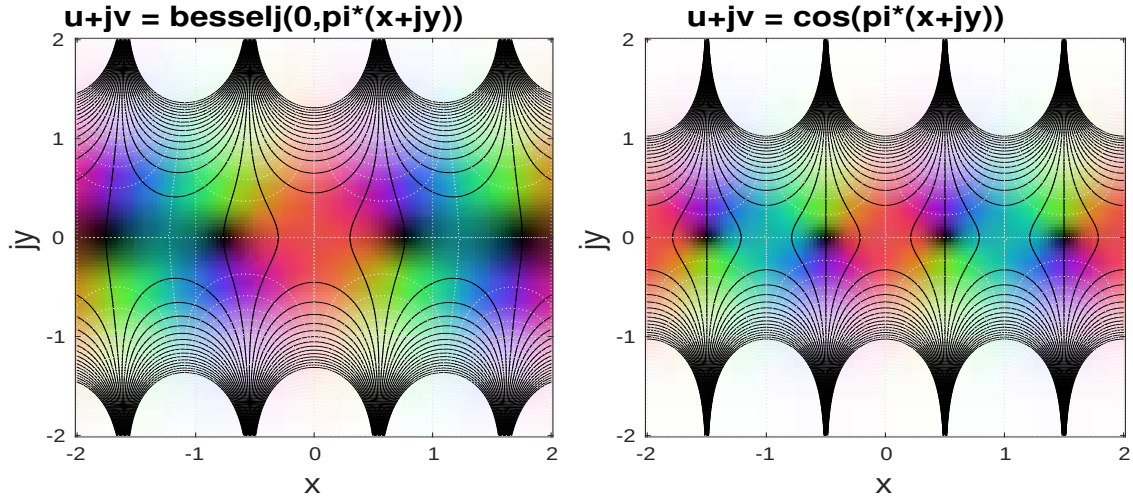


Figure 1.30: *Left: Colorized plot of $w(z) = J_o(\pi z)$. The first zero is at 2.405, and thus appears at $0.7655 = 2.405/\pi$, somewhat larger than the root of $\cos(\pi/2)$. Right: Note the similarity to $w(z) = sin(z)$. The $\mathcal{LT}$'s have similar characteristics, as documented in Table F.3 (p. 210).*

The inverse Laplace transform of $F(s) = 1/(s + 1)$ has a residue of 1 at $s = -1$, thus that is

the only contribution to the integral. More demanding cases are Laplace transform pairs

$$\frac{1}{\sqrt{t}}u(t) \leftrightarrow \sqrt{\frac{\pi}{s}} \quad \text{and} \quad J_o(t)u(t) \leftrightarrow \frac{1}{\sqrt{s^2+1}},$$

and more on p. 210. Many of these are easily proved in the forward direction, but are much more difficult in the inverse direction, due to the properties at $t = 0$, unless of course the residue theorem (Eq. 1.106, p. 119) is invoked. The last $\mathcal{LT}$-pair helps us understand the basic nature of the Bessel functions $J_0(z)$, and $H_0^{(1)}(z^2)$, with a branch cut along the negative axis (see Fig. H.2, p. 259).

Some open questions: Without the use of the CRT (Eq. 1.106) it is difficult to see how to evaluate the inverse Laplace transform of $1/s$ directly. For example, how does one show that the above integral is zero for negative time (or that it is 1 for positive time)? The CRT neatly solves this difficult problem by the convergence of the integral for negative and positive time. Clearly the continuity of the integral at $\omega \to \infty$ plays an important role. Perhaps the Riemann sphere plays a role in this, that has not yet been explored.

1.4.9 Lec 31: Properties of the LT (e.g., Linearity, convolution, time-shift, modulation, etc.)

As shown in the table of Laplace transforms, there are integral (i.e., integration, not integer) relationships, or properties, that are helpful to identify. The first of these is a definition, not a property:[83]

$$f(t) \leftrightarrow F(s).$$

When taking the LT, the time response is given in lower case (e.g., $f(t)$) and the frequency domain transform is denoted in upper case (e.g., $F(s)$). It is required, but not always explicitly specified, that $f(t < 0) = 0$, that is, the time function must be *causal* (P1: Section 1.3.15).

Linearity: A key property so basic that it almost is forgotten, is the linearity property of the LT. These properties are summarized as P2 of Section 1.3.15, p. 101).

Convolution property: One of the most basic and useful properties is that the product of two LTs in frequency, results in convolution in time

$$f(t) \star g(t) = \int_0^t f(\tau)g(t-\tau)d\tau \leftrightarrow F(s)G(s),$$

where we use the $\star$ operator as a shorthand for the convolution of two time functions.

A key application of convolution is filtering, which takes on many forms. The most basic filter is the *moving average*, the moving sum of data samples, normalized by the number of samples. Such a filter has very poor performance. It also introduces a delay of half the length of the average, which may or may not constitute a problem, depending on the application. Another important example is a low-pass filter that removes high frequency noise, or a notch filter that removes line-noise (i.e., 60 [Hz] in the US, and its 2nd and 3d harmonics, 120 and 180 [Hz]). Such noise is typically a result of poor grounding and ground loops. It is better to solve the problem at its root than to remove it with a notch filter. Still, filters are very important in engineering.

[83] Put this notional property in Appendix A.

By taking the LT of the convolution we can derive this relationship

$$\begin{aligned}\int_0^\infty [f(t) \star g(t)]e^{-st}dt &= \int_{t=0}^\infty \left[\int_0^t f(\tau)g(t-\tau)d\tau\right] e^{-st}dt \\ &= \int_0^t f(\tau)\left(\int_{t=0}^\infty g(t-\tau)e^{-st}dt\right)d\tau \\ &= \int_0^t f(\tau)\left(e^{-s\tau}\int_{t'=0}^\infty g(t')e^{-st'}dt'\right)d\tau \\ &= G(s)\int_0^t f(\tau)e^{-s\tau}d\tau \\ &= G(s)F(s).\end{aligned}$$

We first encountered this relationship in Section 1.3.5 (p. 76)) in the context of multiplying polynomials, which was the same as convolving their coefficients. Hopefully the parallel is obvious. In the case of polynomials, the convolution was discrete in the coefficients, and here it is continuous in time. But the relationships are the same.

Time-shift property: When a function is time-shifted by time T_o, the LT is modified by e^{sT_o}, leading to to the property

$$f(t-T_o) \leftrightarrow e^{-sT_o}\,F(s).$$

This is easily shown by applying the definition of the LT to a delayed time function.

Time derivative: The key to the eigen-function analysis provided by the LT is the transformation of a time derivative on a time function, that is,

$$\frac{d}{dt}f(t) \leftrightarrow sF(s).$$

Here s is the eigen-value corresponding to the time derivative of e^{st}. Given the definition of the derivative of e^{st} with respect to time, this definition seems trivial. Yet that definition was not obvious to Euler. It needed to be extended to the space of complex analytic function e^{st}, which did not happen until at least Riemann (1851).

Given a differential equation of order K, the LT results in a polynomial in s, of degree K. It follows that this LT property is the corner-stone of why the LT is so important to scalar differential equations, as it was to the early analysis of Pell's equation and the Fibonacci sequence, as presented in earlier chapters. This property was first uncovered by Euler. It is not clear if he fully appreciated its significance, but by the time of his death, it certainly would have been clear to him. Who first coined the terms *eigen-value* and *eigen-function*? The word *eigen* is a German word meaning *of one*.

Initial and final value theorems: There are much more subtle relations between $f(t)$ and $F(s)$ that characterize $f(0^+)$ and $f(t \to \infty)$. While these properties can be very important in certain application, they are beyond the scope of the present treatment. These relate to so-called *initial value theorems*. If the system under investigation has potential energy at $t = 0$, then the voltage (velocity) need not be zero for negative time. An example is a charged capacitor or a moving mass. These are important situations, but better explored in a more in-depth treatment.

1.4.10 Solving differential equations: Method of Frobenius

Many differential equations may be solved by assuming a power series (i.e., Taylor series) solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n \tag{1.109}$$

with coefficients $c_n \in \mathbb{C}$. The *method of Frobenius* is quite general (Greenberg, 1988, p. 193).

Example: When a solution of this form is substituted into the differential equation, a recursion relation in the coefficients results. For example, if the equation is

$$y''(x) = \lambda^2 y(x)$$

the recursion is $c_n = c_{n-1}/n$. The resulting equation is

$$y(x) = e^{\lambda x} = x^0 \sum_{n}^{\infty} \frac{1}{n!} x^n,$$

namely $c_n = 1/n!$, thus $nc_n = 1/(n-1)! = c_{n-1}$.

Exercise: Find the recursion relation for Bessel's equation of order ν

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2) y(x) = 0.$$

Solution: If we assume a complex analytic solution of the form Eq. 1.109, we find the Bessel recursion relation for coefficients c_k to be

$$c_k = -\frac{1}{k(k+2\nu)} c_{k-2}$$

(Greenberg, 1988, p. 231).

1.4.11 Lec 32: Review for Exam III

1.5 Vector Calculus (Stream 3b)

1.5.1 Lec 33 Properties of Fields and potentials

Before we can define the vector operations $\nabla(), \boldsymbol{\nabla}\cdot(), \boldsymbol{\nabla}\times(), \nabla^2()$, we must define the objects they operate on, *scalar*[84] and *vector*[85] *fields*.[86] The word *field* has two very different meanings, a mathematical definition, which defines an algebraic structure,[87] and a physical one, discussed next.

Ultimately we wish to integrate in $\in \mathbb{R}^3, \mathbb{R}^n$ and $\in \mathbb{C}^n$. Integration is quantified by several fundamental theorems of calculus, each about integration.[88]

Scalar fields: We use the term *scalar field* interchangeably with *analytic* in a connected region of the spatial vector $\boldsymbol{x} = [x, y, z]^T \in \mathbb{R}^3$. In mathematics, functions that are piece-wise differentiable are called *smooth*, which is quite different from analytic. Since an analytic function may be written as a power series, it is both *single valued* and infinitely differentiable. A smooth function has at least one or more derivatives, but need not be single valued.

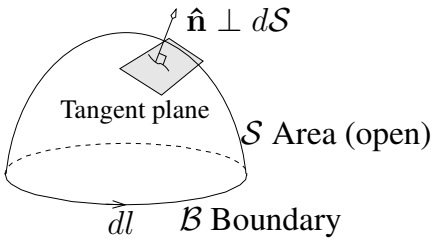
Example: The function $tu(t)$ is smooth, and has one derivative $d/dt\,(tu(t)) = u(t)$, but does not have a second derivative at $t = 0$. Thus $tu(t)$ is not analytic at $t = 0$. The function $\phi(r) = \pm\sqrt{x^2 + y^2 + z^2}$ is analytic everywhere except at $r = 0$, yet is double-valued, thus requiring a branch cut to fully describe it.

Initially we simplify the field by limiting the definition to an *analytic surface* $\mathcal{S}(\boldsymbol{x})$, as shown in Fig. 1.31, having height $z(x, y) \in \mathbb{R}$, as a function of $x, y \in \mathbb{R}^2$ (a plane)

$$z(x, y, t) = \phi(x, y, t),$$

where $z(x, y, t)$ describes a surface that is analytic in $\boldsymbol{x}$. We must allow the field to optionally be a single-valued function of time $t \in \mathbb{R}$, since that is the nature of the solutions of the equations we wish to solve. Examples will be given below.

For example, picture of a smooth single-valued mountain (Fig. 1.31), having isoclines (lines on a surface with constant slope).[89]



This figure shows an open surface $\mathcal{S}(\boldsymbol{x})$, formed by bifurcating the closed volume $\mathcal{V}$ with a plane (not shown), creating the boundary $\mathcal{B}$. Only the upper half of the bifurcated volume is shown. A unit vector $\hat{\boldsymbol{n}}$ is defined as perpendicular ($\perp$) to the shaded tangent plane.

Figure 1.31: *Figure defining the analytic open surface $\mathcal{S}$, having boundary $\mathcal{B}$.*

[84] https://en.wikipedia.org/wiki/Scalar_field
[85] https://en.wikipedia.org/wiki/Vector_field
[86] https://www.grc.nasa.gov/www/k-12/airplane/vectors.html
[87] https://en.wikipedia.org/wiki/Field_(mathematics)
[88] https://en.wikipedia.org/wiki/Line_integral
[89] https://en.wikipedia.org/wiki/Euler_method

Vector fields: A *vector field* is composed of three scalar fields. For example, the electric field used in Maxwell's equations $\boldsymbol{E}(\boldsymbol{x},t) = [E_x, E_y, E_z]^T$ [V/m] has three components, each of which is a scalar field. When the magnetic flux vector $\boldsymbol{B}(\boldsymbol{x})$ is static (P5, p. 101), the potential $\phi(\boldsymbol{x})$ [V] uniquely defines $\boldsymbol{E}(\boldsymbol{x},t)$, via the gradient.

$$\boldsymbol{E}(\boldsymbol{x},t) = -\nabla\phi(\boldsymbol{x},t). \quad [V/m] \tag{1.110}$$

The electric force on a charge q is $\boldsymbol{F} = q\boldsymbol{E}$.

Scalar potentials: The above discussion shows the importance of the potential in defining a vector field (Eq. 1.110). The difference between a potential and a scalar field is that potentials have units, and thus have a physical meaning. Scalar potentials (i.e., voltage $\phi(\boldsymbol{x},t)$ [V], temperature $T(\boldsymbol{x},t)$ [°C] and pressure $\varrho(\boldsymbol{x},t)$ [Pascals]) are examples of physical scalar fields. All potentials are composed of scalar fields, but not all scalar fields are potentials. For example $E_y(\boldsymbol{x},t) = \hat{\mathbf{y}}\cdot\boldsymbol{E}(\boldsymbol{x},t)$ [V/m], the $\hat{\mathbf{y}}$ component of $\boldsymbol{E}$, is not a potential. While $\nabla\boldsymbol{E}_y$ is mathematically reasonable, as the gradient of one component of a vector field, it has no physical meaning (as best I know).

Example: The step function $u(t) \leftrightarrow 1/s$ is not analytic at $t = 0$ in time. In the Laplace frequency domain, $1/s$ is not complex analytic at $s = 0$, due to the pole.

Vector potentials: Vector potentials, like scalar potentials, are vector fields with a physical meaning (have physical units). Thus they are more complicated than scalar potentials because they are composed of three scalar fields, rather than one. But in every other way the are just as important. This follows from the *fundamental theorem of vector calculus* or simply *Helmholtz' decomposition theorem.*

If you found it useful to set up (analyze) problems using a potential, such as voltage, and then take the gradient to find $\boldsymbol{E}(\boldsymbol{x},t)$, the same utility holds when using the vector potential to determine the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$. When operating on a scalar potential we use a gradient, whereas for the vector potential, we must operate with the curl. In Eq. 1.110 we assumed that the magnetic flux vector $\boldsymbol{B}(\boldsymbol{x})$ was static, thus $\boldsymbol{E}(\boldsymbol{x},t)$ is the gradient of the time-dependent voltage $\phi(\boldsymbol{x},t)$. However when the magnetic field is dynamic (*not* static), Eq. 1.110 is not valid due to *magnetic induction*: A voltage induced into a loop of wire is proportional to the time-varying flux cutting across that loop of wire. This is known as *Ampere's law*. In the static case the induced voltage is zero. Lets explore just how useful the potentials can be in terms of quantifying Maxwell's equations. When the magnetic field is time-varying, Eq. 1.110 must be extended to include both the scalar $\phi(\boldsymbol{x},t)$ and vector potentials $\boldsymbol{A}(\boldsymbol{x},t)$

$$\boldsymbol{E}(\boldsymbol{x},t) = -\nabla\phi(\boldsymbol{x},t) - \frac{\partial\boldsymbol{A}}{\partial t}, \quad \text{[V/m]} \tag{1.111}$$

(Sommerfeld, 1952, p. 146), (Feynman, 1970b, p. 18-10).

The magnetic flux $\boldsymbol{B}(\boldsymbol{x},t)$ may also be written in terms of potential

$$\boldsymbol{B}(x,t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x},t) + \epsilon_o\frac{\partial}{\partial t}\nabla\phi, \quad \text{[Wb/m]} \tag{1.112}$$

where the term in red is a wild guess on an additional electrical current term (that needs justification) (Jackson, 1967, p. 179-181), (Feynman, 1970b, p. 18-9).[90] Equation 1.112 is equivalent

[90] To be consistent with the Helmholtz theorem, I boldly speculate that the expression for $\boldsymbol{B}$ is missing a term, that depends on the electrical potential $\phi(x,t)$, such as $\nabla\partial_t\Phi$. Obviously the units must agree, and $\boldsymbol{\nabla}\cdot B = 0$ must hold. The units of $\boldsymbol{B}$ are [webers/m] and $\boldsymbol{A}$ are [Wb/m^2]. The units of permittivity ϵ_o are [Farads/m].

to

$$\boldsymbol{H}(x,t) = \frac{1}{\mu_o}\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{x},t)+\frac{\epsilon_o}{\mu_o}\frac{\partial}{\partial t}\nabla\phi. \quad \text{[A/m]}$$

Thus the electric field strength includes both the scalar potential $\phi(\boldsymbol{x},t)$ and magnetic flux vector potential $\boldsymbol{A}(\boldsymbol{x},t)$ components, while the magnetic field strength only depends on the magnetic potential.[91]

To better understand the difference between scalar and vector potentials, I recommend the first chapters of Feynman (1970b, Chapter 1.1). The two discussions have a similar presentation style.

1.5.2 Lec 34: Gradient ∇, divergence $\nabla\cdot$, curl $\boldsymbol{\nabla}\times$, and Laplacian ∇^2

There are three key vector differential operators that are required for understanding linear partial differential equations, such as the wave and diffusion equations. All of these begin with the ∇ operator:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$$

The official name of this operator is *nabla*. It has three basic uses: 1) the *gradient* of a scalar field, the 2) *divergence* of a vector field, and 3) the *curl* of a vector field. If properly noted, the shorthand notation $\nabla\phi(\boldsymbol{x},t) = (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_x + \hat{\mathbf{z}}\partial_x)\phi(\boldsymbol{x},t)$ is convenient.

Table 1.4: *The three vector operators manipulate scalar and vector fields, as indicated here. The* gradient *converts scalar fields into vector fields. The divergence maps vector fields to scalar fields. Finally the curl maps vector fields to vector fields. It is helpful to have a name for second-order operators (e.g., DoG, GoD: mnemonics defined in Sect. 1.5.13, p. 160).*

Name	Input	Output	Operator	Mnemonic
Gradient	Scalar	**Vector**	$\nabla()$	grad
Divergence	**Vector**	Scalar	$\nabla\cdot()$	div
Curl	**Vector**	**Vector**	$\nabla\times()$	curl
Laplacian	Scalar	Scalar	$\boldsymbol{\nabla}\cdot\nabla = \nabla^2()$	DoG
Vector Laplacian	**Vector**	**Vector**	$\nabla\boldsymbol{\nabla}\cdot = \boldsymbol{\nabla}^2()$	GoD

Basic differential vector operator definitions: The basic definitions of each of the vector operators are summarized in Table 1.4.

Gradient: The *gradient* transforms a scalar field into vector field. In $\mathbb{R}^3$ the gradient of a scalar field $\nabla\phi(\boldsymbol{x})$ is defined as

$$\nabla\phi(\boldsymbol{x}) = \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\phi(\boldsymbol{x}), = \frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}}. \tag{1.113}$$

The gradient may be factored into a unit vector $\hat{\boldsymbol{n}}$, as defined in Fig. 1.31, defining the direction of the gradient, and the gradient's length $||\nabla()||$, defined in terms of the norm. Thus the gradient of

[91] As will be discussed in Section 1.5.14, based on a symmetry argument, it seems likely that Eq. 1.112 should include a current, such as a capacitive $\nabla\dot{\phi}(\boldsymbol{x},t)$.

$\phi(\boldsymbol{x})$ may be written in "polar coordinates" as $\nabla\Phi(\boldsymbol{x}) = ||\nabla(\Phi)||\,\hat{\boldsymbol{n}}$, useful for defining $\hat{\boldsymbol{n}}$ as

$$\hat{\boldsymbol{n}} = \frac{\nabla(\Phi(\boldsymbol{x}))}{||\nabla(\Phi)||}.$$

Important examples of the use of the gradient include the electric field vector $\boldsymbol{E}(\boldsymbol{x}) = -\nabla\phi(\boldsymbol{x})$ [V/m], which is the gradient of a voltage [V], and the force density $\boldsymbol{f}(\boldsymbol{x}) = -\nabla\varrho(\boldsymbol{x})$ [N/m], which is the gradient of a pressure [Pa].

Divergence: The *divergence* of a vector field results in a scalar field. For example, the divergence of the electric field flux vector $\boldsymbol{D}(x)$ [Col/m^2] equals the scalar field charge density $\rho(\boldsymbol{x})$ [Col/m^3]

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{x}) \equiv \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\cdot\boldsymbol{D}(\boldsymbol{x}) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho(\boldsymbol{x}). \tag{1.114}$$

Thus it is analogous to the scalar (dot) product between two vectors.

When working with guided waves (narrow tubes of flux), when the diameter is small compared with the wavelength (P10, p. 102), the divergence is

$$\nabla\cdot\boldsymbol{D}(\boldsymbol{x}) = \nabla_r D_r = \frac{1}{A(r)}\frac{\partial}{\partial r}A(r)D_r(r), \tag{1.115}$$

where r is the distance down the horn (range variable), $A(r)$ is the area of the iso-response surface as a function of range r, and $D_r(r)$ is the radial component of vector $\boldsymbol{D}$ as a function of range r. For example, in spherical coordinates the area $A(r) = A_o r^2$ is proportional to the square of the range. This expression reduces to the radial component of the divergence of $\boldsymbol{D}(\boldsymbol{x})$ in spherical coordinates. In cylindrical coordinates $A(r) = A_o r$, and in rectangular coordinates the area $A = \pi r_o^2$ is independent of the range r. A general and detailed derivation of these cases may be found in Section 1.5.6, p. 142.

Curl: The *curl* transforms a vector field into a vector field. For example, the curl of the *magnetic intensity* $\boldsymbol{H}(\boldsymbol{x})$ [A/m] vector is equal to the vector *current density* $\boldsymbol{C}(x)$ [A/m^2]:

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{x}) \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \boldsymbol{C}(\boldsymbol{x}). \tag{1.116}$$

The notation $|\cdot|$ indicates the determinant (Appendix A, p. 183), ∂_x is shorthand for $\partial/\partial x$ and $\boldsymbol{H} = [H_x, H_y, H_z]^T$.

Laplacian: The *Laplacian* ($\nabla^2 \equiv \boldsymbol{\nabla}\cdot\nabla$) is

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}, \tag{1.117}$$

and takes a scalar field onto a scalar field.

Example: Starting from a scalar field, the gradient produces a vector, which is then operated on by the divergence to take the output of the gradient back to a scalar field. One of the classic

cases is the Laplacian of a voltage field. The gradient of the scalar voltage $\Phi(\boldsymbol{x})$ [V] results in the electric field vector $\boldsymbol{E}$

$$\boldsymbol{E}(\boldsymbol{x}) = [E_x(\boldsymbol{x}), E_y(\boldsymbol{x}), E_z(\boldsymbol{x})]^T = -\nabla V(\boldsymbol{x}), \quad \text{[V/m]}$$

which in free space is proportional to the electric flux $\boldsymbol{D} = \epsilon_o \boldsymbol{E}$ [C/m^2], the divergence of which gives the charge density $\rho(\boldsymbol{x})$ [C/m^3]. Here ϵ_o [F/m] is the *vacuum permittivity*, which is $\approx 8.8542 \times 10^{-12}$ [F/m].

Example: The simplest example of a scalar field is the voltage between two very large (think ∞) conducting parallel planes, or plates (large so that we may ignore the edge effects). In this case the voltage varies linearly (the voltage is complex analytic) between the two plates. For example

$$V(x, y, z) = V_o(1 - x) \tag{1.118}$$

is a scalar field. At $x = 0$ the voltage is V_o and at $x = 1$ the voltage is zero. Between 0 and 1 the voltage varies linearly. Thus $V(x, y, z)$ defines a *scalar field.*

If the same setup is used, but the two plates were 1×1 [cm^2], with a 1 [mm] air gap, there will be a small "fringe" effect at the boundary that would (slightly) modify the ideal fields. This effect can be made small by changing the air gap to area ratio, so that the sides do not significantly impact the capacitor's value.

Example: A second classic example is an acoustic pressure field $\varrho(\boldsymbol{x}, t)$ [Pa], which defines a vector force density $\mathbf{f}(\boldsymbol{x}, t) = -\nabla\varrho(\boldsymbol{x}, t)$ [N/m^2] (Eq. 1.133, p. 140). When this force density [N/m^2] is integrated over an area, the net radial force [N] is

$$F_r = -\int_{\mathcal{S}} \nabla\varrho(\boldsymbol{x})||d\mathcal{S}||. \quad \text{[N]} \tag{1.119}$$

An inflated balloon with a static internal pressure of 3 [atm], in an ambient pressure of 1 [atm] (sea level), forms a sphere due to the elastic nature of the rubber, which acts as a stretched spring under tension. The net force on the surface of the balloon is its area times the pressure drop of 2 atm across the surface. Thus the static pressure is

$$\varrho(\boldsymbol{x}) = 3u(r_o - r) + 1, \quad \text{[Pa]}$$

where $u(r)$ is a step function of the radius $r = ||\boldsymbol{x}|| > 0$, centered at the center of the balloon, having radius r_o.

Taking the gradient gives the negative[92] of the radial force density (i.e., perpendicular to the surface of the balloon)

$$-f_r(r) = \nabla\varrho(\boldsymbol{x}) = \frac{\partial}{\partial r} 3u(r_o - r) + 1 = -2\delta(r_o - r). \quad \text{[Pa]}$$

This describes a static pressure that is 3 atmospheres inside the balloon, and 1 atmosphere [atm] (1 [atm] = 10^5 [Pa]) outside. Note that the net positive force density is the negative of the gradient of the static pressure.

Taking the divergence of this radial force gives the Laplacian of the scalar pressure field

$$\nabla^2\varrho(\boldsymbol{x}) = \boldsymbol{\nabla}\cdot\nabla\varrho(\boldsymbol{x}) = -\boldsymbol{\nabla}\cdot\mathbf{f}(x).$$

[92] The force is pointing out, stretching the balloon.

Laplacian operator in N dimensions

In general it may be shown that in $N = 1, 2, 3$ dimensions (Sommerfeld, 1949, p. 227)

$$\nabla_r^2 \mathcal{P} \equiv \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial \mathcal{P}}{\partial r} \right). \tag{1.120}$$

For each value of N, the area $A(r) = A_o r^{N-1}$. This will turn out to be useful when working with the Laplacian in 1, 2, and 3 dimensions.

Example: When $N = 3$ (i.e., spherical geometry)

$$\nabla_r^2 \mathcal{P} \equiv \frac{1}{r^2} \partial_r r^2 \partial_r \mathcal{P} \tag{1.121}$$

$$= \frac{1}{r} \frac{\partial^2}{\partial r^2} r\mathcal{P} \tag{1.122}$$

resulting in the general d'Alembert solutions (Eq. 1.95 p. 109) for the spherical wave equation

$$\mathcal{P}^{\pm}(r, s) = \frac{1}{r} e^{\mp \kappa(s) r}.$$

for the spherical geometry.

Exercise: Prove this last result by expanding Eq. 1.121, 1.122 using the chain rule. **Solution:** Expanding Eq. 1.121:

$$\begin{aligned} \frac{1}{r^2} \partial_r r^2 \partial_r \mathcal{P} &= \frac{1}{r^2} \left(2r + r^2 \partial_r \right) \partial_r \mathcal{P} \\ &= \frac{2}{r} \mathcal{P}_r + \mathcal{P}_{rr}. \end{aligned}$$

Expanding Eq. 1.122:

$$\begin{aligned} \frac{1}{r} \partial_{rr} r\mathcal{P} &= \frac{1}{r} \partial_r \left(\mathcal{P} + r\mathcal{P}_r \right) \\ &= \frac{1}{r} \left(\mathcal{P}_r + \mathcal{P}_r + r\mathcal{P}_{rr} \right) \\ &= \frac{2}{r} \mathcal{P}_r + \mathcal{P}_{rr}. \end{aligned}$$

Summary: The Laplacian in spherical coordinates (Eq. 1.121, p. 131) simplifies to

$$\nabla^2 \varrho(\boldsymbol{x}) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \varrho(\boldsymbol{x}) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \varrho(\boldsymbol{x}).$$

Since $\nabla^2 = \boldsymbol{\nabla} \cdot \nabla$, it follows that the net force $\boldsymbol{f}(\boldsymbol{x}) = [F_r, 0, 0]^T$, (Eq. 1.119) in spherical coordinates has a radial component F_r, and angular components of zero. Thus the force across a balloon may be approximated by a delta function across the thin sheet of stretched rubber.

Example: The previous example may be extended in an interesting way to the case of a rigid hose, a rigid tube, terminated on the right in an elastic medium (the above example of a balloon),

for example an automobile tire. On the far left let's assume there is a pump injecting the fluid into the hose. Consider two different fluids: air and water. Air may be treated as a compressible fluid, whereas water is incompressible. However such a classification is a relative, being determined by the relative compliance of the balloon (i.e., tire) at the relatively rigid pump and hose.

This is a special case of a more general situation: When the fluid is treated as incompressible (rigid) the speed of sounds becomes infinite, and the wave equation is not the best describing equation, and the motion is best approximated using Laplace's equation. This is the transition from short to long wavelengths, from wave propagation, with delay, to quasi-statics, having no apparent delay.

This example may be modeled as either an electrical or mechanical system. If we take the electrical analog, the pump is a current source, injecting charge (Q) into the hose, which being rigid cannot expand (has a fixed volume). The hose may be modeled as a resistor, and the tire as a capacitor C, which fills with charge as it is delivered via the resistor, from the pump. A capacitor obeys the same law as a spring $F = KV$, or in electrical terms, $Q = CV$. Here V is the voltage, which acts as a force F, Q is the charge, which acts like the mass of the fluid. The charge is conserved, just as the mass of the fluid is conserved, meaning they cannot be created or destroyed. The flow of the fluid is called the ***flux***, which is the general term for the mass or charge current. The two equations may be rewritten directly in terms of the force F, V and flow, as an electrical current $I = dQ/dt$ of mass flux $J = dM/dt$, giving two impedance relations:

$$I = \frac{d}{dt}CV \quad \text{[A]} \tag{1.123}$$

for the electrical analogue, and

$$J = \frac{d}{dt}CF. \quad \text{[kgm/m}^2\text{]} \tag{1.124}$$

It is common to treat the stiffness of the balloon, which acts as a spring with compliance C (stiffness $K = 1/C$), in which case the equations reduce to the same equation, in terms of an impedance Z, typically defined in the frequency domain as the ratio of the generalized force over the generalized flow

$$Z(s) = \frac{1}{sC}. \quad \text{[ohms]}.$$

In the case of the mechanical system $Z_m(s) \equiv F/J$, and for the electrical system $Z_e(s) \equiv V/I$. It is conventional to use the unit [ohms] when working with any impedance. It is convenient to use a uniform terminology for different physical situations and forms of impedance, greatly simplifying the notation.

While the two systems are very different in their physical realization, they are mathematically equivalent, forming a perfect analogue. The formula for the impedance is typically expressed in the Laplace frequency domain, which of course is the Laplace transform of the time variables. In the frequency domain Ohm's law becomes Eq. 1.124 for the case of a spring and Eq. 1.123 for the capacitor.

The final solution of this system is solved in the frequency domain. The impedance seen by the source is the sum of the resistance R added to the impedance of the load, giving

$$Z = R + \frac{1}{sC}.$$

The solution is simply the relation between the force and the flow, as determined by the action of the source on the load $Z(s)$. The final answer is given in terms of the voltage across the compliance in

terms of the voltage V_s (or current I_s) due to the source. Once the algebra is done, in the frequency domain, the voltage across the compliance V_c divided by the voltage of the source is given as

$$\frac{V_c}{V_{\text{source}}} = \frac{R}{R + 1/sC}.$$

Thus the problem reduces to some algebra in the frequency domain. The time domain response is found by taking the inverse Laplace transform, which in this case has a simple pole at $s_p = 1/RC$. Cauchy's residue formula gives the final answer, which describes how the voltage across the compliance builds exponentially with time, from zero to the final value. Given the voltage, the current may also be computed as a function of time. This then represents the entire process of either blowing up a balloon, or charging a capacitor, the difference only being the physical notation, the math being identical.

Note that the differential equation is first-order in time, which in frequency means the impedance has a single pole. This means that the equation for the charging of a capacitor, or pumping up a balloon, describes a *diffusion process*. If we had taken the impedance of the mass of the fluid in the hose into account, we would have a lumped-parameter model of the wave equation, with a second-order system. This is mathematically the same as the homework problem about train cars (masses) connected together by springs (Fig. H.1 p. 254).

Example: The voltage

$$\phi(\boldsymbol{x}, t) = e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t - x/c) \leftrightarrow \frac{1}{s} e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} \quad [\text{V}] \tag{1.125}$$

is an important case since it represents one of d'Alembert's two solutions (Eq. 1.95, p. 109) of the wave equation (Eq. 1.24, p. 57), as well as an eigen-function of the gradient operator ∇. From the definition of the scalar (dot) product of two vectors (Fig. 1.16, p. 80),

$$\boldsymbol{\kappa} \cdot \boldsymbol{x} = \kappa_x x + \kappa_y y + \kappa_z z = ||\boldsymbol{\kappa}||\ ||\boldsymbol{x}|| \cos\theta_{\kappa x},$$

where $||\boldsymbol{\kappa}|| = \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}$ and $||\boldsymbol{x}|| = \sqrt{x^2 + y^2 + z^2}$ are the lengths of vectors $\boldsymbol{\kappa}$, and $\boldsymbol{x}$ and $\theta_{\kappa x}$ is the angle between them. As before, $s = \sigma + \omega\jmath$ is the Laplace frequency.

To keep things simple let $\boldsymbol{\kappa} = [\kappa_x, 0, 0]^T$ so that $\boldsymbol{\kappa} \cdot \boldsymbol{x} = \kappa_x x\hat{\mathbf{x}}$. We shall soon see that $||\boldsymbol{\kappa}|| = 2\pi/\lambda$ follows from the basic relationship between a wave's radian frequency $\omega = 2\pi f$ and its wavelength λ

$$\omega\lambda = c_o. \tag{1.126}$$

As frequency increases, the wavelength becomes shorter. This key relationship may have been first researched by Galileo (c.1564), followed by (c.1627) Mersenne[93] (Fig. 1.2, p. 23).

Exercise: Show that Eq. 1.125 is an eigen-function of the gradient operator ∇. **Solution:** Taking the gradient of $\phi(\boldsymbol{x}, t)$ gives

$$\begin{aligned} \nabla e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t) &= -\nabla\boldsymbol{\kappa} \cdot \boldsymbol{x}\, e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t) \\ &= -\boldsymbol{\kappa}\, e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t), \end{aligned}$$

[93] http://www-history.mcs.st-and.ac.uk/Biographies/Mersenne.html

> "In the early 1620s, Mersenne listed Galileo among the innovators in natural philosophy whose views should be rejected. However, by the early 1630s, less than a decade later, Mersenne had become one of Galileo's most ardent supporters."
>
> –D Garber, Perspect. Sci. 12 (2) (2004), 135-163.

or in terms of $\phi(\boldsymbol{x}, t)$

$$\nabla\phi(\boldsymbol{x}, t) = -\boldsymbol{\kappa}\,\phi(\boldsymbol{x}, t) \leftrightarrow -\frac{s}{c}e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}}.$$

Thus $\phi(\boldsymbol{x}, t)$ is an eigen-function of ∇, having the vector eigen-value $\boldsymbol{\kappa}$. As before, $\nabla\phi$ is proportional to the current since ϕ is a voltage, and the ratio, i.e., the eigen-value may be thought of as a mass. In general the units provide the physical interpretation of the eigen-values and their spectra. A famous example is the Rydberg spectrum of the Hydrogen atom.

Exercise: Compute $\hat{\boldsymbol{n}}$ for $\phi(\boldsymbol{x}, t)$ (Eq. 1.125). **Solution:** $\hat{\boldsymbol{n}} = \boldsymbol{\kappa}/||\boldsymbol{\kappa}||$. This represents a unit vector in the direction of the current.

Exercise: If the sign of $\boldsymbol{\kappa}$ is negative, what are the eigen-vectors and eigen-values of $\nabla\phi(\boldsymbol{x}, t)$?
Solution:

$$\begin{aligned}\nabla e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}}u(t) &= -\boldsymbol{\kappa}\cdot\nabla(\boldsymbol{x})e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}}u(t)\\ &= -\boldsymbol{\kappa}\; e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}}u(t).\end{aligned}$$

Nothing changes other than the sign of $\boldsymbol{\kappa}$. Physically this means the wave is traveling in the opposite direction, corresponding to the forward and retrograde d'Alembert waves.

Prior to this section, we have only considered the Taylor series in one variable, such as for polynomials $P_n(x), x \in \mathbb{R}$ (Sect. 1.3.1, Eq.1.27 p. 59) and $P_n(s), s \in \mathbb{C}$ (Sect. 1.4.2, Eq.1.46 p. 71). The generalization from real to complex analytic functions led to the Laplace transform, an the hosts of integration theorems (FTCC, Cauchy I, II, III). What is in store when we generalize from one spatial variable ($\mathbb{R}$) to three ($\mathbb{R}^3$)?

Exercise: If $\boldsymbol{E}(\boldsymbol{x}, t) = E_x\hat{\mathbf{x}}$, express $\boldsymbol{E}(\boldsymbol{x}, t)$ in terms of the voltage potential $\phi(\boldsymbol{x}, t)$ [V].
Solution: The electric field strength may be found from the voltage as

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\nabla\phi(\boldsymbol{x}, t) = -\hat{\mathbf{x}}\frac{\partial}{\partial x}\phi(\boldsymbol{x}, t). \quad [\text{V/m}]$$

Exercise: Find the velocity $v(t)$ of an electron in a field $\boldsymbol{E}$. **Solution:** From Newton's 2nd law, $-qE = m_e\dot{v}(t)$ [Nt], where m_e is the mass of the electron. Thus we must solve this first-order differential equation to find $v(t)$. This is easily done in the frequency domain $v(t) \leftrightarrow V(\omega)$.

Role of Potentials: Note that the scalar fields (e.g., temperature, pressure, voltage) are all scalar potentials, summarized in Fig. 1.3 (p. 91). In each case the gradient of the potential results in a vector field, just as in the electric case above (Eq. 1.110).

It is important to understand the physical meaning of the gradient of a potential, which is typically a generalized force (electric field, acoustic force density, temperature flux), which in turn generates a flow (current, velocity, heat flux). *The ratio of the gradient over the flow determines the impedance.*

1. *Example 1:* The voltage drop across a resistor causes a current to flow, as described by Ohm's law. Taking the difference in voltage between two points is a crude form of gradient when the frequency f [Hz] is low, such that the wavelength is much larger than the distance between the two points. This is the essence of the *quasi-static approximation* P10 (102).

2. *Example 2:* The gradient of the pressure gives rise to a force density in the fluid medium (air, water, oil, etc.), which causes a flow (velocity vector) in the medium.

3. *Example 3:* The gradient of the temperature also causes a flow of heat, which is proportional to the thermal resistance, given Ohm's law for heat (Feynman, 1970c, p. 3-7).[94]

4. *Example 4:* Nernst potential: When a solution contains charged ions, it is called an electro-chemical Nernst potential $N(\boldsymbol{x},t)$.[95] This electro-chemical field is similar to a voltage or temperature field, the gradient of which defines a virtual force on the charged ions.

Thus in the above examples there is a potential, the gradient of which is a force, which when applied to the medium (an impedance) causes a flow (flux or current) proportional to that impedance, due to the medium. This is a very general set of concepts, worthy of some thought. In every case there is a force and a flow. The product of the force and flow is a power, while the ratio may be modeled using 2x2 ABCD impedance matrices (Eq. 1.68, p. 89).

Exercise: Show that the integral of Eq. 1.110 is an anti-derivative. **Solution:** The solution uses the definition of the anti-derivative, defined by the FTC (Eq. 1.86, p. 106):

$$\begin{aligned}
\phi(\boldsymbol{x},t)-\phi(\boldsymbol{x}_o,t) &= \int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \boldsymbol{E}(\boldsymbol{x},t)\cdot d\boldsymbol{x} \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \nabla\phi(\boldsymbol{x},t)\cdot d\boldsymbol{x} \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\phi(\boldsymbol{x},t)\cdot d\boldsymbol{x} \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \left(\hat{\mathbf{x}}\frac{\partial\phi}{\partial x}+\hat{\mathbf{y}}\frac{\partial\phi}{\partial y}+\hat{\mathbf{z}}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{\mathbf{x}}dx+\hat{\mathbf{y}}dy+\hat{\mathbf{z}}dz) \\
&= -\int_{x_o}^{x}\frac{\partial\phi}{\partial x}dx-\int_{y_o}^{y}\frac{\partial\phi}{\partial y}dy-\int_{z_o}^{z}\frac{\partial\phi}{\partial z}dz \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} d\phi(\boldsymbol{x},t) \\
&= -\Big(\phi(\boldsymbol{x},t)-\phi(\boldsymbol{x}_o,t)\Big).
\end{aligned}$$

This may be verify by taking the gradient of both sides

$$\nabla\phi(\boldsymbol{x},t)-\nabla\cancelto{0}{\phi(\boldsymbol{x}_o,t)} = -\nabla\int_{\boldsymbol{x}_o}^{\boldsymbol{x}}\boldsymbol{E}(\boldsymbol{x},t)\cdot d\boldsymbol{x} = \boldsymbol{E}(\boldsymbol{x},t).$$

Applying the FTC (Eq. 1.86, p. 106), the anti-derivative must be $\phi(\boldsymbol{x},t)=E_x x\hat{\mathbf{x}}+0\hat{\mathbf{y}}+0\hat{\mathbf{z}}$. This very same point is made by Feynman (1970c, p. 4-1, Eq. 4.28).

Given that the force on a charge is proportional to the gradient of the potential, from the above exercise showing that the integral of the gradient only depends on the end points, the work done in moving a charge only depends on the limits of the integral, which is the definition of a *conservative field*, but which only holds in the ideal case where $\boldsymbol{E}$ is determined by Eq. 1.110, i.e., the medium has no friction (i.e., there are no other forces on the charge).

[94] https://en.wikipedia.org/wiki/Thermal_conduction#Fourier's_law
[95] https://en.wikipedia.org/wiki/Nernst_equation

The *conservative* field: An important question is: "When is a field conservative?" A field is conservative when the work done by the motion is independent of the path of motion. Thus the conservative field is related to the FTC, which states that the integral of the work only depends on the end points.

A more complete answer must await the introduction of the *fundamental theorem of vector calculus*, discussed in Sect. 1.5.14 (Eq. 1.183, p. 163). A few specific examples provide insight:

Example: The gradient of a scalar potential, such as the voltage (Eq. 1.110), defines the electric field, which drives a current (flow) across a resistor (impedance). When the impedance is infinite, the flow will be zero, leading to zero power dissipation. When the impedance is lossless, the system is conservative.

Example: At audio frequencies the viscosity of air is quite small and thus, for simplicity, it may be taken as zero. However when the wavelength is small (e.g., at 100 [kHz] $\lambda = c_o/f = 345/10^5 = 3.45$ [mm]) the lossless assumption breaks down, resulting in a significant propagation loss.[96] When the viscosity is taken into account, the field is lossy, thus the field is no longer conservative.

Example: If a temperature field is a time-varying constant (i.e., $T(\boldsymbol{x}, t) = T_o(t)$), there is no "heat flux," since $\nabla T_o(t) = 0$. When there is no heat flux (i.e., flux, or flow), there is no heat power, since the power is the product of the force times the flow.

Example: The force of gravity is given by the gradient of Newton's gravitational potential (Eq. 1.22, p. 57)

$$F = -\nabla \phi_g(r).$$

Historically this was the first conservative field, used to explain the elliptic orbits of the planets around the sun.

1.5.3 Lec 35 (I): Partial differential equations and field evolution:

In all cases the space operator is the Laplacian ∇^2, the definition of which depends on the dimensionality of the waves, thus on the coordinate system being used. There are three main categories of *partial differential equations* (PDEs): parabolic, elliptic and hyperbolic, distinguished by the order of the derivative with respect to time:

1. *Diffusion equations* (Eq. 1.131), describe the evolution of the scalar temperature $T(\boldsymbol{x}, t)$ (a scalar potential), gradients of solution concentrations (i.e., ink in water) and Brownian motion. Diffusion is first-order in time, which is categorized as *parabolic* (first-order in time, second-order in space). When these equations are Laplace transformed, diffusion has a single real root, resulting in a real solution (e.g., $T \in \mathbb{R}$). There is no wave-front for the case of the diffusion equation. As soon as the source is turned on, the field is non-zero at every point in the bounded container.

2. *Poisson's equation:* In the steady state the diffusion equation degenerates to either Poisson's or Laplace's equation, which are classified as *elliptic equations* (2nd order in space, 0th order in time). Like the diffusion equation, the evolution has a wave velocity that is functionally infinite.

3. *Wave equations*

[96] https://en.wikipedia.org/wiki/Laminar_flow#Examples

(a) *scalar wave equations* (Eq. 1.23) describe the evolution of a scalar potential field such as pressure $\varrho(\boldsymbol{x}, t)$ (sound), or the displacement of a string or membrane under tension. The wave equation is second-order in time. When transformed into the frequency domain, the solution has pairs of complex conjugate roots, leading to two real solutions (i.e, $\varrho(\boldsymbol{x}, t \in \mathbb{R})$. The wave equation is classified as *hyperbolic* (second-order in time and space).

(b) *vector wave equations* (i.e., Maxwell's equations) describe the propagation of electric $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{D}(\boldsymbol{x}, t)$ and magnetic $\boldsymbol{B}(\boldsymbol{x}, t)$ and $\boldsymbol{H}(\boldsymbol{x}, t)$ vector fields.

These three classifications, elliptic, parabolic and hyperbolic, are simply labels, with little mathematical utility (the categories fail to generalize in any useful way).

Related partial differential equations

The Laplacian ∇^2: We first discussed the Laplacian as a 2D operator in Section 1.4.2 (p. 107), when we studied complex analytic functions. Then an approximation for horns was presented as Eq. 1.5.2 p. 129. In 3D rectangular coordinates it is defined as

$$\nabla^2 T(\boldsymbol{x}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(\boldsymbol{x}). \tag{1.127}$$

In summary, the Laplacian operator is ubiquitous in mathematical physics, starting with simple complex analytic functions (Laplace's equation) and progressing to Poisson's equation, the diffusion equation, and finally the wave equation. Only the wave equation expresses delay. The diffusion equation "wave" has an instantaneous spread (the effective "wave" velocity is infinite, i.e., it's not a wave).

Historically the wave equation was seen to be related to several other important partial differential equations, as the theory evolved, resulting in the following:

1. *Fourier diffusion equation*

$$\nabla^2 T(\boldsymbol{x}, t), = D_o \frac{\partial T(\boldsymbol{x}, t)}{\partial t} \leftrightarrow s D_o T(\boldsymbol{x}, s) \tag{1.128}$$

which describes, for example, the temperature $T(\boldsymbol{x}, t) \leftrightarrow T(\boldsymbol{x}, \omega)$, as proposed by Fourier in 1822, or the diffusion of two miscible liquids (Fick, 1855) and Brownian motion (Einstein, 1905). The diffusion equation is not a wave equation since the temperature wave front propagates instantaneously.

2. *Poisson's equation*

$$\nabla^2 \Phi(\boldsymbol{x}, t) = \rho(x, t), \tag{1.129}$$

which holds for gravitational fields, or the voltage around a charge.

3. *Laplace's equation*

$$\nabla^2 \Phi(\boldsymbol{x}, t) = 0, \tag{1.130}$$

which describes, for example, the voltage inside a closed chamber with various voltages on the walls, or the steady state temperature within a closed container, given a specified temperature distribution on the walls.

Each of these three equations has properties that may be simply explained, and visualized, for $N = 1, 2, 3$ geometries, and all contain the Laplacian $\nabla^2(\cdot) = \boldsymbol{\nabla} \cdot \nabla(\cdot)$ (p. 131).

Solution evolution: The partial differential equation defines the "evolution" of the scalar field (pressure $\varrho(\boldsymbol{x}, t)$ and temperature $T(\boldsymbol{x}, t)$, or vector field ($\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{H}$), as functions of space $\boldsymbol{x}$ and time t. There are two basic categories of field evolution, *diffusion* and *propagation*.

1. *Diffusion*: The simplest and easiest PDE example, easily visualized, is a *static*[97] (time invariant) scalar temperature field $T(\boldsymbol{x})$ [°C]). Just like an impedance or admittance, a field has regions where it is analytic, and for the same reasons, $T(\boldsymbol{x}, t)$ satisfies Laplace's equation

$$\nabla^2 T(\boldsymbol{x}, t) = 0.$$

   Since there is no current when the field is static, such systems are lossless, and thus are conservative.

   When $T(\boldsymbol{x}, t)$ depends on time (is not static), it is described by the *diffusion equation*

$$\nabla^2 T(\boldsymbol{x}, t) = \kappa \frac{\partial}{\partial t} T(\boldsymbol{x}, t), \tag{1.131}$$

   a rule for how $T(\boldsymbol{x}, t)$ evolves with time from its initial state $T(\boldsymbol{x}, 0)$. Constant κ is called the *thermal conductivity* which depends on the properties of the fluid in the container, with $s\kappa$ being the admittance per unit area. The conductivity is a measure of how the heat gradients induce heat currents $\mathbf{J} = \kappa \nabla T$, analogous to Ohm's Law for electricity.

   Note that if $T(\boldsymbol{x}, \infty)$ reaches *steady state* $\mathbf{J} = 0$ as $t \to \infty$, it evolves into a static state, thus $\nabla^2 T = 0$. This depends on what is happening at the boundaries. When the wall temperature of a container is a function of time, then so will the internal temperature continue to change, but with a delay, that depends on the thermal conductivity κ.

   Such a system is analogous to an electrical resistor-capacitor series circuit, connected to a battery. The wall temperature and the voltage on the battery represent the potential driving the system, the thermal conductivity κ and the electrical resistor are analogous, and the fluid (like the electrical capacitor), are being heated (charged) by the heat (charge) flux. In both cases Ohm's law defines the ratio of the potential and the flux. How this happens can only be understood once the solution to the equations has been established.

2. *Propagation* Pressure and electromagnetic waves are described by a scalar potential (pressure) (Eq. 1.23, p. 57) and a vector potential (electromagnets) (Eq. 1.188, p. 167) resulting in scalar and vector *wave equations*.

All these partial differential equations, scalar and vector wave equations, and the diffusion equation, depend on the Laplacian ∇^2, which we first saw with the Cauchy-Riemann conditions (Eq. 1.94, p. 108).

The vector Taylor series: Next we shall expand the concept of the one-dimensional Taylor series, a function of one variable, to $\boldsymbol{x} \in \mathbb{R}^3$. Just as we generalized the derivative with respect to the real frequency variable $\omega \in \mathbb{R}$ to the complex analytic frequency $s = \sigma + \omega\jmath \in \mathbb{C}$, here we generalize the derivative with respect to $x \in \mathbb{R}$, to the vector $\boldsymbol{x} \in \mathbb{R}^3$.

[97] Postulate (P3), p. 101.

Since the scalar field is analytic in $\boldsymbol{x}$, it is a perfect place to start. Assuming we have carefully defined the Taylor series 1.40 (p. 68) in one and two (Eq. 1.91, p. 108) variables, the Taylor series of $f(\boldsymbol{x})$ in $\boldsymbol{x} = 0 \in \mathbb{R}^3$ may be defined as

$$f(\boldsymbol{x} + \delta\boldsymbol{x}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x}) \cdot \delta\boldsymbol{x} + \frac{1}{2!} \sum_k \sum_l \frac{\partial^2 f(\boldsymbol{x})}{\partial x_k \partial x_l} \delta x_k \delta x_l + \text{HOT}. \tag{1.132}$$

From this definition it is clear that the gradient is the generalization of the second term in the 1D Taylor series expansion.

Vector field: A *vector field* is an ordered set of scalar fields. For example

$$\boldsymbol{A}(\boldsymbol{x}) = [\phi(\boldsymbol{x}), \psi(\boldsymbol{x}), \theta(\boldsymbol{x})]^T$$

is a vector field in $\mathbb{R}^3$ when each of the three functions is differentiable (i.e., analytic). For example, $\boldsymbol{A}(\boldsymbol{x}) = [x, xy, xyz]^T$ is a legal vector field (the components are analytic in $\boldsymbol{x}$).

Taking an example from Maxwell's equations, the magnetic flux vector is given by

$$\boldsymbol{B}(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{A}(\boldsymbol{x}, t).$$

We shall see that this is always true because the magnetic charge $\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{x}, t)$ is always 0.

To verify that a field is a potential, check out the units [V, A, °C]. However a proper mathematical definition is that the potential must be an analytic function of $\boldsymbol{x}$ and t, so that one may operate on it with $\nabla()$ and $\boldsymbol{\nabla} \times ()$. The divergence of a scalar field is not a legal operation.

An important discussion of vector potentials, with extensive examples, may be found in Feynman (1970c, p. 14-1 to 14.3). If you need to master vector potentials, read Ch. 14.1 of Feynman (1970c).

Summary: For every potential $\phi(\boldsymbol{x}, t)$ there exists a force density $\boldsymbol{f}(\boldsymbol{x}, t) = -\nabla\phi(\boldsymbol{x}, t)$, proportional to the potentials, which drives a generalized flow $\boldsymbol{u}(\boldsymbol{x}, t)$. If the normal component of the force and flow are averaged over a surface, the mean-force and volume-flow (i.e, volume-velocity for the acoustic case) are define. In such cases the impedance is the net force through the surface force over the net flow, and Gauss's Law and quasi-statics (P10) come into play (Feynman, 1970a)

Assuming linearity linear (P2, p. 101), the product of the force and flow is the power, and the ratio (force/flow) is an impedance (Fig. 1.3, p. 91). This impedance statement is called either Ohm's law, Kirchhoff's laws, Laplace's law, or Newton's laws. In the simplest of cases, they are all linearized (proportional) complex relationships between a force and a flow. Very few impedance relationships are inherently linear over a large range of force or current, but for physically useful levels, they can be treated as linear. Nonlinear interactions require a more sophisticated approach, typically involving numerical methods.

Note that it is the *difference* in the potential (i.e., voltage, temperature, pressure) that is proportional to the flux. This can be viewed as a major simplification of the gradient relationship, justified by the quasi-static assumption P10 (p. 102). is the physical basis of the fundamental theorem of algebra (p. 73), since the roots of the impedance are key to the finding the eigen-modes of the system equations.

In electrical circuits it is common to define a zero potential *ground* point that all voltages use as the referenced potential. This results in abstracting away (hiding) the difference in voltage. The ground is a useful convention, as a simplifying rule, but it obscures the physics, and obscures the

fact that the voltage is *not* the force. Rather the force is the voltage drop, referenced to the ground, which is defined as zero volts. It seems misleading (more precisely it is wrong) to state Ohm's law as the voltage over the current, since Ohm's law actually says that the *voltage drop* (i.e., voltage gradient) over the current defines the impedance. Like a voltage, the pressure is the potential, the gradient of which is a force density, which drives the flow. More on this in section 1.5.14 (p. 160), where we introduce the *fundamental theorem of vector calculus* (aka Helmholtz' decomposition theorem), which generalizes Ohm's law to include circulation (e.g., angular momentum, vorticity and the related EM magnetic effects). To understand these generalizations in flow one needs to understand compressible and rotational fields, complex analytic functions, and a lot more history of mathematical-physics (Table 1.7, p. 164).

1.5.4 Lec 35 (II): Scalar Wave Equations (Acoustics)

In this section we discuss the general solution to the wave equation. The wave equation has two forms: scalar waves (acoustics) and vector waves (electromagnetics). These have an important mathematical distinction, but have a similar solution space, one scalar and the other vector. To understand the differences we start with the scalar wave equation.

The scalar wave equation: A good starting point for understanding PDEs is to explore the scalar wave equation (Eq. 1.23, p. 57). Thus, we shall limit our analysis to acoustics, the classic case of scalar waves. Acoustic wave propagation was first analyzed mathematically by Isaac Newton (electricity had yet to be discovered) in his famous book *Principia* (1687), in which he first calculated the speed of sound based on the conservation of mass and momentum.

Early history: The study of wave propagation begins at least as early as Huygens (ca. 1678), followed soon after (ca. 1687) by Sir Isaac Newton's calculation of the speed of sound (Pierce, 1981). To obtain a wave, one must include two basic components: the stiffness of air, and its mass. These two equations shall be denoted (1) *Newton's 2nd law* ($F = ma$) and (2) *Hooke's law* ($F = kx$), respectively. In vector form these equations are (1) *Euler's equation* (i.e., conservation of momentum density)

$$-\nabla\varrho(\boldsymbol{x},t) = \rho_o \frac{\partial}{\partial t}\boldsymbol{u}(\boldsymbol{x},t) \leftrightarrow \rho_o s\mathcal{V}(\boldsymbol{x},s), \tag{1.133}$$

which assumes the density ρ_o is independent of time and position $\boldsymbol{x}$, and (2) the *continuity equation* (i.e., conservation of mass density)

$$-\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x},s) = \frac{1}{\eta_o P_o}\frac{\partial}{\partial t}\varrho(\boldsymbol{x},t) \leftrightarrow \frac{s}{\eta_o P_o}\mathcal{P}(\boldsymbol{x},s) \tag{1.134}$$

(Pierce, 1981, page 15). Here $P_o = 10^5$ [Pa], is the barometric pressure, $\eta_o = 1.4$ and $\eta_o P_o$ is the dynamic (adiabatic) stiffness. Combining Eqs. 1.133 and 1.134 (removing $\boldsymbol{u}(\boldsymbol{x},t)$) results in the 3-dimensional (3D) scalar pressure wave equation

$$\nabla^2\varrho(\boldsymbol{x},t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(\boldsymbol{x},t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(\boldsymbol{x},s) \tag{1.135}$$

with $c_o = \sqrt{\eta_o P_o/\rho_o}$ is the sound velocity. Because the merged equations describe the pressure, which is a scalar field, this is an example of the *scalar wave equation*

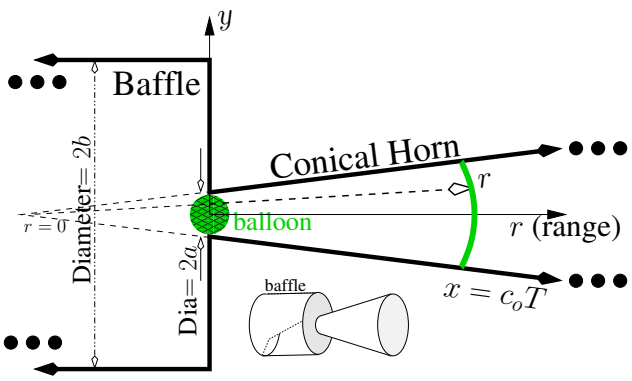


Figure 1.32: *Experimental setup showing a large pipe on the left terminating the wall containing a small hole with a balloon, shown in green. At time $t = 0$ the balloon is pricked and a pressure pulse is created. The baffle on the left is meant to represent a semi-∞ long tube having a large radius compared to the horn input diameter $2a$, such that the acoustic admittance looking to the left ($A/\rho_o c_o$ with $A \to \infty$) is very large compared to the horn's throat admittance (Eq. 1.153). At time T the outbound pressure pulse $p(r,T) = \delta(t - x/c_o)/r$ has reached a radius $x = r - r_o = c_o T$ where $r = x$ is the location of the source at the throat of the horn and r is measured from the vertex.*

Exercise: Show that Eqs. 1.133 and 1.134 can be reduced to Eq. 1.135. **Solution:** Taking the divergence of Eq. 1.133 gives

$$-\boldsymbol{\nabla}\cdot\nabla\varrho(\boldsymbol{x},t) = \rho_o \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x},t). \tag{1.136}$$

Note that $\boldsymbol{\nabla}\cdot\nabla = \nabla^2$. Next, substituting Eq. 1.134 into the above relation results in the scalar wave equation Eq. 1.135, since $c_o = \sqrt{\eta_o P_o/\rho_o}$.

1.5.5 Lec 36a: The Webster horn equation (I)

There is an important generalization of the problem of loss-less plane-wave propagation in 1-dimensional (1D) uniform tubes (e.g., transmission line theory). By allowing the area $A(r)$ of the horn to vary along the *range* axis r (the direction of wave propagation), as depicted in Fig. 1.32 for a *conical horn* (i.e., $A(r) = A_o(r/r_o)^2$), general solutions to the wave equation may be explored. Classic applications of horns include vocal tract acoustics, loudspeaker design, cochlear mechanics, the hydrogen atom, and cases having wave propagation in periodic media (Brillouin, 1953).

For the 1D scalar wave equation (guided waves, aka, acoustic horns), the *Webster Laplacian* is

$$\nabla_r^2\,\varrho(r,t) = \frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\,\varrho(r,t). \tag{1.137}$$

The Webster Laplacian is based on the *quasi-static approximation* (P10: p. 101) which requires that the frequency lies below the critical value $f_c = c_o/2d$, namely that a half wavelength is greater than the horn diameter d (i.e., $d < \lambda/2$).[98] For the case of the adult human ear canal, $d = 7.5$ [mm] and $f_c = (343/2 \cdot 7.5) \times 10^{-3} \approx 22.87$ [kHz].

The term on the right of Eq. 1.137, which is identical to Eq. 1.120 (p. 131), is also the Laplacian for thin tubes (e.g., rectangular, spherical, and cylindrical coordinates). Thus the Webster horn "wave" equation is

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\varrho(r,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(r,t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(r,s) \tag{1.138}$$

[98] This condition may be written several ways, the most common being $ka < 1$, where $k = 2\pi/\lambda$ and a is the horn radius. This may be expressed in terms of the diameter as $\frac{2\pi}{\lambda}\frac{d}{2} < 1$, or $d < \lambda/\pi < \lambda/2$. Thus $d < \lambda/2$ may be a more precise metric, by the factor $\pi/2 \approx 1.6$. This is call this the *half-wavelength assumption* a synonym for the quasi-static approximation.

where $\varrho(r,t) \leftrightarrow \mathcal{P}(r,s)$ is the average pressure (Hanna and Slepian, 1924; Mawardi, 1949; Morse, 1948), Olson (1947, p. 101), Pierce (1981, p. 360). Extensive experimental analysis for various types of horns (conical, exponential, parabolic) along with a review of horn theory may be found in Goldsmith and Minton (1924).

The limits of the Webster horn equation: It is frequently (i.e., always) stated that the Webster horn equation (WHEN) is fundamentally limited, thus is an approximation that only applies to frequencies much lower than f_c. However in all these discussions it is assumed that the area function $A(r)$ is the horn's cross-sectional area, not the area of the iso-pressure wave-front (Morse, 1948; Shaw, 1970; Pierce, 1981).

In the next section it is shown that this "limitation" may be totally avoided (subject to the $f < f_c$ quasi-static limit (P10, p. 102)), *making the Webster horn theory an "exact" solution for the lowest order "plane-wave" eigen-function.* The nature of the quasi-static approximation is that it "ignores" higher order evanescent modes, which are naturally small since they are in cutoff (evanescent modes do not propagate). This is the same approximation that is required to define an impedance, since every eigen-mode defines an impedance.

To apply this theory, the acoustic variables (eigen-functions) are redefined for the *average pressure* and its corresponding *volume velocity*, each defined on the iso-pressure wave-front boundary (Webster, 1919; Hanna and Slepian, 1924). The resulting impedance is then the ratio of the average pressure over the volume velocity. This approximation is valid up to the frequency where the next mode begins to propagate ($f > f_c$), which may be estimated from the roots of the Bessel eigen-functions (Morse, 1948). Perhaps it should be noted that these ideas, that come from acoustics, apply equally well to electromagnetics, or any other wave phenomena described by eigen-functions.

The best known examples of wave propagation are electrical and acoustic *transmission lines*. Such systems are loosely referred to as the *telegraph* or *telephone equations*, referring back to the early days of their discovery (Brillouin, 1953; Heaviside, 1892; Campbell, 1903b; Feynman, 1970a). In acoustics, guided waves are called horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names reflect the historical development, to a time when the mathematics and the applications were running in close parallel.

1.5.6 Lec 36b: Webster horn equation (II): Derivation

Here we transform the acoustic equations Eq. 1.133 and 1.134 (p. 140) into their equivalent integral form Eq. 1.138 (p. 141). This derivation is similar (but not identical) to that of Hanna and Slepian (1924) and Pierce (1981, p. 360).

Conservation of momentum: The first step is an integration of the normal component of Eq. 1.133 (p. 140) over the iso-pressure surface $\mathcal{S}$, defined by $\nabla p = 0$

$$-\int_{\mathcal{S}} \nabla p(\mathbf{x},t) \cdot d\mathbf{A} = \rho_o \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{u}(\mathbf{x},t) \cdot d\mathbf{A}.$$

The *average pressure* $\varrho(x,t)$ is defined by dividing by the total area

$$\varrho(x,t) \equiv \frac{1}{A(x)} \int_{\mathcal{S}} p(x,t)\, \hat{\boldsymbol{n}} \cdot d\boldsymbol{A}. \tag{1.139}$$

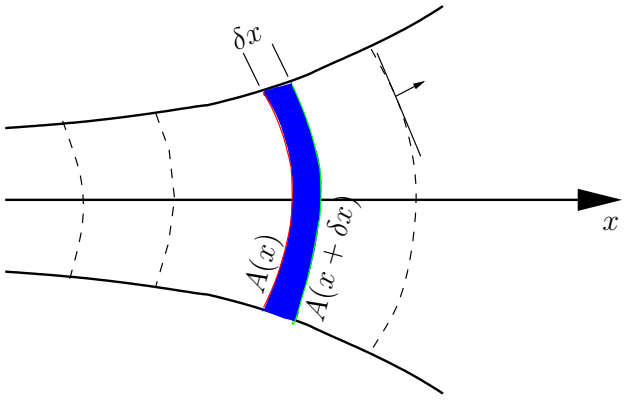


Figure 1.33: *Derivation of horn equation using Gauss's law: The divergence of the velocity* $\nabla \cdot \mathbf{u}$*, within* δx*, shown as the filled shaded region, is integrated over the enclosed volume. Next the divergence theorem is applied, transforming the integral to a surface integral normal to the surface of propagation. This results in the difference of the two volume velocities* $\delta\nu = \nu(x+\delta x) - \nu(x) = [\mathbf{u}(x+\delta x)\cdot\mathbf{A}(x+\delta x) - \mathbf{u}(x)\cdot\mathbf{A}(x)]$*. The flow is always perpendicular to the iso-pressure contours.*

From the definition of the gradient operator

$$\nabla p = \frac{\partial p}{\partial x}\hat{\boldsymbol{n}}, \tag{1.140}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the iso-pressure surface $\mathcal{S}$. Thus the left side of Eq. 1.133 reduces to $\partial\varrho(x,t)/\partial x$.

The integral on the right side defines the *volume velocity*

$$\nu(x,t) \equiv \int_S \mathbf{u}(x,t)\cdot d\boldsymbol{A}. \tag{1.141}$$

Thus the integral form of Eq. 1.133 becomes

$$-\frac{\partial}{\partial x}\varrho(x,t) = \frac{\rho_o}{A(x)}\frac{\partial}{\partial t}\nu(x,t) \leftrightarrow \mathscr{Z}(x,s)\mathcal{V} \tag{1.142}$$

where

$$\mathscr{Z}(s,x) = s\rho_o/A(x) = sM(x) \tag{1.143}$$

and $M(x) = \rho_o/A(x)$ [kgm/m^5] is the per-unit-length mass density of air.

Conservation of mass: Integrating Eq. 1.134 (p. 140) over the volume $\mathcal{V}$ gives

$$-\int_{\mathcal{V}} \nabla\cdot\boldsymbol{u}\, dV = \frac{1}{\eta_o P_o}\frac{\partial}{\partial t}\int_{\mathcal{V}} p(\boldsymbol{x},t)dV.$$

Volume $\mathcal{V}$ is defined by two iso-pressure surfaces between x and $x+\delta x$ (Fig. 1.33). On the right-hand side we use our definition for the *average pressure* (i.e., Eq. 1.139), integrated over the thickness δx.

Applying Gauss's law to the left-hand side,[99] and using the definition of ϱ (on the right), in the limit $\delta x \to 0$, gives

$$\frac{\partial\nu}{\partial x} = -\frac{A(x)}{\eta_o P_o}\frac{\partial\varrho}{\partial t} \leftrightarrow -\mathscr{Y}(x,s)\mathcal{P} \tag{1.144}$$

where

$$\mathscr{Y}(s,x) = sA(x)/\eta_o P_o = sC(x).$$

$C(x) = A(x)/\eta_o P_o$ [m^4/N] is the per-unit-length compliance of the air. These two equations Eq. 1.142 and 1.144 accurately characterize the Webster plane-wave mode up to the frequency where the higher order eigen-modes begin to propagate (i.e, $f > f_c$).

[99] As shown in Fig. 1.33, we convert the divergence into the difference between two volume velocities, namely $\nu(x+\delta x) - \nu(x)$, and $\partial\nu/\partial x$ as the limit of this difference over δx, as $\delta x \to 0$.

Speed of sound c_o: In terms of $M(x)$ and $C(x)$, the speed of sound and the acoustic admittance are

$$c_o = \frac{1}{\sqrt{C(x)M(x)}} = \sqrt{\frac{\eta_o P_o}{\rho_o}} = \frac{1}{\sqrt{\textit{stiffness} \cdot \textit{mass}}}. \tag{1.145}$$

Characteristic admittance $\mathcal{Y}(x)$: Since the horn equation (Eq. 1.138) is 2d order, it has pressure eigen-function solutions $\mathcal{P}^+$ and $\mathcal{P}^-$ and their corresponding velocity eigen-functions $\mathcal{V}^+$ and $\mathcal{V}^-$, related through Eq. 1.142, which defines the *characteristic admittance* $\mathcal{Y}(x)$

$$\mathcal{Y}(x) = \sqrt{\frac{\textit{stiffness}}{\textit{mass}}} = \sqrt{\frac{C(x)}{M(x)}} = \frac{A(x)}{\rho_o c_o} = \frac{\mathcal{V}^+}{\mathcal{P}^+} = \frac{\mathcal{V}^-}{\mathcal{P}^-} \tag{1.146}$$

(Campbell, 1903a, 1910, 1922). The *characteristic impedance* $\mathcal{Z}(x) = 1/\mathcal{Y}(x)$. Based on physical requirements that the admittance must be positive, thus only the positive square root is allowed.

Since the horn (Eq. 1.138) is loss less, $\mathcal{Y}(x)$ must be real and positive. If losses are introduced, the *propagation function* $\kappa(s)$ (p. 110) and the characteristic impedance $\mathcal{Y}(x, s)$ would become complex analytic functions of the Laplace frequency s (Kirchhoff, 1974; Mason, 1928; Ramo *et al.*, 1965; Pierce, 1981, p. 532-4).

One must be carefully in the definition the area $A(x)$. This area is *not* the cross-sectional area of the horn, rather it is the wave-front area, as discussed in Section 1.5.7 (p. 144). Since $A(x)$ is independent of frequency, it is independent the wave direction.

1.5.7 Matrix formulation of WHEN (III)

Newton's conservation of momentum law (Eq. 1.133), along with conservation of mass (Eq. 1.134), are modern versions of Newton's starting point for accurately calculating the horn lowest order plane-wave eigen-mode. Following the simplification of averaging the normal component of the *particle velocity* over the iso-pressure wave front, Eqs. 1.142, 1.144 may be rewritten as a 2x2 matrix in the acoustic variables, average pressure $\mathcal{P}(r, \omega)$ and volume velocity $\mathcal{V}(r, \omega)$ (here we replace the range-variable x by r)

$$-\frac{d}{dr}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix} = \begin{bmatrix}0 & sM(r)\\ sC(r) & 0\end{bmatrix}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix}, \tag{1.147}$$

where $M(r) = \rho_o/A(r)$ and $C(r) = A(r)/\eta_o P_o$ are the unit-length mass and compliance of the horn (Ramo *et al.*, 1965, p. ???). The acoustic variables $\mathcal{P}_c(r, \omega)$ and $\mathcal{V}(r, \omega)$ are sometimes referred to as *conjugate variables*.[100]

To obtain the Webster horn pressure equation Eq. 1.138 (p. 141) from Eq. 1.147 take the partial derivative of the top equation

$$-\frac{\partial^2 \mathcal{P}}{\partial r^2} = s\frac{\partial M(r)}{\partial r}\mathcal{V} + sM(r)\frac{\partial \mathcal{V}}{\partial r}.$$

Use the lower equation to remove $\partial\mathcal{V}/\partial r$

$$\frac{\partial^2 \mathcal{P}}{\partial r^2} + s\frac{\partial M(r)}{\partial r}\mathcal{V} = s^2 M(r)C(r)\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P},$$

[100] `https://en.wikipedia.org/wiki/Conjugate_variables_(thermodynamics)` The product of conjugate variables defines an *intensity* while their ratio defines an *impedance* (Pierce, 1981, p. 37-41).

and the upper equation a second time to remove $\mathcal{V}$. Thus Eq. 1.147 reduces to

$$\frac{\partial^2}{\partial r^2}\mathcal{P}(r,s) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\mathcal{P}_r = \frac{s^2}{c_o^2}\mathcal{P}(r,s). \tag{1.148}$$

Equations of this form may be directly integrated by parts by use of the chain rule

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\mathcal{P}(r,s) = \frac{\partial^2}{\partial r^2}\mathcal{P}(r,s) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\mathcal{P}_r(r,s), \tag{1.149}$$

where the integration factor is the horn area function $A(r)$.

Merging Eqs. 1.148 and 1.149 results in the Webster horn equation (Eq. 1.138, p. 141):

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\mathcal{P}(r,s) = \frac{s^2}{c_o^2}\mathcal{P}(r,s).$$

Equations having this integrated form are known as *Sturm-Liouville equations*. This important class of differential equations follow from the use of separation of variables on the Laplacian, in any (i.e., every) separable coordinate system (Morse and Feshbach, 1953, Ch. 5.1, p. 494-523).

Summary: Applying Gauss's law to the 3D wave equation (Eq. 1.135, p. 140) results in a 1D Webster horn equation (WHEN, Eq. 1.138, p. 141), which is a non-singular Sturm-Liouville equation, where the area function is the integration factor $A(r)$.

Thus Eqs. 1.135 and 1.147 are equivalent to the WHEN (Eq. 1.138).

1.5.8 Lec 37a: d'Alembert's eigen-vector superposition principle

Since the Webster horn equation (Eq. 1.138) is second order in time, it has two unique pressure eigen-functions $\mathcal{P}^+(r,s)$ and $\mathcal{P}^-(r,s)$. The general solution may always be written as the superposition of pressure eigen-functions, with amplitudes determined by the boundary conditions.

Based on d'Alembert's superposition principle, the pressure $\mathcal{P}$ and velocity $\mathcal{V}$ may be decomposed in terms of the pressure eigen-functions $\mathcal{P}^+$ and $\mathcal{P}^-$

$$\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix} = \begin{bmatrix}1 & 1\\ \mathcal{Y}(r) & -\mathcal{Y}(r)\end{bmatrix}\begin{bmatrix}\mathcal{P}^+(r,\omega)\\ \mathcal{P}^-(r,\omega)\end{bmatrix}. \tag{1.150}$$

This equation has several applications.

Generalized admittance/impedance: The *generalize admittance*[101] $Y_{in}(r,s)$ looking into the horn is

$$Y_{in}(r,s) \equiv \frac{\mathcal{V}(r,\omega)}{\mathcal{P}(r,\omega)} = \frac{\mathcal{V}^+ - \mathcal{V}^-}{\mathcal{P}^+ + \mathcal{P}^-} = \frac{\mathcal{V}_+}{\mathcal{P}_+}\left(\frac{1-\mathcal{V}^-/\mathcal{V}^+}{1+\mathcal{P}^-/\mathcal{P}^+}\right) = \mathcal{Y}(r)\frac{1-\Gamma(r,s)}{1+\Gamma(r,s)}. \tag{1.151}$$

Here we have factored out the forward traveling eigen-function $\mathcal{V}^+$ and $\mathcal{P}^+$, and re-expressed Y_{in} in terms of two ratios, the *characteristic admittance* $\mathcal{Y}(r,s)$ (Eq. 1.146) and the reflectance $\Gamma(r,s)$. $Y_{in}(s)$ depends on the entire horn. In the case of a finite length horn, it depends on the terminating

[101] It is "generalize" in the sense that it is not a Brune, rational function, impedance.

admittance. When the horn is terminated, reflections occur, resulting in the horn having poles and zeros at frequencies $s_k \in \mathbb{C}$, where $\Gamma(r, s_k) = \pm 1$.

The *reflectance* is defined as

$$\Gamma(r,s) \equiv \frac{\mathcal{V}^-(r,\omega)}{\mathcal{V}^+(r,\omega)} = \frac{\mathcal{P}^-(r,\omega)}{\mathcal{P}^+(r,\omega)}, \tag{1.152}$$

which follows by a rearrangement of terms in Eq. 1.146. The magnitude of the reflections depends $|\Gamma|$, which must be between 0 and 1. Alternatively this equation may be obtained by solving Eq. 1.151 for $\Gamma(r, s)$.

Horn radiation admittance: A horn's acoustic *radiation admittance* $Y^{\pm}_{rad}(r, s)$ is the input admittance (Eq. 1.151) when there is no terminating load[102]

$$Y^{\pm}_{rad}(r,s) = \lim_{r\to\infty} Y^{\pm}_{in}(r,s) = -\lim_{r=\to\infty} \frac{A(r)}{s\rho_o}\frac{d}{dr}\ln \mathcal{P}^{\pm}(r,s). \tag{1.153}$$

The input admittance becomes the radiation admittance when the horn is infinite in length, namely it is the input admittance for an eigen-function. A table of properties is given in Table 1.5 for four different simple horns.

Table 1.5: *Table of horns and their properties for $N = 1$, 2 or 3 dimensions, along with the exponential horn (EXP). In this table the horn's range variable is x, having* area $A(x)$, *radius* $r_o(x) = \sqrt{A(x)/\pi}$. $F(x)$ *is the coefficient on* $\mathcal{P}_x$, $\kappa(s) \equiv s/c_o$, *where* c_o *is the speed of sound and* $s = \sigma + \omega\jmath$ *is the Laplace frequency. The range variable* x *may be rendered dimensionless (see Fig. 1.34) by normalizing it as* $x \equiv (\xi - \xi_o)/(L - \xi_o)$, *with* ξ *the linear distance along the horn axis, from* $x = \xi_o$ *to* L *corresponding to* $x = 0$ *to* 1. *The horn's eigen-functions are* $\mathcal{P}^{\pm}(x,\omega) \leftrightarrow \varrho^{\pm}(x,t)$. *When* $\pm$ *is indicated, the outbound solution corresponds to the negative sign. Eigen function* $H^{\pm}_o(x,s)$ *are outbound and inbound Hankel functions. The last column is the* radiation admittance *normalized by the* characteristic admittance $\mathcal{Y}(x) = A(x)/\rho_o c_o$.

N	Name	radius	Area/A_o	$F(x)$	$\mathcal{P}^{\pm}(x,s)$	$\varrho^{\pm}(x,t)$	$Y^{\pm}_{in}/\mathcal{Y}$
1D	uniform	1	1	0	$e^{\pm\kappa(s)x}$	$\delta(t \mp x/c)$	1
2D	parabolic	$\sqrt{x/x_o}$	x/x_o	$1/x$	$H^{\pm}_o(-\jmath\kappa(s)x)$	—	$-\jmath x H^{\pm}_1/H^{\pm}_o$
3D	conical	x	x^2	$2/x$	$e^{\pm\kappa(s)x}/x$	$\delta(t \mp x/c)/x$	$1 \pm c/sx$
EXP	exponential	e^{mx}	e^{2mx}	$2m$	$e^{-\left(m\pm\sqrt{m^2+\kappa^2}\right)x}$	$e^{-mx}E(t)$	—

1.5.9 Lec37b: Complex-analytic $\Gamma(s)$ and $Z_{in}(s)$

When defining the complex reflectance $\Gamma(s)$ as a function of the complex frequency $s = \sigma + j\omega$, a *very* important assumption has been made: even though $\Gamma(s)$ is defined by the ratio of two functions of real (radian) frequency ω, like the impedance, the reflectance must be *causal* (postulate P1, p. 101). Namely $\Gamma(s) \leftrightarrow \gamma(t)$ is zero for $t < 0$. The same holds for the time-domain impedance $\zeta(t) \leftrightarrow Z_{in}(s)$. That $\gamma(t)$ and $\zeta(t)$ are causal is required by the physics.

The forward and retrograde waves are functions of frequency ω, as they depend on the source pressure (or velocity) and the point of horn excitation. The reflectance is a transfer function (thus the source term cancels) that depends only on the Thévenin impedance (or reflectance) looking into the system (at any position r).

[102] To compute the radiation impedance $Y^{\pm}_{rad}$ one must know the eigen-functions $\mathcal{P}^{\pm}(r,s)$.

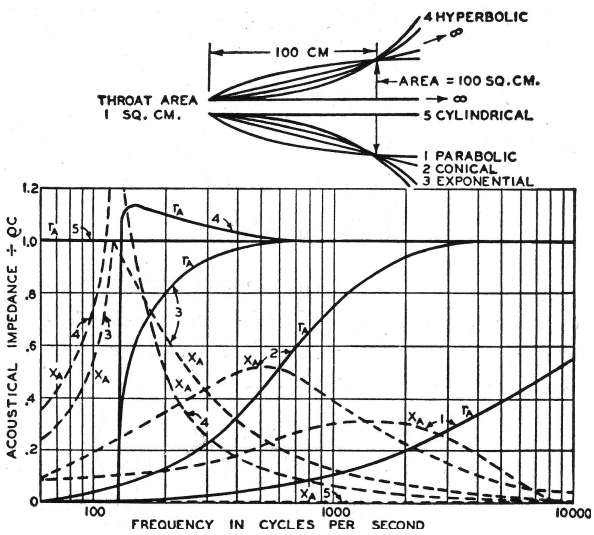


Figure 1.34: *Throat acoustical resistance r_A and acoustical reactance x_A, frequency characteristics of infinite eigen-functions of the parabolic, conical, exponential, hyperbolic and cylindrical horns,having a throat area of 1 square centimeter. Note how the "critical" frequency (defined here as the frequency where the reactive and real parts of the radiation impedance are equal) of the horn reduces dramatically with the type of horn. For the uniform horn, the reactive component is zero, so there is no cutoff frequency. For the parabolic horn (1) the cutoff is around 3 kHz. For the conical horn (2) the cutoff is at 0.6 [kHz]. For the exponential horn (3) the critical frequency is around 0.18 [kHz], which is one 16th that of the parabolic horn. For each horn the cross-sectional area is defined as 100 [cm^2] at a distance of 1 [m] from the throat (Olson, 1947, p. 101).*

To specify $\Gamma(r,s)$ we invert d'Alembert's supperposition equation (Eq. 1.150)

$$\begin{bmatrix}\mathcal{P}^+(r,s)\\ \mathcal{P}^-(r,s)\end{bmatrix} = \frac{1}{2\mathcal{Y}(r)}\begin{bmatrix}\mathcal{Y}(r) & 1\\ \mathcal{Y}(r) & -1\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & \mathcal{Z}(r)\\ 1 & -\mathcal{Z}(r)\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}. \tag{1.154}$$

The reflectance is defined as the ratio of the two pressure eigen-functions

$$\Gamma(r,s) = \frac{\mathcal{P}^-}{\mathcal{P}^+} = \frac{\mathcal{P}-\mathcal{Z}\mathcal{V}}{\mathcal{P}+\mathcal{Z}\mathcal{V}} = \frac{Z_{in}-\mathcal{Z}}{Z_{in}+\mathcal{Z}}, \tag{1.155}$$

which is equivalent to Eq. 1.151.

Given some experience with $Z_{in}(r,s)$ and $\Gamma(r,s)$, one soon appreciates the advantage of working with the reflectance over the radiation impedance/admittance $Z_{rad}(s)$ (aka immittance). The impedance has complicated properties, all of which are difficult to verify, whereas the reflectance is easily understood (it is closer to the physics). For example, we know that for a physical passive impedance $\Re Z \geq 0$. The corresponding property for the reflectance is $|\Gamma(\omega)| \leq 1$, with equality when the input resistance is zero.

Exercise:

1. Show that $\Re Y_{in}(s) \geq 0$ if and only if $|\Gamma \leq 1|$. Hint: Use Eq 1.155 (or 1.151).

2. Showing that the unit circle in the $\Gamma(s)$ plane maps onto the $\omega\jmath$ axis in the impedance plane.

Solution: To prove this take the real part of $Y_{in}(s)$ (Eq. 1.151) and show that it is greater than zero if $|\Gamma(s)| \leq 1$

$$\begin{aligned}\frac{2}{\mathcal{Y}(r)}\Re Y_{in}(s) &= \frac{1-\Gamma}{1+\Gamma} + \frac{1-\Gamma^*}{1+\Gamma^*}\\ &= \frac{(1-\Gamma)(1+\Gamma^*)+(1+\Gamma)(1-\Gamma^*)}{|1+\Gamma|^2}\\ &= \frac{2(1-|\Gamma|^2)}{|1+\Gamma|^2} \geq 0.\end{aligned}$$

In conclusion:

1. if $|\Gamma| < 1$, then $\Re Z_{in} > 0$.
2. if $|\Gamma| = 1$, then $\Re Z_{in} = 0$.

1.5.10 Lec 37c Finite length horns

For a horn of finite length L the acoustic variables $\mathcal{P}(x,s), \mathcal{V}(x,s)$ may be expressed in terms of pressure eigen-functions. If we define the forward wave $\mathcal{P}^+(x,\omega)$ as launched from $x=0$ and the retrograde wave $\mathcal{P}^-(x,\omega)$ as launched from $x=L$, we may write the pressure and velocity as

$$\begin{bmatrix}\mathcal{P}(x)\\ \mathcal{V}(x)\end{bmatrix} = \begin{bmatrix}\mathcal{P}^+(x) & \mathcal{P}^-(x-L)\\ \mathcal{Y}(x)\mathcal{P}^+(x) & -\mathcal{Y}(x)\mathcal{P}^-(x-L)\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \tag{1.156}$$

Here $\alpha(x,\omega)$ scales the forward wave and $\beta(x,\omega)$ scales the retrograde wave. Thus the reflectance $\Gamma(L,\omega) = \beta/\alpha$ is defined at the site of reflection ($x=L$). Typically the *characteristic admittance* $\mathcal{Y}(x) = A(x)/\rho_o c_o$ only depends on both the location x, and not on the Laplace frequency s. This formula may not be correct if the horn has losses ($Y_c \in \mathbb{C}$), as discussed in Kirchhoff (1868); Mason (1927, 1928); Robinson (2017).

To evaluate the coefficients $\alpha(\omega)$ and $\beta(\omega)$ we must invert Eq. 1.156. α, β are determined at the cite of reflection $x=L$.

Proof read red

Notation: Adopting subscript notation: $\mathcal{P}_x^{\pm} \equiv \mathcal{P}^{\pm}(x)$, $\mathcal{V}_x^{\pm} \equiv \mathcal{V}^{\pm}(x)$. $\mathcal{Y}_x = \mathcal{Y}(x)$ and inverting Eq. 1.156 gives

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_x = \frac{-1}{\Delta_x}\begin{bmatrix}-\mathcal{Y}_x\mathcal{P}^-(x-L) & -\mathcal{P}^-(x-L)\\ -\mathcal{Y}_x\mathcal{P}^+(x) & \mathcal{P}^+(x)\end{bmatrix}_x\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_x \tag{1.157}$$

where the determinant is

$$\Delta_x = -2\mathcal{Y}_x\mathcal{P}_x^+\mathcal{P}^-(x-L).$$

Simplifying with $x=L$ gives

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L = \frac{1}{\mathcal{Y}_L\mathcal{P}_L^+\mathcal{P}^-(L-L)}\begin{bmatrix}\mathcal{Y}_L\mathcal{P}^-(L-L) & \mathcal{P}^-(L-L)\\ \mathcal{Y}_L\mathcal{P}_L^+ & -\mathcal{P}_L^+\end{bmatrix}_L\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.158}$$

Defining $\mathcal{P}_L^- = \mathcal{P}^-(x-L)|_{x=L}$ and $\mathcal{Z}_L = 1/\mathcal{Y}_L$, it simplifies to

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L = \frac{1}{\mathcal{P}_L^+\mathcal{P}_L^-}\begin{bmatrix}\mathcal{P}_L^- & \mathcal{Z}_L\mathcal{P}_L^-\\ \mathcal{P}_L^+ & -\mathcal{Z}_L\mathcal{P}_L^+\end{bmatrix}_L\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.159}$$

Typically the eigen-functions at $x=0$ are normalize to 1 (i.e., $\mathcal{P}_0^+ = 1$ and $\mathcal{P}_L^- = 1$), thus

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L = \frac{1}{\mathcal{P}_L^+}\begin{bmatrix}1 & \mathcal{Z}_L\\ \mathcal{P}_L^+ & -\mathcal{Z}_L\mathcal{P}_L^+\end{bmatrix}_L\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L \tag{1.160}$$

This is a general expression for the eigenfunction amplitudes α, β at the reflection site $x=L$, where the two mix for the first time. The reflection coefficient is given by the ratio of β/α, which depends on the load impedance

$$Z_{load}(x=L,s) = -\mathcal{P}_L/\mathcal{V}_L.$$

The sign of $\mathcal{V}_L$ must be negative to satisfy the definition of every ABCD matrix (i.e., the output velocity must equal the input velocity of the next cell).

Substituting Eq. 1.160 into Eq. 1.156 results in an expression for the input acoustic variables at $x = 0$ in terms of those at $x = L$:

agree with ven below.

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \begin{bmatrix}\mathcal{P}_0^+ & \mathcal{P}_{-L}^- \\ \mathcal{Y}_0\mathcal{P}_0^+ & -\mathcal{Y}_0\mathcal{P}_{-L}^-\end{bmatrix} \times \frac{1}{\mathcal{P}_L^+}\begin{bmatrix}1 & \mathcal{Z}_L \\ \mathcal{P}_L^+ & -\mathcal{Z}_L\mathcal{P}_L^+\end{bmatrix}_L \begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.161}$$

Thus

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \frac{1}{\Delta_L}\begin{bmatrix}-\mathcal{Y}^-(L)\mathcal{P}_L^- - \mathcal{Y}^+(L)\mathcal{P}_L^+ & P_L^+ - P_L^- \\ -\mathcal{Y}^+(0)\mathcal{Y}^-(L)\mathcal{P}_L^- + \mathcal{Y}^-(0)\mathcal{Y}^+(L)\mathcal{P}_L^+ & -\mathcal{Y}^+(0)\mathcal{P}_L^- + \mathcal{Y}^-(0)\mathcal{P}_L^+\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_L, \tag{1.162}$$

It may be more visual to leave this expression in terms of $\Gamma(s)$ than to substitute Eq. 1.157 into Eq. 1.156.

1.5.11 Lec 37d Three examples of horns

Here we give three examples of the horn equation for the 1) uniform ($A(x) = A_o$), 2) Conical (spherical) ($A(r) = A_o r^2$) and 3) Exponential ($A(r) = A_o e^{2mr}$) horns. The input impedance for each of these horns is shown in the classic figure reproduced here, from Olson (1947, p. 101).

Summary of four classic horns: The radiation impedance $Z_{rad}(r, \omega)$ for five different horns is shown in Fig. 1.34. The same information is given in Table 1.5 is a summary of the properties for four different horns, as numerically identified in the figure 5-uniform: 1-parabolic, 2-conical and 3-exponential horns.

1) The uniform horn

The 1D wave equation [$A(r) = A_o$]

$$\frac{d^2}{dr^2}\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}.$$

Solutions: The two eigen-functions of this equation are the two d'Alembert waves (Eq. 1.95, p. 109)

$$\varrho(x,t) = \varrho^+(t - x/c) + \varrho^-(t + x/c) \;\leftrightarrow\; \mathcal{P}^+(x,s) + \mathcal{P}^-(x,s)$$

where $\mathcal{P}^\pm(x,s)$ are Laplace transform pairs representing the causal forward and retrograde traveling wave pressure amplitudes. It is convenient to normalize $\mathcal{P}^\pm(x = 0, s) = 1$. Doing so gives the normalized primary solutions

$$\varrho^+(x,t) = \delta(t - x/c_o) \leftrightarrow \mathcal{P}^+(0,s)e^{-\kappa(s)x}$$

and

$$\varrho^-(x,t) = \delta(t + x/c_o) \leftrightarrow \mathcal{P}^-(0,s)e^{\kappa(s)x}$$

where $\kappa(s) = s/c_o = \jmath\omega/c$ is called the wave-evolution function, propagation constant, or wave number. Note that for the uniform horn $\omega/c_o = 2\pi/\lambda$. When the area is not constant, λ is a complex function of frequency, resulting in a complex input impedance (admittance), internal standing waves and wave propagation loss.

The characteristic admittance (Eq. 1.146) is independent of direction. The signs must be "physically chosen," with the velocity $\mathcal{V}^\pm$ *into* the port, to assure that $\mathcal{Y} > 0$, for both waves.

Applying the boundary conditions: The general solution is compactly formulated as an ABCD matrix (i.e., Section 1.3.9, p. 89), starting from

$$\begin{bmatrix}\mathcal{P}(x)\\ \mathcal{V}(x)\end{bmatrix} = \begin{bmatrix}e^{-\kappa x} & e^{\kappa x}\\ \mathcal{Y}e^{-\kappa x} & -\mathcal{Y}e^{\kappa x}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_x, \tag{1.163}$$

where α, β are the relative weights on the two unknown eigen-functions, to be determined by the boundary conditions at $x = 0, L$, and $\kappa = s/c$, $\mathcal{Y} = 1/\mathcal{Z} = A_o/\rho_o c$.

Solving Eq. 1.163 for α and β (with $\mathcal{Z} = 1/\mathcal{Y}$ and determinant $\Delta = -2\mathcal{Y}$), at $x = L$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L = \frac{-1}{2\mathcal{Y}}\begin{bmatrix}-\mathcal{Y}\,e^{\kappa L} & -e^{\kappa L}\\ -\mathcal{Y}e^{-\kappa L} & e^{-\kappa L}\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L = \frac{1}{2}\begin{bmatrix}e^{\kappa L} & \mathcal{Z}e^{\kappa L}\\ e^{-\kappa L} & -\mathcal{Z}e^{-\kappa L}\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.164}$$

We may uniquely determine these two weights given the pressure and velocity at the boundary $x = L$, which is typically determined by the load impedance $(\mathcal{P}_L/\mathcal{V}_L)$.

Once the weights have been determined, they may be substituted back into Eq. 1.163, to determine the pressure and velocity amplitudes at any point $0 \le x \le L$.

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_x = \frac{1}{2}\begin{bmatrix}e^{-\kappa x} & e^{\kappa x}\\ \mathcal{Y}e^{-\kappa x} & -\mathcal{Y}e^{\kappa x}\end{bmatrix}\begin{bmatrix}e^{\kappa L} & \mathcal{Z}e^{\kappa L}\\ e^{-\kappa L} & -\mathcal{Z}e^{-\kappa L}\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.165}$$

Multiplying these out gives the final transmission matrix

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_x = \frac{1}{2}\begin{bmatrix}e^{\kappa(L-x)} + e^{-\kappa(L-x)} & \mathcal{Z}(e^{\kappa(L-x)} - e^{-\kappa(L-x)})\\ \mathcal{Y}(e^{\kappa(L-x)} - e^{-\kappa(L-x)}) & e^{\kappa(L-x)} + e^{-\kappa(L-x)}\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.166}$$

Applying the last boundary condition, we evaluate Eq. 1.164 to obtain the ABCD matrix at the input $(x = 0)$ (Pipes, 1958)

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \begin{bmatrix}\cosh(\kappa L) & \mathcal{Z}\sinh(\kappa L)\\ \mathcal{Y}\sinh(\kappa L & \cosh(\kappa L)\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.167}$$

Exercise: Evaluate this expression in terms of the load impedance. **Solution:** Since $Z_{load} = P_L/V_L$,

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \begin{bmatrix}\mathcal{Z}_{load}\cosh(\kappa L) & -\mathcal{Z}\sinh(\kappa L)\\ \mathcal{Z}_{load}\mathcal{Y}\sinh(\kappa L & -\cosh(\kappa L)\end{bmatrix}. \tag{1.168}$$

Impedance matrix: Expressing Eq. 1.168 as an impedance matrix gives (algebra required)

$$\begin{bmatrix}\mathcal{P}_o\\ \mathcal{P}_L\end{bmatrix}. = \frac{\mathcal{Z}}{\sinh(\kappa L)}\begin{bmatrix}\cosh(\kappa L) & 1\\ 1 & \cosh(\kappa L)\end{bmatrix}\begin{bmatrix}\mathcal{V}_o\\ \mathcal{V}_L\end{bmatrix}.$$

Input admittance Y_{in}: Given the input admittance of the horn, it is possible to determine if it is uniform, without further analysis. Namely, if the horn is uniform and infinite in length, the input impedance at $x = 0$ is

$$Y_{in}(0, s) \equiv \frac{\mathcal{V}(0, \omega)}{\mathcal{P}(0, \omega)} = \mathcal{Y},$$

since $\alpha = 1$ and $\beta = 0$. That is, for an infinite uniform horn, there are no reflections.

When the horn is terminated with a fixed impedance Z_L at $x = L$, one may substitute pressure and velocity measurements into Eq. 1.164 to find α and β, and given these, one may calculate the *reflectance* at $x = L$ (see Eq. **??**, **??**)

$$\Gamma_L(s) \equiv \left.\frac{\mathcal{P}^-}{\mathcal{P}^+}\right|_{x=L} = \frac{\beta}{\alpha} = \frac{\mathcal{P}(L,\omega) - \mathcal{Z}\mathcal{V}(L,\omega)}{\mathcal{P}(L,\omega) + \mathcal{Z}\mathcal{V}(L,\omega)} = \frac{Z_L - \mathcal{Z}}{Z_L + \mathcal{Z}}$$

given sufficiently accurate measurements of the throat pressure $\mathcal{P}(0,\omega)$, velocity $\mathcal{V}(0,\omega)$, and the characteristic impedance of the input $\mathcal{Z} = \rho_o c/A(0)$.

2) Conical horn

For each horn we must find the natural normalization from the range variable to the normalized range variable x. For the conical horn the radius is proportional to the range variable r, thus

$$A(r) = 4\pi \sin^2(\Theta/2)\, r^2. \quad [\mathrm{m}^2]$$

The angle Θ is a measure of the solid (cone) angle. When $\Theta = \pi$ we have the case of the entire sphere, so the solid angle is 4π [steradian] and the area is $4\pi r^2$. The formula for the area may be simplified by defining $A_\theta \equiv 4\pi \sin^2(\Theta/2)\, r_o^2 \ \ [m^2]$, resulting in the more convenient relation

$$A(r) = A_\theta\, (r/r_o)^2. \quad [\mathrm{m}^2].$$

This is a bit tricky because A_θ is not a constant since it depends on the place where the area was normalized, in this case r_o.

Using the conical horn area $A(r) \propto r^2$ in Eq. 1.138, p. 141 [or Eq. 1.147 (p. 144)] results in the spherical wave equation (Appendix 1.5.2 p. 131)

$$\mathcal{P}_{rr}(r,\omega) + \frac{2}{r}\mathcal{P}_r(r,\omega) = \kappa^2 \mathcal{P}(r,\omega). \tag{1.169}$$

Here $F(r) = \partial_r \ln A(r) = \frac{2}{r}$ (see Table 1.5, p. 146). Remember to apply the steradian scale factor A_θ.

3) Exponential horn: The case of the *exponential horn*

$$\mathcal{P}^\pm(r,\omega) = e^{-mr} e^{\mp j\sqrt{\omega^2-\omega_c^2}\, r/c} \tag{1.170}$$

is of special interest because the radiation impedance is purely reactive below the horn's cutoff frequency ($\omega < \omega_c = mc_o$), as may be seen from curves 3 and 4 of Fig. 1.34, since no energy can radiate from an open horn below ω_c, because

$$\kappa(s) = -m \pm \frac{\jmath}{c_o}\sqrt{\omega^2 - \omega_c^2}$$

becomes purely real for $\omega < \omega_c$ (non-propagating evanescent waves).

1.5.12 Lec 38: Solution methods

There are two distinct mathematical methods used to describe physical systems: lumped models (i.e., quasi-statics) and differential equations. We shall describe these methods for the case of the *scalar wave equation*, which describes the evolution of a scalar field, such as the average pressure or voltage, or equivalently, the flow (e.g., the volume velocity).

1. **Lumped-element method:** A system may be represented in terms of *lumped elements*, such as electrical inductors, capacitors and resistors, or their mechanical counterparts, masses, springs and dashpots. Such systems are represented by transmission-matrices rather than by differential equations, the number of which is equal to the number of elements in the network. When the system of lumped element *networks* contains only resistors and capacitors, it does not support waves, and is related to the diffusion equation in its solution. Depending on the elements in the system of equations, there can be an overlap between a diffusion process and scalar waves, represented as transmission lines, both modeled as lumped networks of 2x2 matrices (Section 1.3.9, Eq. 1.68, p. 89).

   When lumped elements are used, the equations accurately approximate the transmission line equations below a critical frequency f_c which depends on the density of model-elements. When the wavelength is longer than the physical distance between the elements (one per matrix), the approximation is equivalent to a transmission line. As the frequency increases, the wavelength eventually becomes equal to ($f = f_c$), and then shorter than the element spacing, where the quasi-static (lumped element) model breaks down. This is under the control of the modeling process, as more elements are required to represent higher frequencies (shorter wavelengths). If the nature of the solution at high frequencies is desired, the lumped parameter model fails and one must use the differential equation method. However for many (perhaps most) problems, lumped elements are easy to use, and accurate, for frequencies below the cutoff (where the wavelength approaches the element spacing). These relations are elegantly explained in Brillouin (1953).

2. **Separable coordinate systems:** Classically PDEs are often solved by a technique called *separation of variables*, which is limited to a few coordinate systems such as rectangular, cylindrical and spherical coordinates (Morse, 1948, p. 296-7). Even a slight deviation from separable specific coordinate systems represents a major barrier to further understanding, blocking insight into more general cases. These few separable coordinate systems are special cases, which have high degrees of symmetry, while the wave equation is not tied to a specific coordinate system. Thus lumped-parameter methods (quasi-statics) provides solutions over a much wider class of geometries.

   When the coordinate system is separable the resulting PDE is called a Sturm-Liouville equation, and its eigen-functions are the basis functions for solutions to these equations. Webster horn theory (Webster, 1919; Morse, 1948; Pierce, 1981) is a generalized Sturm-Louville equation which adds physics to the mathematical 19th century approach of Sturm-Liouville, in the form of the area-function of the horn.

   As is common in mathematical physics, it is the physical applications, not the mathematics, that make a theory powerful. Mathematics provides rigor, while physics provides a physical rational. Both are important: however, the relative importance depends on ones view point, and the nature of the problem being solved.

Eigen-solutions $\varrho^{\pm}(r,t)$

Because the wave equation (Eq. 1.135) is 2nd order in time, there are two causal independent eigen-solutions of the homogeneous (i.e., un-driven) Webster horn equation: an *outbound* (right-traveling) $\varrho^{+}(r,t)$ and an *inbound* (left-traveling) $\varrho^{-}(r,t)$ wave. The causal eigen-solutions may be Laplace transformed

$$\varrho^{\pm}(r,t) \leftrightarrow \mathcal{P}^{\pm}(r,s) = \int_{0^-}^{\infty} \varrho^{\pm}(r,t)e^{-st}dt,$$

into the frequency-domain, also sometimes called the time-independent representation.

These eigen-functions may be normalized so that $\mathcal{P}^{\pm}(r_o,s) = 1$, where r_o is the source excitation reference point. Other normalizations are possible.

These eigen-functions critically depend on the area function $A(r)$. Because the characteristic impedance of the wave in the horn changes with location, the must be local reflections due to the area variations. Thus there is a basic relationship between the area change $dA(r)/dr$, the eigen-functions $\mathcal{P}^{\pm}(r,s)$ and their eigen-modes.[103]

Complex vs. real frequency: We shall continue to maintain the distinction that functions of ω are Fourier transforms and functions of Laplace frequency s correspond to Laplace transforms, which, because they are causal, *are* necessarily complex analytic in s in the *right half-plane* (RHP) *region of convergence* (ROC). This distinction is critical since we typically describe impedance $Z(s)$ and admittance $Y(s)$, which must be causal, in terms of their poles and zeros, as complex analytic functions in s.[104] Likewise the eigen-functions of Eq. 1.138 are complex analytic and thus causal.

Eigen-function solutions for plane-waves: Huygens (1690) was the first to gain insight into wave propagation, today known as "Huygens's principle." While his concept showed a deep insight, we now know it was seriously flawed as it ignored the backward traveling wave (Miller, 1991). In 1747 d'Alembert, published the first correct solution for the plane-wave scalar wave equation

$$\varrho(x,t) = f(t - x/c_o) + g(t + x/c_o), \tag{1.171}$$

where $f(\cdot)$ and $g(\cdot)$ are quite general functions of their argument. That this is the solution may be easily shown by use of the chain rule, by taking partials with respect to x and t. Surprisingly, this is the solution even when the functions are not differentiable. For example, $u(t \mp x/c_o)$ and $\delta(t \mp x/c_o)$ are valid solutions, even though the proof is more difficult.

In terms of the physics, d'Alembert's general solution describes two arbitrary wave-forms $f(\cdot)$, $g(\cdot)$, traveling at a speed c_o, one forward and one reversed. Thus his solution is quite easily visualized. As previously discussed, Newton (1687) was the first to calculate the speed of sound c_o, albeit with an error of $\sqrt{1.4}$. This error was not corrected for over 200 years, following the creation of thermodynamics and the concept of an adiabatic process.

[103] These relationships will be explored further in Chapter J.

[104] When an analytic function of complex variable s includes the pole it is called a *Laurent series* in s. For example, the impedance of a capacitor C is $Z_c(s) = 1/sC$, which is analytic in s everywhere other than $s = 0$. The capacitor has a voltage time response given by the integral of the current, i.e., $v(t) = \frac{1}{C}\int^t i(t)dt = \frac{1}{C}u(t) \star i(t)$, where $u(t)$ is the Heaviside step function and $\star$ represents convolution.

Exercise: By the use of the chain rule, prove that d'Alembert's formula satisfies the 1D wave equation. **Solution:** Taking a derivative with respect to t and r gives

- $\partial_t \varrho(r,t) = -c_o f'(r - c_o t) + c_o g'(r + c_o t)$
- $\partial_r \varrho(r,t) = f'(r - c_o t) + g'(r + c_o t)$,

and a second derivative gives

- $\partial_{tt} \varrho(r,t) = c_o^2 f''(r - c_o t) + c_o^2 g''(r + c_o t)$
- $\partial_{rr} \varrho(r,t) = f''(r - c_o t) + g''(r + c_o t)$.

From these last two equations we have the 1D wave equation

$$\partial_{rr} \varrho(r,t) = \frac{1}{c_o^2} \partial_{tt} \varrho(r,t),$$

having solutions Eq. 1.171, in disagreement with Huygens's 1690 "principle" (Miller, 1991).

Example: Assuming $f(\cdot), g(\cdot)$ are $\delta(\cdot)$, find the Laplace transform of the solution. **Solution:** Using Table F.3 (p. 210) of Laplace transforms on Eq. 1.171 gives

$$\varrho(x,t) = \delta(t - x/c_o) + \delta(t + x/c_o) \leftrightarrow e^{-sx/c_o} + e^{sx/c_o}. \tag{1.172}$$

Note that the delay is $T_o = \pm x/c_o$.

3D d'Alembert spherical eigen-functions: The d'Alembert solution generalizes to 3D waves by changing the area function in Eq. 1.138 with the 3D area $A(r) = A_o r^2$ (Table 1.5, p. 146)[105]

$$\nabla_r^2 \varrho(r,t) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \varrho(r,t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \varrho(r,t).$$

Multiplying by r results in the general spherical (3D) d'Alembert wave equation solution

$$\varrho(r,t) = \frac{f(t - r/c_o)}{r} + \frac{g(t + r/c_o)}{r},$$

for arbitrary wave-forms $f(\cdot)$ and $g(\cdot)$. These are the eigen-functions for the spherical scalar wave equation.

1.5.13 Lec 39: Integral forms of $\nabla()$, $\boldsymbol{\nabla} \cdot ()$ and $\boldsymbol{\nabla} \times ()$

The *vector wave equation* describes the evolution of a vector field, such as Maxwell's electric field vector $\boldsymbol{E}(\boldsymbol{x}, t)$. When these fields are restricted to a one dimensional domain they are known as *guided waves* constrained by *wave guides*.

Nathan suggested $\cdot\, v$.

There are two definitions for each operator: the point, or differential form, and the integral limit form. The integral form gives an intuitive view of what the operator does, and in the limit, converges to the differential form. These two definitions are discussed next.

The divergence and the curl each have fundamental theorems: Gauss's law (divergence theorem) and Stokes's law (curl theorem). Without the use of these very fundamental vector calculus theorems, Maxwell's equations cannot be understood. The history of these important theorems is discussed in the caption of Fig. 1.13, p. 56.

[105] This form of the spherical Laplacian is discussed in Appendix 1.5.2, Eq. 1.120, p. 131.

Gradient: $\boldsymbol{E} = -\nabla\phi(x, y, z)$ [V/m]

As briefly summarized on page 128, the differential definition of the gradient maps $\mathbb{R}^1 \underset{\nabla}{\mapsto} \mathbb{R}^3$. For example, the electric field strength is the gradient of the voltage

$$\boldsymbol{E}(\boldsymbol{x}) = -\left[\partial_x,\, \partial_y,\, \partial_z\right]^T \phi(\boldsymbol{x}) = -\left[\frac{\partial\phi}{\partial x},\, \frac{\partial\phi}{\partial y},\, \frac{\partial\phi}{\partial z}\right]^T (\boldsymbol{x}).$$

The negative sign is optional.

The gradient is the slope of the tangent plane of the potential $\phi(\boldsymbol{x})$ at $\boldsymbol{x}$ pointing in the direction of the maximum slope. The gradient (i.e., $\boldsymbol{E}(\boldsymbol{x})$) is $\perp$ of the constant potential contours of $\phi(\boldsymbol{x})$ (the iso-potential lines), since derivatives along any iso-contours must be zero.

As a specific example, consider the paraboloid $z = 1 - (x^2 + y^2)$ as the potential, with iso-potential circles of constant z that have radius of zero at $z = 1$, and unit radius at $z = 0$. The negative gradient

$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla z(x, y) = 2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + 0\hat{\mathbf{z}})$$

is $\perp$ to the circles of constant radius (constant z), and thus points in the direction of the radius.

If one were free-fall skiing this surface, they would be the first one down the hill. Normally skiers try to stay close to the iso-clines (not in the direction of the gradient), so they can stay in control. If you ski an iso-cline, you must walk, since there is no pull due to gravity.

The integral definition of the gradient: The gradient may be defined in terms of the integral

$$\nabla\phi(x, y, z) \equiv \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\int_{\mathcal{S}} \phi(x, y, z)\, \hat{\mathbf{n}}\, d|\mathcal{S}|}{|\mathcal{S}|} \right\}, \tag{1.173}$$

over a closed surface $\mathcal{S}$, having area $|\mathcal{S}|$ and volume $||\mathcal{S}||$, centered at (x, y, z)(Greenberg, 1988, p. 773).[106] Here $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface $\mathcal{S}$

$$\hat{\boldsymbol{n}} = \frac{\nabla\phi}{||\nabla\phi||}.$$

The dimensions of Eq. 1.173 are in the units of the potential times the area, divided by the volume, as needed for a gradient (e.g., [V/m]).

The natural way to define the surface and volume is to place the surface on the iso-potential surfaces, forming either a cube or pill-box shaped volume. As the volume goes to zero, so must the area. One must avoid *irregular volumes* such that the area is finite as the volume goes to zero.(Greenberg, 1988, footnote p. 762).

A well-known example is the potential

$$\phi(x, y, z) = \frac{Q}{\epsilon_o\sqrt{x^2 + y^2 + z^2}} = \frac{Q}{\epsilon_o R} \quad \text{[V]}$$

around a point charge Q [SI Units of Coulombs]. The constant ϵ_o is the *permittivity* [farad/m^2].

[106] See further discussions on pages Greenberg (1988, pp. 778, 791, 809).

How does this work? To better understand what Eq. 1.173 means, consider a three-dimensional Taylor series expansion of the potential in $\boldsymbol{x}$ about the limit point $\boldsymbol{x}_o$ To compute the *higher order terms* (HOT) one needs the *Hessian* matrix[107]

$$\phi(\boldsymbol{x}) \approx \phi(\boldsymbol{x}_o) + \nabla\phi(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{x}_o) + \text{HOT}.$$

We could define the gradient using this relationship as

$$\nabla\phi(\boldsymbol{x}_o) = \lim_{\boldsymbol{x}\to\boldsymbol{x}_o} \frac{\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_o)}{\boldsymbol{x} - \boldsymbol{x}_o}.$$

For this definition to apply, $\boldsymbol{x}$ must approach $\boldsymbol{x}_o$ along $\hat{\boldsymbol{n}}$.

The natural form for the surface $|\mathcal{S}|$ is to lie along the iso-potential surfaces as much as possible, so that the integral is a constant (the potential) times the area. The remainder of the surface must be perpendicular to these iso-potential surfaces, in the direction of the gradient, or maximum change of the potential. The secret to the integral definition is in taking the limit. As the volume $||\mathcal{S}||$ shrinks to zero, the HOT terms are small, and the integral reduces to the first-order term in the Taylor expansion, since the constant term integrates to zero. Such a construction is used in the proof of the Webster horn equation (1.5.6, p. 142; Fig. 1.33, p. 143).

Divergence: $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho$ [Col/m^3]

As briefly summarized on page 129, the differential definition of the gradient which maps $\mathbb{R}^3 \underset{\nabla\cdot}{\mapsto} \mathbb{R}^1$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} \equiv [\partial_x,\, \partial_y,\, \partial_z] \cdot \boldsymbol{D} = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right] = \rho(x, y, z).$$

The divergence is a direct measure of the flux (flow) of the vector field it operates on ($\boldsymbol{D}$), coming from $\boldsymbol{x}$. A vector field is said to be *incompressible* if the divergence of that field is zero. It is therefore *compressible* when the divergence is non-zero. Compared to air, water is considered to be incompressible. However at very low frequencies, air can also be considered as incompressible. Thus the definition of compressible depends on the wavelength in the medium, so the terms must be used with some awareness of the circumstances.

Models of the electron: It is helpful to consider the physics of the electron, a negatively charged particle that is frequently treated as a single point in space. If the size were truly zero, there could be no magnetic spin moment. One size estimate is the Lorentz radius, 2.810^{-15} [m]. One could summarize the Lorentz radius as follows: *Here lie many unsolved problems in physics*. More specifically, at dimensions of the Lorentz radius, what exactly is the structure of the electron?

Ignoring the difficulties, if one integrates the charge density of the electron over the Lorentz radius and places the total charge at a single point, then one may make a grossly oversimplified model of the electron. For example, the *electric displacement* ($\boldsymbol{D} = \epsilon_o\boldsymbol{E}$) (flux density) around a point charge is

$$\boldsymbol{D} = -\epsilon_o\nabla\phi(R) = -Q\nabla\left\{\frac{1}{R}\right\} = -Q\,\delta(R).$$

This is a formula taught in many classic texts, but one should remember how crude a model of an electron it is. But it does describe the electric flux in an easily remembered form. However,

[107] $H_{i,j} = \partial^2(\boldsymbol{x})\phi/\partial x_i\partial x_j$, which will exist if the potential is analytic in $\boldsymbol{x}$ at $\boldsymbol{x}_o$.

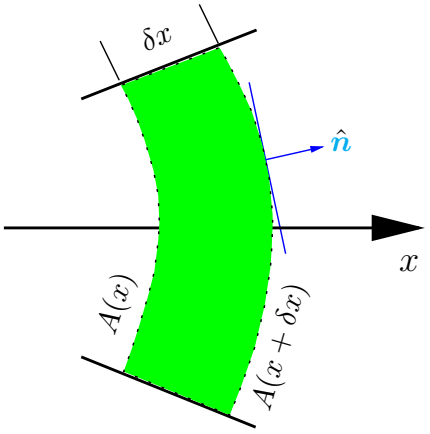


Figure 1.35: *Derivation of Gauss's law: The divergence of the velocity* $\nabla \cdot \mathbf{u}$*, within* δx*, shown as the filled shaded region, is integrated over the closed volume. Then the divergence theorem is applied, transforming the integral to a surface integral normal to the surface, resulting in the difference of the two volume velocities* $\delta\nu = \nu(x+\delta x) - \nu(x) = [\mathbf{u}(x+\delta x) \cdot \boldsymbol{A}(x+\delta x) - \mathbf{u}(x) \cdot \boldsymbol{A}(x)]$*, as the limit of* $\delta x \to 0$*. Because the ends are in the direction of* $\mathbf{u}$*, the contribution to the volume velocity from the ends is zero.*

computationally, it is less nice, due to the delta function. The main limitation of this model is that the electron has a magnetic dipole moment (aka, *spin*), which a simple point charge model does not capture. When placed in a magnetic field, due to the dipole, the electron will align itself with the field.

Divergence and Gauss's law

Like the gradient, the divergence of a vector field may be defined as the surface integral of a *compressible* vector field, as a limit as the volume enclosed by the surface goes to zero. As for the case of the gradient, for this definition to make sense, the surface $\mathcal{S}$ must be a closed, defining volume $\mathcal{V}$. The difference is that the surface integral is over the normal component of the vector field being operated on. Specifically (Greenberg, 1988, p. 762-763),

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \lim_{|\mathcal{S}| \to 0} \left\{ \frac{\int_{\mathcal{S}} \boldsymbol{D} \cdot \hat{\mathbf{n}}\, d|\mathcal{S}|}{|\mathcal{S}|} \right\} = \rho(x, y, z). \tag{1.174}$$

As with the case of the gradient we have defined the surface as $\mathcal{S}$, its area as $|\mathcal{S}|$ and the volume within as $||\mathcal{S}||$. As the area $|\mathcal{S}|$ goes to zero, so does the volume $||\mathcal{S}||$. This is a necessary condition for the integral to converge to the divergence.

Here $\hat{\mathbf{n}}$ is a unit vector normal to the surface $\mathcal{S}$. The limit, as the volume goes to zero, defines the total flux across the surface. Thus the surface integral is a measure of the total flux $\perp$ to the surface. It is helpful to compare this formula with that for the gradient Eq. 1.173.

Gauss's law: The above definitions resulted in a major breakthrough in vector calculus, the fundamental theorem of vector calculus (*aka, Gauss's law*):

> *The surface integral over the normal component of the flux (i.e., the total flux) is equal to the volume integral over the divergence of the flux.*

For the electrical case this is equivalent to the observation that the total flux across the surface is equal to the net charge enclosed by the surface. Since the volume integral over charge density $\rho(x, y, z)$ is total charge enclosed Q_{enc} [Col],

$$Q_{enc} = \iiint_{||\mathcal{S}||} \boldsymbol{\nabla} \cdot \boldsymbol{D}\, ||d\mathcal{S}|| = \iint_{|\mathcal{S}|} \boldsymbol{D} \cdot \hat{\boldsymbol{n}}\, d|\mathcal{S}|. \tag{1.175}$$

When the surface integral over the normal component of $\boldsymbol{D}(\boldsymbol{x})$ is zero, the charge density $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho(\boldsymbol{x})$ and total charge, are both zero.

Taking the derivative with respect to time, Eq. 1.175 evaluates the total component of the current, normal to the surface:

$$\mathcal{I} = \iint_{|\mathcal{S}|} \boldsymbol{D}\cdot\hat{\boldsymbol{n}}\, d|\mathcal{S}| = \dot{Q}_{enc} = \iiint_{||\mathcal{S}||} \dot{\rho}_{enc}\, d||\mathcal{S}||. \tag{1.176}$$

As summarized by Feynman (1970c, p. 13-2):

> *The current leaving the closed surface $|\mathcal{S}|$ equals the rate of the charge leaving that volume $||\mathcal{S}||$, defined by that surface.*

Of course the surface must be closed to define the volume, a necessary condition for Gauss's law. This reduces to a common sense summary that can be grasped intuitively, an example of the beauty in Feynman's understanding.

Integral definition of the curl: $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{C}$

As briefly summarized on page 129 (p. 129), the differential definition of the curl maps $\mathbb{R}^3 \underset{\boldsymbol{\nabla}\times}{\mapsto} \mathbb{R}^3$. For example, the curl of the *magnetic field strength* $\boldsymbol{H}(\boldsymbol{x})$ is equal to the total current $\boldsymbol{C}$

$$\boldsymbol{\nabla}\times\boldsymbol{H} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \boldsymbol{C}. \quad \text{[A]}$$

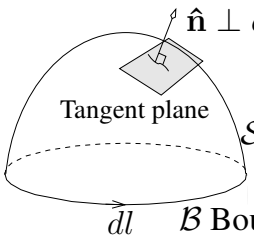
As we shall see in Sect. 1.5.15 (p. 165), the curl and the divergence are both key when writing out Maxwell's four equations. Without a full understanding of these two differential operators ($\boldsymbol{\nabla}\cdot$, $\boldsymbol{\nabla}\times$), there is no hope of understanding Maxwell's basic result, typically viewed as the most important equations of mathematical physics, and the starting point for Einstein's relativity theories. Some will say that quantum mechanics falls outside the realm of MEs, but this is at least open to debate, if not hotly debated.

The curl is a measure of the rotation of a vector field. If this were water, it would correspond to the angular momentum of the water, such as water going down the drain, as in a whirlpool, or with wind, a tornado. A spinning top is another an excellent example, given a spinning solid body. While a top (aka gyroscope) will fall over if not spinning, once it is spinning, it can stably stand on its pointed tip. These systems are stable due to conservation of angular momentum: Once something is spinning, it will continue to spin.

Example: When $\boldsymbol{H} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$, $\boldsymbol{\nabla}\times\boldsymbol{H} = 2\hat{\mathbf{z}}$, and thus has a constant *rotation*; when $\boldsymbol{H} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$, $\boldsymbol{\nabla}\times\boldsymbol{H} = \mathbf{0}$ has a curl of zero, and thus is *irrotational*. There are rules that precisely govern when a vector field is rotational versus irrotational, and compressible versus incompressible. These classes are dictated by Helmholtz's theorem, the third fundamental theorem of vector calculus (Eq. 1.183, p. 163).

Curl and Stokes's law: As in the cases of the gradient and divergence, the curl also may be written in integral form, allowing for the physical interpretation of its meaning.

Surface integral definition of $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{C}$ where the current $\boldsymbol{C}$ is $\perp$ to the rotation plane of $\boldsymbol{H}$. Stokes's law states that the *open surface integral* over the normal component of the curl of the



$$\boldsymbol{\nabla}\times\boldsymbol{H} \equiv \lim_{|\mathcal{S}|\to 0}\left\{\frac{\int_{\mathcal{S}}\hat{\boldsymbol{n}}\times\boldsymbol{H}\,d|\mathcal{S}|}{|\mathcal{S}|}\right\}$$

$$\mathcal{I}_{enc} = \iint_{|\mathcal{S}|}(\boldsymbol{\nabla}\times\boldsymbol{H})\cdot\hat{\boldsymbol{n}}\,d|\mathcal{S}| = \oint_{\mathcal{B}}\boldsymbol{H}\cdot d\boldsymbol{l} \quad [\text{A}]$$

Figure 1.36: *The integral definition of the curl is related to that of the divergence (Greenberg, 1988, p. 774), as an integration over the tangent to the surface,* **except:** *1) the curl is defined as the* cross product $\hat{\boldsymbol{n}}\times\boldsymbol{H}$ *[A/m^2], of* $\hat{\boldsymbol{n}}$ *with the current density* $\boldsymbol{H}$*, and 2) the surface is open, leaving a boundary* $\mathcal{B}$ *along the open edge. As with the divergence, which leads to Gauss's law, this definition leads to a second fundamental theorem of vector calculus:* Stokes's law *(aka the* curl theorem*).*

magnetic field strength ($\hat{\boldsymbol{n}}\,\dot{\boldsymbol{\nabla}}\times\boldsymbol{H}$ [A/m^2]) is equal to the line integral $\oint_{\mathcal{B}}\boldsymbol{H}\cdot dl$ along the boundary $\mathcal{B}$. As summarized in Fig. 1.36, Stokes's law is

$$\mathcal{I}_{enc} = \iint_{|\mathcal{S}|}(\boldsymbol{\nabla}\times\boldsymbol{H})\cdot\hat{\boldsymbol{n}}\,d|\mathcal{S}| = \oint_{\mathcal{B}}\boldsymbol{H}\cdot d\boldsymbol{l} \quad [\text{A}], \tag{1.177}$$

namely

> *The line integral of* $\boldsymbol{H}$ *along the open surface's boundary* $\mathcal{B}$ *is equal to the total current enclosed* $\mathcal{I}_{enc}$ [A].

Table 1.6: *Summary of the fundamental theorems of integral calculus, each of which deals with integration. There are at least two main theorems related to scalar calculus, and three more for vector calculus.*

Name	Mapping	p.	Description
Leibniz (FTC)	$\mathbb{R}^1 \to \mathbb{R}^0$	105	Area under a real curve
FTCC	$\mathbb{R}^1 \to \mathbb{R}^0$	105	Area under a complex curve
Cauchy's Theorem	$\mathbb{C}^1 \to \mathbb{C}^0$	119	Close integral over analytic region is zero
Cauchy's Integral Formula	$\mathbb{C}^1 \to \mathbb{C}^0$	119	Residue integration and complex analytic functions
residue Theorem	$\mathbb{C}^1 \to \mathbb{C}^0$	119	Residue integration and complex analytic functions

Summing it up: As mentioned earlier (Fig. 1.13, p. 56), the history of the discovery and proof of this theorem is both complex and interesting.

Since integration is a linear process (sums of smaller elements), one may tile, or tessellate, the surface, breaking it up into smaller surfaces and their boundaries, the sum of which is equal to the integral over the original boundary. This is an important concept, which leads to the proof of Stokes's law.

The integral formulations of Gauss's and Stokes's laws use $\hat{\boldsymbol{n}}\cdot\boldsymbol{D}$ and $\boldsymbol{H}\times\hat{\boldsymbol{n}}$ in the integrands. The key distinction between the two laws naturally follows from the properties of the scalar $(\boldsymbol{A}\cdot\boldsymbol{B})$ and vector $(\boldsymbol{A}\times\boldsymbol{B})$ products, as discussed in Sect. 1.3.6, p. 77, and detailed in Fig. 1.16, p. 80. To fully appreciate the differences between Gauss's and Stokes's laws, these two types of vector products must be mastered.

Paraphrasing Feynman (1970c, 3-12),

1. $V_2 = V_1 + \int_1^2 \nabla\Phi \cdot d\mathcal{S}$
2. $\oint \boldsymbol{D}\,\hat{\boldsymbol{n}}\,d|\mathcal{S}| = \oint \boldsymbol{\nabla}\cdot\boldsymbol{D}\,d||\mathcal{S}||$
3. $\oint_{\mathcal{B}} \boldsymbol{E}\,d\boldsymbol{l} = \oint_{|\mathcal{S}|}(\boldsymbol{\nabla}\times\boldsymbol{E})\cdot\hat{\boldsymbol{n}}\;d|\mathcal{S}|$

1.5.14 Lec 40 Second-order operators: Terminology

Besides the above first-order vector derivatives, second-order combinations exist, the most common being the scalar Laplacian $\boldsymbol{\nabla}\cdot\nabla() = \nabla^2()$ (Table 1.4, p. 128; Appendix 1.5.2, p. 131).

ook "Div, grad, curl and at"

There are other important second-order combinations of ∇, enough that we need a memory aid to remember them. Thus I define mnemonics DoC, DoG, CoG CoC and GoD as follows:

1. DoG: Divergence of the gradient ($\boldsymbol{\nabla}\cdot\nabla = \nabla^2$), i.e., Laplacian,
2. DoC*: Divergence of the curl ($\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times$),
3. CoG*: Curl of the gradient ($\boldsymbol{\nabla}\times\nabla$),
4. CoC: Curl of the curl ($\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times$), and
5. GoD: Gradient of the Divergence (GoD) $\nabla\boldsymbol{\nabla}\cdot$ and the vector Laplacian GoD* $\boldsymbol{\nabla}^2$.

DoC*() and CoG*() are special because they are always zero:

$$\boldsymbol{\nabla}\times\nabla\phi = 0; \qquad \nabla\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = 0,$$

a property that makes them useful in proving the *fundamental Theorem of Vector Calculus* (Helmholtz' decomposition, Eq. 1.183, p. 163). A third key vector identity CoC may be expanded as

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A} = \nabla(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A}, \tag{1.178}$$

defining the vector Laplacian (i.e., GoD*=GoD-CoC): $\boldsymbol{\nabla}^2() = \nabla\boldsymbol{\nabla}\cdot() - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times()$.

When using second-order differential operators one must be careful with the order of operations, which can be subtle in a few cases. Most of this is common sense. For example, don't operate on a scalar field with $\boldsymbol{\nabla}\times$, and don't operate on a vector field with ∇. The vector Laplacian GoD* must *n*ot be thought of as $\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{A})$, rather it acts as the Laplacian on each vector component $\boldsymbol{\nabla}^2\boldsymbol{A} = \nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$.

Exercise: Show that GoD and GoD* differ. **Solution:** Use CoC on a $\boldsymbol{A}$ to explore this relationship.

Helmholtz's decomposition

We may now restate everything defined above in terms of two types of vector fields that decompose every analytic vector field. The *irrotational field* is defined as one that is "curl free." An *incompressible field* is one that is "diverge free." According to Helmholtz's decomposition, every analytic vector field may be decomposed into independent rotational and a compressible components. Another name for *Helmholtz decomposition* is the *fundamental theorem of vector calculus* (FTVC); Gauss's and Stokes's laws, along with Helmholtz's decomposition, form the three key

fundamental theorems of vector calculus. Images of Helmholtz and Kirchhoff are provided in Fig. 1.37, p. 163.

A *magnetic solenoidal field* is one that is generated by a solenoidal coil, and is an excellent approximation, uniform inside the coil. As a result, the curl of a solenoidal field is, to a good approximation, zero. I recommend you know this term (it is widely used), but suggest the preferred term *irrotational*. Strictly speaking, the term "solenoidal field" only applies to a magnetic field produced by a solenoid, thus the term is specific to that case.

The decomposition of differentiable vector fields: This theorem is easily stated (and proved), but less easily appreciated (Heras, 2016). A physical description facilitates: *Every vector field may be split into two independent parts: dilation and rotation.* We have seen this same idea appear in vector algebra, where the scalar and cross products of two vectors are perpendicular (Fig. 1.16, p. 80). Also think of linear and angular momentum, which are independent in that they represent different ways of absorbing and delivering kinetic energy, leading to independent degrees of freedom. Thus the idea of the linear and rotational parts being independent is a common theme, rooted in geometry. In the same sense, a vector field may be split into a dilation and rotational parts, which are independent (but can interact under certain conditions). An object with mass can be moving along a path, and independently be rotating. The two modes of motion define different types of kinetic energy: translational (compressional) and rotational. In some real sense, Helmholtz decomposition quantifies these degrees of freedom: one DoF for translation and three DoFs for rotation. Each eigen-mode of vibration can be viewed as a DoF. The role of the FTVC is especially powerful when applied to Maxwell's Eqs.

Solving Maxwell's equations

The application of FTVC is not as obvious as it might seem when applied to Maxwell's equations, due to DoC and Cog. In Section 1.5.1 we defined the electric and magnetic field strengths in terms of the scalar and vector potentials $\phi(\boldsymbol{x}, t)$ and $\boldsymbol{A}(\boldsymbol{x}, t)$, repeated here for convenience. The electric field strength is (Eq. 1.111, p. 127)

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial}{\partial t}\boldsymbol{A} \qquad (1.111)$$

while the magnetic field strength is (Eq. 1.112, p. 127)

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \qquad (1.112)$$

which are the solutions to Maxwell's equations in terms of the two potentials (Sommerfeld, 1952, p. 146). Note these relations are invariant to the addition of a constant to each potential, and by the application of DoC and CoG, they are equivalent to Maxwell's equations.

Helmholtz theorem and the potential representation: Taking the curl of Eq. 1.111, and using CoG=0, recovers Maxwell's electric equation

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E} &= -\cancelto{0}{\boldsymbol{\nabla}\times\nabla\Phi} + \boldsymbol{\nabla}\times\frac{\partial\boldsymbol{A}}{\partial t}\\ &= -\frac{\partial\boldsymbol{B}}{\partial t},\end{aligned} \qquad (1.179)$$

while taking the divergence along with DoC=0

$$\cancelto{0}{\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\boldsymbol{E}} = -\frac{\partial\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{x},t)}{\partial t} = 0, \tag{1.180}$$

requires that $\boldsymbol{\nabla}\cdot\boldsymbol{B}$ is independent of time, and therefore that

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}(x) = 0. \tag{1.181}$$

We would like to recover Maxwell's magnetic equation $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{C}$ from the potential solution. Taking the curl of Eq. 1.112 gives

$$\frac{1}{\mu_o}\boldsymbol{\nabla}\times\mu_o\boldsymbol{H} = \boldsymbol{C} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A} = \nabla\boldsymbol{\nabla}\cdot\boldsymbol{A} - \boldsymbol{\nabla}^2\boldsymbol{A}.$$

This says that the current $\boldsymbol{C}$ only depends on $\boldsymbol{A}$, which follows directly from Eq. 1.112 forward.

Since $\boldsymbol{A}$ must satisfy the wave equation,

$$\boldsymbol{\nabla}^2\boldsymbol{A} = \frac{1}{c_o^2}\boldsymbol{A} - \boldsymbol{C},$$

which requires that

$$\nabla\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{c_o^2}\boldsymbol{A}.$$

Taking the divergence of Eq. 1.111 gives an expression for $\boldsymbol{\nabla}\cdot\boldsymbol{A}$:

$$\frac{1}{\epsilon_o}\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho/\epsilon_o = -\nabla^2\phi - \partial_t\boldsymbol{\nabla}\cdot\boldsymbol{A}.$$

Here ϵ_0 [Col/m] and μ_o [H/m].

Exercise: Starting from the values of the speed of light $c_o = 3\times10^8$ [m/s] and the characteristic resistance of light waves $r_o = 377$ [ohms], use the formula for $c_0 = 1/\sqrt{\mu_o\epsilon_o}$ and $r_o = \sqrt{\epsilon_o/\mu_o}$ to find values for ϵ_o and μ_o. **Solution:** Squaring $c_o^2 = 1/\mu_o\epsilon_o$ and $r_o^2 = \mu_o/\epsilon_o$ we may solve for the two unknowns: $c_o^2 r_o^2 = \frac{1}{\cancel{\mu_o}\epsilon_o}\frac{\cancel{\mu_o}}{\epsilon_o} = 1/\epsilon_o^2$, thus $\epsilon_o = 1/c_o r_o = \frac{10^{-8}}{3\cdot377} = 8.84\times10^{-12}$ [Fd/m]. Likewise $\mu_o = r_o/c_o = (377/3)\times10^{-8} \approx 125.67\times10^{-8}$. The value of μ_o is defined in the international SI standard as $4\pi10^{-7} \approx 12.56610^{-7}$ [H/m].

In conclusion, Eq. 1.111, along with DoC=0 and CoG=0, give Maxwell's Eq. 1.179 and Eq. 1.181 result. It would appear that Eq. 1.111 is the key. This equation defines the magnetic component of the field, expressed in terms of its vector potential, in the same way as Eq. 1.110 describes $\boldsymbol{E}(x,t)$ in terms of the scalar potential (voltage). Does the same argument apply for Eq. 1.112?

Exercise: Take the divergence of Maxwell's equation for the magnetic intensity

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{x},t) = \boldsymbol{J}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{x},t)$$

and explain what results. **Solution:** The divergence of the curl is always zero (DoC=0), thus

$$\cancelto{0}{\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\boldsymbol{H}}(\boldsymbol{x},t) = \boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\rho(\boldsymbol{x},t) = 0, \tag{1.182}$$

which is conservation of charge.

evaluation

Helmholtz's decomposition is expressed as the linear sum of a scalar potential $\phi(x, y, z)$ (think voltage) and a vector potential (think magnetic vector potential). Specifically

$$\boldsymbol{E}(\boldsymbol{x}, s) = -\nabla\phi(\boldsymbol{x}, s) + \nabla \times \boldsymbol{A}(x, s), \tag{1.183}$$

where ϕ is the scalar and $\boldsymbol{A}$ is the vector potential, as a function of the Laplace frequency s. Of course this decomposition is general (not limited to the electro-magnetic case). It applies to linear fluid vector fields, which include most liquids and air. When the rotational and dilation become coupled, this relation must break down.[108]

Figure 1.37: *Left: von Helmholtz portrait taken from the English translation of his 1858 paper "On integrals of the hydrodynamic equations that correspond to Vortex motions" (in German) (Helmholtz, 1978). Right: Gustav Kirchhoff.*

To show how this relationship splits the vector fields $\boldsymbol{E}$ into its two parts, we need DoC and CoG, the two key vector identities that are always zero for analytic fields: the *curl of the gradient* (CoG)

$$\boldsymbol{\nabla} \times \nabla\phi(\boldsymbol{x}) = 0, \tag{1.184}$$

and the *divergence of the curl*[109] (DoC)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0. \tag{1.185}$$

The above identities are easily verified by working out a few specific examples, based on the definitions of the three operators, gradient, divergence and curl, or in terms of the operator's integral definitions, defined in Sect. 1.5.13, p. 154. The identities have a physical meaning, as stated above: every vector field may be split into its translational and rotational parts. If $\boldsymbol{E}$ is the electric field [V/m], ϕ is the voltage and $\boldsymbol{A}$ is the induced rotational part, induced by a current. We shall explore this in our discussion of Maxwell's equations in Sect. 1.5.15 and Chapter J.

By applying these two identities to Helmholtz's decomposition, we can better appreciate the theorem's significance. It is a form of proof actually, once you have satisfied yourself that the vector identities are true. In fact one can work backward using a physical argument, that rotational

[108] The nonlinear Navier–Stokes equations may be an example.

[109] Helmholtz was the first person to apply mathematics in modeling the eye and the ear (Helmholtz, 1863a).

momentum (rotational energy) is independent of the translational momentum. Once these forces are made clear, the vector operations all take on a very well defined meaning, and the mathematical constructions, centered around Helmholtz's theorem, begin to provide some common-sense meaning. One could conclude that the physics is simply related to the geometry via the scalar and vector product.

Specifically, if we take the divergence of Eq. 1.183, and use the DoG, then

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \boldsymbol{\nabla}\cdot\{-\nabla\phi + \cancelto{0}{\boldsymbol{\nabla}\times\boldsymbol{A}}\} = -\boldsymbol{\nabla}\cdot\nabla\phi = -\nabla^2\phi,$$

since the DoG zeros the vector potential $\boldsymbol{A}(x,y,z)$. If instead we use the CoG, then

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}\times\{\cancelto{0}{-\nabla\phi} + \boldsymbol{\nabla}\times\boldsymbol{A}\} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A} = \nabla(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A},$$

since the CoG zeros the scalar field $\phi(x,y,z)$. The last expression requires GoD.

Table 1.7: *The four possible classifications of scalar and vector potential fields: rotational/irrotational. compressible/incompressible. Rotational fields are generated by the vector potential (e.g., $\boldsymbol{A}(\boldsymbol{x},t)$), while compressible fields are generated by the scalar potentials (e.g., voltage $\phi(\boldsymbol{x},t)$, velocity ψ, pressure $\varrho(\boldsymbol{x},t)$ or temperature $T(\boldsymbol{x},t)$).*

Field: $\boldsymbol{v}(\boldsymbol{x},t)$	Compressible $\boldsymbol{\nabla}\cdot\boldsymbol{v}\neq 0$	Incompressible $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$
Rotational $\boldsymbol{\nabla}\times\boldsymbol{v}\neq 0$	$\boldsymbol{v}=\nabla\phi+\boldsymbol{\nabla}\times\boldsymbol{\omega}$ Vector wave Eq. (EM) $\boldsymbol{\nabla^2 v}=\frac{1}{c^2}\ddot{\boldsymbol{v}}$	$\boldsymbol{v}=\boldsymbol{\nabla}\times\boldsymbol{w}$ Lubrication theory Boundary layers
Irrotational Conservative $\boldsymbol{\nabla}\times\boldsymbol{v}=0$	Acoustics $\boldsymbol{v}=\nabla\psi$ $\nabla^2\varrho(\boldsymbol{x},t)=\frac{1}{c^2}\ddot{\varrho}(\boldsymbol{x},t)$	Statics $\nabla^2\phi=0$ Laplace's Eq. ($c\to\infty$)

The four categories of linear fluid flow: The following is a summary of the four cases for fluid flow, as summarized in Fig. 1.7:

1,1 Compressible and rotational fluid (general case): $\nabla\phi\neq 0$, $\boldsymbol{\nabla}\times\boldsymbol{w}\neq 0$. This is the case of wave propagation in a medium where viscosity cannot be ignored, as in the case of acoustics close to the boundaries, where viscosity contributes to losses (Batchelor, 1967).

1,2 Incompressible, rotational, fluid (Lubrication theory): $\boldsymbol{v}=\boldsymbol{\nabla}\times\boldsymbol{w}\neq 0, \boldsymbol{\nabla}\cdot\boldsymbol{v}=0, \nabla^2\phi=0$. In this case the flow is dominated by the walls, while the viscosity and heat transfer introduce shear. This is typical of lubrication theory.

2,1 Fluid compressible irrotational flow (acoustics): $\boldsymbol{v}=\nabla\phi$, $\boldsymbol{\nabla}\times\boldsymbol{w}=0$. Here losses (viscosity and thermal diffusion) are small (assumed to be zero). One may define a velocity potential ψ, the gradient of which gives the air particle velocity, thus $\boldsymbol{v}=-\nabla\phi$. Thus for an irrotational fluid $\boldsymbol{\nabla}\times\boldsymbol{v}=0$ (Greenberg, 1988, p. 826). This is the case of the conservative field, where $\int\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dR$ only depends on the end points, and $\oint_{\mathcal{B}}\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,dR=0$. When a fluid may be treated as having no viscosity, it is typically assumed to be irrotational, since it is the viscosity that introduces the shear (Greenberg, 1988, p. 814). A *fluid's angular velocity* is $\Omega=\frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{v}=0$, thus irrotational fluids have zero angular velocity ($\Omega=0$).

2,2 Incompressible and irrotational fluid (statics): $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$ thus $\boldsymbol{v} = \nabla\phi$ and $\nabla^2\phi = 0$. An example of such a case is water in a small space at low frequencies, where the wavelength is long compared to the size of the container, the fluid may be treated as incompressible. When $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$, the effects of viscosity may be ignored, as it is the viscosity that creates the shear leading to rotation. This is the case of modeling the cochlea, where losses are ignored and the quasi-static limit is justified.

In summary, each of the cases is some sort of approximation that best applies in the low frequency limit. This is why it is called *quasi-static*, meaning low, but not zero frequency, where the wavelength is large compared with the dimensions (e.g., diameter).

1.5.15 Lec 41 Maxwell's Equations: The unification of electricity and magnetism

Once you have mastered the three basic vector operations, the gradient, divergence and curl, you are ready to appreciate Maxwell's equations. Like the vector operations, these equations may be written in integral or vector form. An important difference is that with Maxwell's equations, we are dealing with well defined physical quantities. The scalar and vector fields take on meaning, and units. Thus to understand these important equations, one must master the names of the four fields $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{B}, \boldsymbol{D}$.

Figure 1.38: *A solenoid is a uniform coil of wire. When a current is passed through the wire, a uniform magnetic field intensity $\boldsymbol{H}$ is created. From a properties point of view, this coil is indistinguishable from a permanent bar magnet, having north and south poles. Depending on the direction of the current, one end of a finite solenoidal coil is the north pole of the magnet, and the other end is the south pole. The uniform field inside the coil is called* solenoidal, *a confusing synonym for irrotational. (Figure from Wikipedia.)*

Field strengths $\boldsymbol{E}, \boldsymbol{H}$: As summarized in Fig. 1.8 there are two field strengths, the electric $\boldsymbol{E}$, with units of [V/m] and the magnetic $\boldsymbol{H}$ having units of [A/m]. Their the ratio $|\boldsymbol{E}|/|\boldsymbol{H}|$ is in [ohms].

To understand the meaning of $\boldsymbol{E}$, if two conducting plates are placed 1 [m] apart, with 1 [V] across them, the electric field is $\boldsymbol{E} = 1$ [V/m]. If a charge (i.e., and electron) is placed in an electric field, it feels a force $\boldsymbol{f} = q\boldsymbol{E}$, where q is the magnitude of the charge [Col].

To understand the meaning of $\boldsymbol{H}$, consider the solenoid made of wire, as shown in Fig. 1.38, which carries a current of 1 [A]. The magnetic field $\boldsymbol{H}$ inside such a solenoid is uniform and is pointed along the long axis, with a direction that depends on the polarity of the applied voltage (i.e., direction of the current in the wire).

Flux: Flux is a flow, such as the mass flux of water flowing in a pipe [kg/s], driven by a force (pressure drop) across the ends of the pipe, or the heat flux in a thermal conductor, having a temperature drop across it (i.e., a window or a wall). The flux is the same as the flow, be it current, mass or heat. In Maxwell's equations there are also two fluxes, the electric flux $\boldsymbol{D}$, and the magnetic flux $\boldsymbol{B}$. The flux density units for $\boldsymbol{D}$ are [A/m^2] (flux in [A]) and the magnetic flux $\boldsymbol{B}$ is measured in Webers [Wb] [A/m^2]) or [Tesla] (Henry-amps/area) [H-A/m^2].

Table 1.8: *The variables of Maxwell's equations have names (e.g., EF, MI) and units (in square brackets [SI Units]). The units are required to gain a full understanding of each of the four variable and their corresponding equation. For example, Eq.* EF *has units [V/m]. When you integrate* $\boldsymbol{E}$ *from* $x = a, b$*, you obtain the voltage difference between those two points. The* speed of light in vacuo *is* $c = 3 \times 10^8 = 1/\sqrt{\mu_o \epsilon_o}$ *[m/s], and the* characteristic resistance *of light* $r_o = 377 = \sqrt{\mu_o/\epsilon_o}$ *[Ω] (i.e., ohms). The dot over a vector is shorthand for the partial with respect to time (i.e.,* $\dot{\boldsymbol{B}} = \partial \boldsymbol{B}/\partial t$*).*

Symbol	Name	Units	Maxwell's Eq.
$\boldsymbol{E}$	EF: Electric Field strength	[V/m]	$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$
$\boldsymbol{D} = \epsilon_o \boldsymbol{E}$	ED: Electric Displacement (flux density)	[Col/m^2]	$\nabla \cdot \boldsymbol{D} = \rho$
$\boldsymbol{H}$	MF: Magnetic Field strength	[A/m]	$\nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}$
$\boldsymbol{B} = \mu_o \boldsymbol{H}$	MI: Magnetic Induction (flux density)	[Wb/m^2]	$\nabla \cdot \boldsymbol{B} = 0$

Maxwell's equations

As shown in Fig. 1.8, Maxwell's equations consist of two curl equations, operating on the field strengths EF $\boldsymbol{E}$ and MF $\boldsymbol{H}$, and two divergence equations, operating of the field fluxes ED $\boldsymbol{D}$ and MI $\boldsymbol{B}$. Stokes's law may be applied to the curl equations and Gauss's law may be used on the divergence equations. This should be logically obvious.

Example: When a static current is flowing in a wire in the $\hat{\mathbf{z}}$ direction, the magnetic flux is determined by Stokes's theorem (Fig. 1.36). Thus just outside of the wire we have

$$\mathcal{I}_{enc} = \iint_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{H}) \cdot \hat{\boldsymbol{n}} \; d|\mathcal{S}| = \oint_{\mathcal{B}} \boldsymbol{H} \cdot d\boldsymbol{l}. \quad [\text{A}] \tag{1.186}$$

For this simple geometry, of the current in a wire is related to $\boldsymbol{H}(\boldsymbol{x}, t)$ by

$$\mathcal{I}_{enc} = \oint_{\mathcal{B}} \boldsymbol{H} \cdot d\boldsymbol{l} = H_\theta 2\pi r.$$

Here H_θ is perpendicular to both the radius r and the direction of the current $\hat{\mathbf{z}}$. Thus

$$\boldsymbol{H}_\theta = \frac{\mathcal{I}_{enc}}{2\pi r},$$

and we see that $\boldsymbol{H}$, and thus $\boldsymbol{B} = \mu_o \boldsymbol{H}$, drop off as the reciprocal of the radius r.

Exercise: Explain how Stokes's law may be applied to $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$, and explain what it means. Hint: This is the identical argument given above for the current in a wire, but for the electric case.

Solution: Integrating the left side of equation EF over an open surface results in a voltage (emf) induced in the loop closing the boundary $\mathcal{B}$ of the surface

$$\phi_{\text{induced}} = \iint_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot \hat{\boldsymbol{n}} \; d|\mathcal{S}| = \oint_{\mathcal{B}} \boldsymbol{E} \cdot d\boldsymbol{l} \quad [\text{V}].$$

The emf (electromagnetic force) is the same as the Thévenin source voltage induced by the rate of change of the flux. Integrating the Eq. 1.5.15 over the same open surface $\mathcal{S}$ results in the source of the induced voltage ϕ_{induced}, which is proportional to the rate of change of the flux [webers]

$$\phi_{\text{induced}} = -\frac{\partial}{\partial t} \iint_{\mathcal{S}} \boldsymbol{B} \cdot \hat{\boldsymbol{n}} \; dA = L\dot{\psi}, \quad [\text{Wb/s}].$$

where L is the *inductance* of the wire. The area integral on the left is in [Wb/m^2] resulting in the total flux crossing normal to the surface ψ [Wb].

If we apply Gauss's law to the divergence equations, we find the total flux crossing the closed surface.

Verify

Exercise: Apply Gauss's law to equation ED and explain what it means in physical terms. **Solution:** The area of the normal component of $\boldsymbol{D}$ is equal to the volume integral over the charge density: thus, Gauss's law says that the total charge within the volume Q_{enc}, found by integrating the charge density $\rho(\boldsymbol{x})$ over the volume $\mathcal{V}$, is equal to the normal component of the flux $\boldsymbol{D}$ crossing the surface $\mathcal{S}$

$$Q_{enc} = \iiint_{\mathcal{V}} \boldsymbol{\nabla}\cdot\boldsymbol{D}\, dV = \iint_{\mathcal{S}} \boldsymbol{D}\cdot\hat{\boldsymbol{n}}\, dA.$$

Summary: Maxwell's four equations relate the field strengths to the flux densities. There are two types of variables: field strengths ($\boldsymbol{E}, \boldsymbol{H}$) and flux densities ($\boldsymbol{D}, \boldsymbol{B}$). There are two classes: electric ($\boldsymbol{E}, \boldsymbol{D}$) and magnetic $\boldsymbol{H}, \boldsymbol{B}$. One might naturally view this as a 2x2 matrix, with rows being electric and magnetic strengths, and columns being electric and magnetic and flux densities, defining a total of four variables:

	Strength	Flux
Electric	$\boldsymbol{E}$ [V/m]	$\boldsymbol{D}$ [Col/m^2]
Magnetic	$\boldsymbol{H}$ [A/m]	$\boldsymbol{B}$ [Wb/m^2]

.

Applying Stokes's curl law to the forces induces a Thévein voltage (emf) or Norton current source. Applying Gauss' divergence law to the flows gives the total charge enclosed. The magnetic charge is always zero since $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, because magnetic mono-poles do not exist. However, magnetic dipoles do exist, as in the example of the electron which contains a magnetic dipole.

The wave equation: When Maxwell's equations are combined, the vector wave equation results. There is the famous story that Maxwell introduced the displacement current $\dot{\boldsymbol{D}}$ to make the equations symmetric, allowing him to predict the formula for the speed of light. The displacement current was not verified by the experiments of Faraday and Ampere.

Taking the curl of Maxwell's electric field strength equation for the $\boldsymbol{\nabla}\times\boldsymbol{E}$ (Fig. 1.8, p. 166), and using the expression for $\boldsymbol{\nabla}\times\boldsymbol{B}$ gives

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} = -\boldsymbol{\nabla}\times\dot{\boldsymbol{B}} = -\mu_o\boldsymbol{\nabla}\times\dot{\boldsymbol{H}} = -\mu_o\epsilon_o\ddot{\boldsymbol{E}}. \tag{1.187}$$

Applying the GoD identity (Eq. 1.178) gives

$$\boldsymbol{\nabla}^2\boldsymbol{E} - \frac{1}{\epsilon_o}\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{D}) = \mu_o\epsilon_o\ddot{\boldsymbol{E}}. \tag{1.188}$$

In a charge-free region $\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0$ and since $1/c^2 = \mu_o\epsilon_o$, one obtains the *vector wave equation*

$$\boldsymbol{\nabla}^2\boldsymbol{E}(\boldsymbol{x},t) = \frac{1}{c^2}\ddot{\boldsymbol{E}}(\boldsymbol{x},t) \leftrightarrow \frac{s^2}{c_o^2}\boldsymbol{E}(\boldsymbol{x},\omega). \tag{1.189}$$

It is important to distinguish the vector Laplacian from the scalar Laplacian.

Recall the d'Alembert solutions of the scalar wave equation (Eq. 1.95, p. 109)

$$\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x} - ct) + \boldsymbol{g}(\boldsymbol{x} + ct),$$

where $\boldsymbol{f}, \boldsymbol{g}$ are arbitrary vector fields. This result applies to the vector case since it represents three identical, yet independent, scalar wave equations, in the three dimensions.

In a like manner one may derive the wave equation in terms of $\boldsymbol{H}$

$$\boldsymbol{\nabla}^2\boldsymbol{H}(\boldsymbol{x},t) = \frac{1}{c^2}\ddot{\boldsymbol{H}}(\boldsymbol{x},t) \leftrightarrow \frac{s^2}{c_o^2}\boldsymbol{H}(\boldsymbol{x},\omega). \tag{1.190}$$

This equation does not have the restriction that there is no free charge, because $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. Thus both $\boldsymbol{E}, \boldsymbol{H}$ obey the wave equation (thus they are locked together in space-time if we assume no free charge) (Sommerfeld, 1952).

Poynting's theorem: The energy flux density $\boldsymbol{\mathcal{P}}$ [W/m^2] is perpendicular to $\boldsymbol{E}$ and $\boldsymbol{B}$, denoted as

$$\boldsymbol{\mathcal{P}} = \frac{1}{\mu_o}\boldsymbol{E}\times\boldsymbol{B}.$$

Electrical impedance seen by an electron: Up to now we have only considered the Brune impedance which is a special case with no branch points or branch cuts. We can define impedance for the case of diffusion, as in the case of the diffusion of heat. There is also the diffusion of electrical and magnetic fields at the surface of a conductor, where the resistance of the conductor dominates the dielectric properties, which is called the electrical *skin effect*, where the conduction currents are dominated by the conductivity of the metal rather than the displacement currents. In such cases the impedance is proportional to $\sqrt{s}$, implying that it has a branch cut. Still in this case the real part of the impedance must be positive in the right s half-plane, the required condition of all impedances, such that postulate P3 (p. 101) is satisfied.

Example: When we deal with Maxwell's equations the force is defined by the *Lorentz force*

$$\boldsymbol{f} = q\boldsymbol{E} + q\boldsymbol{v}\boldsymbol{\nabla}\times\boldsymbol{B}) = q\boldsymbol{E} + \boldsymbol{C}\times\boldsymbol{B},$$

which is the force on a charge (e.g., electron) due to the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields. The magnetic field plays a role when the charge has a velocity $\boldsymbol{v}$. When a charge is moving with velocity $\boldsymbol{v}$, it may be viewed as a current $\boldsymbol{C} = q\boldsymbol{v}$.

In this case the impedance in a wire, where the current is constrained, the complex impedance density is

$$Z(s) = \sigma + s\epsilon_o, \qquad [\Omega/\text{m}^2]$$

which when integrated over an area is the impedance in ohms (Feynman, 1970c, p. 13-1). Here σ is the electrical conductivity and ϵ_o is the electrical permittivity. Since $\sigma \gg \omega\epsilon_o$ this reduces to the resistance of the wire, per unit length.

1.5.16 Lec 42 The Quasi-static approximation

There are a number of assumptions and approximations that result in special cases, many of which are classic. These manipulations are all done at the differential equation level, by making assumptions that change the basic equations that are to be solved. These approximations are distinct from assumptions made while solving a specific problem.[110]

[110] https://www.youtube.com/watch?v=_pEiA0-r5A8

A few and important examples include

1. *In vacuo* waves (free-space scalar wave equation)
2. Expressing the vector wave equation in terms of scalar and vector potentials
3. Quasi-statics
   (a) scalar wave equation
   (b) Kirchhoff's low-frequency lumped approximation (LRC networks)
   (c) Transmission line equations (telephone and telegraph equations)

One of the very first insights into wave propagation was due to Huygens (c1640) (Fig. 1.24).

Quasi-statics and its implications: *Quasi-statics* (Postulate (P10), p. 102) is an approximation used to reduce a partial differential equations to a scalar (one-dimensional) equation (Sommerfeld, 1952). Quasi-statics is a way of reducing a three-dimensional problem to a one–dimensional problem. So that it is not miss-applied, it is important to understand the nature of this approximation, which goes to the heart of transmission line theory. The quasi-static approximation states that the wavelength λ is greater than the dimensions of the object Δ (e.g., $\lambda \gg \Delta$). The best known examples, Kirchhoff's current and voltage laws, KCL and KVL, almost follow from Maxwell's equations given the quasi-static approximation (Ramo *et al.*, 1965). These laws, based on ohm's law, state that the sum of the currents at a node must be zero (KCL) and the sum of the voltages around a loop must be zero (KCL).

These well-known laws are the analogues of Newton's laws of mechanics. The sum of the forces at a point is the analogue of the sum of the loop voltages. Voltage ϕ is the force potential, since the electric field $\boldsymbol{E} = -\nabla\phi$. The sum of the currents is the analogue of the vector sum of velocities (mass) at a point, which is zero.

The acoustic wave equation describes how the scalar field pressure $p(\boldsymbol{x}, t)$ and the vector force density potential ($f(\boldsymbol{x}, t) = -\nabla p(\boldsymbol{x}, t)$ [N/m^2]) propagate in three dimensions. The net force is the integral of the pressure gradient over an area. If the wave propagation is restricted to a pipe (e.g., organ pipe), or to a string (e.g., an guitar or lute), the transverse directions may be ignored, due to the quasi-static approximation. What needs to be modeled by the equations is the wave propagation along the pipe (string). Thus we may approximate the restricted three-dimensional wave by a one-dimensional wave.

However if we wish to be more precise about this reduction in geometry ($\mathbb{R}^2 \rightarrow \mathbb{R}$), we need to consider the quasi-static approximation, as it makes assumptions about what is happening in the other directions, and quantifies the effect ($\lambda \gg \Delta$). Taking the case of wave propagation in a tube, say the ear canal, there is the main wave direction, down the tube. But there is also wave propagation in the transverse direction, perpendicular to the direction of propagation. As shown in Table H.2 (p. 273), the key statement of the quasi-static approximation is that the wavelength in the transverse direction is much larger that the radius of the pipe. This is equivalent to saying that the radial wave reaches the walls and is reflected back, in a time that is small compared to the distance propagated down the pipe. Clearly the speed of sound down the pipe and in the transverse direction is the same if the medium is homogeneous (i.e., air or water). Thus the sound reaches the walls and is returned (reflected) to the center line in a time that the axial wave traveled about 1 diameter along the pipe. So if the distance traveled is several diameters, the radial parts of the wave have time to come to equilibrium. So the question one must ask is: What are the properties

of this equilibrium? The most satisfying answer is provided by looking at the internal forces on the air, due to the gradients in the pressure.

The pressure $\varrho(\boldsymbol{x}, t)$ is a potential, thus its gradient is a force density $\mathbf{f}(\boldsymbol{x}, t) = -\nabla\varrho(\boldsymbol{x}, t)$. This equation tells us how the pressure wave evolves as it propagates down the horn. Any curvature in the pressure wave-front induces stresses, which lead to changes (strains) in the local wave velocity, in the directions of the force density. The main force is driving the wave-front forward (down the horn), but there are radial (transverse) forces as well, which tend to rapidly go to zero.

For example, if the tube has a change in area (or curvature), the local forces will create radial flow, which is immediately reflected by the walls, due to the small distance to the walls, causing the forces to average out. After traveling a few diameters, these forces will come to equilibrium and the wave will trend towards a plane wave (or satisfy Laplace's equation if the distortions of the tube are sever). The internal stress caused by this change in area will quickly equilibrate.

There is a very important caveat, however: only at low frequencies, such that $ka < 1$, can the plane wave mode can dominate. At higher frequencies ($ka \geq 1$) where the wavelength is small compared to the diameter, the distance traveled between reflections is much greater than a few diameters. Fortunately the frequencies where this happens are so high that they play no role in frequencies that we care about in the ear canal. This effect is describes as *cross-modes*, which imply some sort of radial standing waves.

Of course such modes exist in the ear canal. However the are much more obvious on the eardrum where the sound wave speed is much slower than that in air (Parent and Allen, 2010; Allen, 2014). Because of the slower speed, the ear drum has low-frequency cross-modes, and these may be seen in the ear canal pressure, and are easily observable in ear canal impedance measurements. Yet they seem to have a negligible effect on our ability to hear sound with high fidelity. The point here is that the cross modes are present, but we call upon the quasi-static approximation as a justification for ignoring them, to get closer to the first-order physics.

Quasi-statics and Quantum Mechanics

It is important to understand the meaning of Planck's constant h, which appears in the relations of both photons (light "particles") and electrons (mass particles). If we could obtain a handle on what exactly Planck's constant means, we might have a better understanding of quantum mechanics, and physics in general. By cataloging the *dispersion relations* (the relation between the wavelength $\lambda(\nu)$ and the frequency ν), between electrons and photons, this may be attainable.

Basic relations from quantum mechanics for photons and electrons include:

1. Photons (mass=0, velocity = c)

   (a) $c = \lambda\nu$: The speed of light c is the product of its wavelengths λ times its frequency ν. This relationship is only for mono-chromatic (single frequency) light.

   (b) The speed of light is
   $$c = \frac{1}{\sqrt{\mu_o\epsilon_o}} = 0.3 \times 10^6 \quad \text{[m/s]}$$

   (c) The *characteristic resistance* of light $r_o = \sqrt{\mu_o/\epsilon_o} = |E|/|H| = 377$ [ohms] is defined as the magnitude of the ratio of the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ field, of a plane wave *in-vacuo*.

   (d) $E = h\nu$: the photon energy is given by Planck's constant $h \approx 6.623 \times 10^{-34}$ [m^2 kgm/s], times the frequency (or bandwidth) of the photon)

2. Electrons (mass = m_e, velocity $V = 0$):

   (a) $E_e = m_e c^2 \approx 0.91 \cdot 10^{-30} \cdot 0.3^2 \cdot 10^{12} = 8.14 \times 10^{-20}$ [J] is the *electron rest energy* (velocity $V = 0$) of every electron, of mass $m_e = 9.1 \times 10^{-31}$ [kgm], where c is the speed of light.

   (b) $p = h/\lambda$: The momentum p of an electron is given by Planck's constant h divided by the wavelength of an electron λ. It follows that the bandwidth of the photon is given by

$$\nu_e = \frac{E_e}{h}$$

   and the wavelength of an electron is

$$\lambda_e = \frac{h}{p_e}.$$

One might reason that QM obeys the quasi-static (long wavelength) approximation. If we compare the velocity of the electron V to the speed of light c, then we see that

$$c = E/p \gg V = E/p = mV^2/V$$

Conjecture on photon energy:

Photons are seen as quantized because they are commonly generated by atoms, which produce light-particles having the difference in two energy (quantum, or eigen-states) levels. The relation $E = h\nu$ does not inherently depend on ν being a fixed frequency. Planck's constant h is the EM energy density over frequency, and $E(\nu_o)$ is the integral over frequency

$$E(\nu_o) = h \int_{-\nu_o}^{\nu_o} d\nu = 2h\nu_o.$$

When the photon is generated by an atom, ν_o is quantized by the energy level difference that corresponds to the frequency (energy level difference) of the photon jump.

1.5.17 Lec 43: Summary and Overview

Summary: Mathematics began as a simple way of keeping track of how many things there were. But eventually physics and mathematics evolved together as tools to help us navigate our environment, both locally and globally, to solve daily problems such as food, water and waste management, to understand the solar system and the stars, to defend ourselves using tools of war, etc.

Based on the historical record of the abacus, one can infer that people precisely understood the concepts of counting, addition, subtraction and multiplication (recursive addition).

There is some evidence that the abacus, a simple counting tool formalizing the addition of very large numbers, was introduced to the Chinese by the Romans, where it was used for trade.

However, this working knowledge of arithmetic did not to show up in written number systems. The Roman numerals were not useful for doing calculations done on the abacus. The final answer would then be expressed in terms of the Roman number system.

According to the known written record, the number zero (null) had no written symbol until the time of Brahmagupta (628 CE). One should not assume the concept of zero was not understood simply because there was no symbol for it in the Roman numeral system. Negative numbers and

zero would be obvious when using the abacus. Numbers between the integers would be represented as *rational numbers* $\mathbb{Q}$ since any number may be approximated with arbitrary accuracy with rational numbers.

Mathematics is the science of formalizing a repetitive method into a set of rules, and then generalizing it as much as possible. Generalizing the multiplication and division algorithm to different types of numbers becomes increasingly more complex as we move from integers to rational numbers, irrational numbers, real and complex numbers and ultimately, vectors and matrices. How do you multiply two vectors, or multiply and divide one matrix by another? Is it subtraction as in the case of two numbers? Multiplying and dividing polynomials (by long division) generalizes these operations even further. Linear algebra is a further important generalization, fallout from the fundamental theorem of algebra, and essential for solving the generalizations of the number systems.

Many of the concepts about numbers naturally evolved from music, where the length of a string (along with its tension) determined the pitch (Stillwell, 2010, pp. 11, 16, 153, 261). Cutting the string's length by half increased the frequency by a factor of 2. One fourth of the length increases the frequency by a factor of 4. One octave is a factor of 2 and two octaves a factor of 4 while a half octave is $\sqrt{2}$. The musical scale was soon factored into rational parts. This scale almost worked, but did not generalize (sometimes known as the *Pythagorean comma*[111]), resulting in today's *well tempered scale*, which is based on 12 equal geometric steps along one octave, or 1/12 octave ($\sqrt[12]{2} \approx 1.05946 \approx 18/17 = 1 + 1/17$).

But the concept of a *factor* was clear. Every number may be written as either a sum or a product (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of a second degree polynomial was understood, which led to a generalization of factoring, since the polynomial, a sum of terms, may be written in factored form. If you think about this a bit, it is sort of an amazing idea that needed to be discovered (Stillwell, 2010, p. ???). This concept led to an important string of theorems on factoring polynomials, and how to numerically describe physical quantities. Newton was one of the first to master these tools with his proof that the orbits of the planets are ellipses, not circles. This led him to expanding functions in terms of their derivatives and power series. Could these sums be factored? The solution to this problem led to calculus.

So mathematics, a product of the human mind, is a highly successful attempt to explain the physical world. All aspects of our lives were impacted by these tools. Mathematical knowledge is power. It allows one to think about complex problems in increasingly sophisticated ways. An equation is a mathematical sentence, expressing deep knowledge. Witness $E = mc^2$ and $\nabla^2\psi = \ddot{\psi}$.

[111] https://en.wikipedia.org/wiki/Pythagorean_comma

1.5.18 Reading List

The above concepts come straight from mathematical physics, as developed in the 17^{th}–19^{th} centuries. Much of this was first developed in acoustics by Helmholtz, Stokes and Rayleigh, following in Green's footsteps, as described by Lord Rayleigh (1896). When it comes to fully appreciating Green's theorem and reciprocity, I have found Rayleigh (1896) to be a key reference. If you wish to repeat my reading experience, start with Brillouin (1953, 1960), followed by Sommerfeld (1952) and Pipes (1958). Second-tier reading contains many items: Morse (1948); Sommerfeld (1949); Morse and Feshbach (1953); Ramo *et al.* (1965); Feynman (1970a); Boas (1987). A third tier might include Helmholtz (1863a); Fry (1928); Lamb (1932); Bode (1945); Montgomery *et al.* (1948); Beranek (1954); Fagen (1975); Lighthill (1978); Hunt (1952); Olson (1947). It would be a mistake to ignore other important physics writings by stalwart authors, J.C. Slater[112] and Landau and Lifshitz,[113] and their impressive series of mathematical physics books.

You must enter at a level that allows you to understand. Successful reading of these books critically depends on what you already know. Rudimentary (high school) level math must be mastered first. Read in the order that helps you best understand the material.

Without a proper math vocabulary, mastery is hopeless. I suspect that one semester of college math can bring you up to speed. This book is my attempt to present this level of understanding.

[112] https://en.wikipedia.org/wiki/John_C._Slater

[113] https://www.amazon.com/Mechanics-Third-Course-Theoretical-Physics/dp/0750628960

1.6 Exercises

EXERCISES NS-1

Topic of this homework: Introduction to MATLAB/OCTAVE (see the *Matlab or Octave tutorial* for help).

Deliverable: Report with charts and answers to questions. Hint: Use LaTeX[114]

Plotting complex quantities in Matlab Plot real, imaginary, magnitude and phase quantities.

1. Consider the functions $f(s) = s^2 + 6s + 25$ and $g(s) = s^2 + 6s + 5$.
   (a) Find the zeros of functions $f(s)$ and $g(s)$ using the command `roots`.
   (b) On a single plot, show the roots of $f(s)$ as red circles, and the roots of $g(s)$ as blue plus signs. The x-axis should display the real part of each root, and the y-axis should display the imaginary part. Use `hold on` and `grid on` when plotting the roots.
   (c) Give your figure the title 'Complex Roots of f(s) and g(s)' using the command `title`. Label the x-axis 'Real Part' and the y-axis 'Imaginary Part' using `xlabel` and `ylabel`. Type `ylim([-10 10])` and `xlim([-10 10])`, to expand the axes.
2. Consider the function $h(t) = e^{j2\pi ft}$ for $f = 5$ and `t=[0:0.01:2]`
   (a) Use `subplot` to show the real and imaginary parts of $h(t)$ as two graphs in one figure. Label the x-axes 'Time (s)' and the y-axes 'Real Part' and 'Imaginary Part'.
   (b) Use `subplot` to plot the magnitude and phase parts of $h(t)$. Use the command `angle` or `unwrap(angle())` to plot the phase. Label the x-axes 'Time (s)' and the y-axes 'Magnitude' and 'Phase (radians)'.

1. Prime numbers in Matlab
   (a) Use the Matlab function `factor` to find the prime factors of 123, 248, 1767, and 999,999.
   (b) Use the Matlab function `isprime` to check if 2, 3 and 4 are prime numbers. What does the function `isprime` return when a number is prime, or not prime? Why?
   (c) Use the Matlab function `primes` to generate prime numbers between 1 and 10^6 and save them in a vector `x`. Plot this result using the command `hist(x)`.
   (d) Now try `[n,bin_centers] = hist(x)`. Use `length(n)` to find the number of bins.
   (e) Set the number of bins to 100 by using an extra input argument to the function `hist`. Show the resulting figure and give it a title and axes labels.
2. `Inf`, `NaN` and logarithms in Matlab

[114] `http://www.overleaf.com`

(a) Try `1/0` and `0/0` in the command window. What are the results? What do these 'numbers' mean in Matlab?

(b) In Matlab, the natural logarithm $\ln(\cdot)$ is computed using the function `log` ($\log_{10}$ and $\log_2$ are computed using `log10` and `log2`). Try `log(0)` in the command window.

(c) Try `log(-1)` in the command window. Do you get what you expect for $\ln(-1)$? Show how Matlab arrives at the answer by considering $-1 = e^{i\pi}$.

(d) (*not graded*) What is a decibel? Look up decibels on the internet.

3. Find the largest prime number that can be stored on an Intel 64 bit computer, which we call $\pi_{\texttt{max}}$. Hint: As explained in the Matlab/Octave command `help flintmax`, the largest positive integer is 2^{53}, however the largest integer that can be factored is $2^{32} = \sqrt{2^{64}}$. Explain the logic of your answer. Hint: help isprime().

```
%Matlab code to find the largest prime in IEEE-floating point
clear variables; close all
clc
format long;
N=2^32; %flintmax says this is the largest integer
disp(sprintf('N %g',N));
 %
for n=1:20
p=isprime(N-n);
if p
    F=factor(N-n)
disp(sprintf('n= %g, N=%g; Factor: %d',n,N,factor(N-n)))
end
end
```

4. Suppose you are interested in primes that are greater than $\pi_{\texttt{max}}$. How can you find them on an Intel computer (i.e., one using IEEE-floating point)?

   (a) Hint 1: Since every prime number greater than 2 is odd, there is no reason to check the even numbers. Thus consider a sieve containing only odd numbers, starting from 3 (not 2). Thus odd integers $n_{\text{odd}} \in \mathbb{N}/2$ contain all the primes other than 2.

5. The following idenity is interesting:

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \\ 1+3+5+7+9 &= 5^2 \\ &\cdots \\ \sum_{n=0}^{N-1} 2n+1 &= N^2. \end{aligned}$$

   Can you find a proof?[115]

[115] This problem came from an exam problem for Math 213, Fall 2016.

EXERCISES NS-2

Topic of this homework: Prime numbers, greatest common divisors, the continued fraction algorithm

Deliverable: Answers to questions.

1. According to the fundamental theorem of arithmetic, every integer may be written as a product of primes.

   (a) Put the numbers $1,000,000$, $1,000,004$ and $999,999$ in the form $N = \prod_k \pi_k^{\beta_k}$ (you may use Matlab to find the prime factors).

   (b) Give a generalized formula for the natural logarithm of a number, $\ln(N)$, in terms of its primes π_k and their multiplicities β_k. Express your answer as a *sum* of terms.

2. Explain why the following 2-line Matlab/Octave program returns the prime numbers π_k between 1 and 100?

```
n=2:100;
k = isprime(n);
n(k)
```

3. Prime numbers may be identified using 'sieves'

   (a) By hand, perform the sieve of Eratosthenes for $n = 1 \ldots 49$. Circle each prime p then cross out each number which is a multiple of p.

   (b) In part (a), what is the highest number you need to consider before only the primes remain?

   (c) Generalize: for $n = 1 \ldots N$, what is the highest number you need to consider before only the primes remain?

   (d) Write each of these numbers as a product of primes:
   22=
   30=
   34=
   43=
   44=
   48=
   49=
   .

4. Find the largest prime $\pi_k \leq 100$? Hint: Do not use matlab other than to check your answer. Hint: Write out the numbers starting with 100 and counting backwards: 100, 99, 98, 97, $\cdots$. Cross off the even numbers, leaving $99, 97, 95, \cdots$. Pull out a factor (only 1 is necessary to show that it is not prime).

5. Find the largest prime $\pi_k \leq 1000$? Hint: Do not use matlab other than to check your answer.

Consider the Euclidean algorithm to find the greatest common divisor (GCD; the largest common prime factor) of two numbers. Note this algorithm may be performed using one of two methods:

Method	Division	Subtraction
On each iteration...	$a_{i+1} = b_i$ $b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$	$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$ $b_{i+1} = \min(a_i, b_i)$
Terminates when...	$b = 0$ (gcd$= a$)	$b = 0$ (gcd$= a$)

The division method (Eq. 2.1, Sect. 2.1.2, Lec 5, Ch. 2) is prefered because the subtraction method is much slower.

1. Understand the Euclidean (GCD) algorithm
   (a) Use the Matlab command `factor` to find the prime factors of $a = 85$ and $b = 15$. What is the greatest common prime factor of these two numbers?
   (b) By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.
   (c) By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?
   (d) Consider the first step of the GCD division algorithm when $a < b$ (e.g. $a = 25$ and $b = 75$). What happens to a and b in the first step? Does it matter if you begin the algorithm with $a < b$ vs. $b < a$?
   (e) Describe in your own words how the GCD algorithm works. Try the algorithm using numbers which have already been separated into factors (e.g. $a = 5 \cdot 3$ and $b = 7 \cdot 3$).
2. Coprimes
   (a) Define the term *coprime*.
   (b) How can the Euclidean algorithm be used to identify coprimes?
   (c) Give at least one application of the Euclidean algorithm.
3. Write a Matlab function, `function x = my_gcd(a,b)`, which uses the Euclidean algorithm to find the GCD of any two inputs `a` and `b`. Test your function on the (a,b) combinations from parts (a) and (b). Include a printout (or handwrite) your algorithm to turn in.

   Hints and advice:

   - Don't give your variables the same names as Matlab functions! Since `gcd` is an existing Matlab/Octave function, if you use it as a variable or function name, you won't be able to use `gcd` to check your `gcd()` function. Try `clear all` to recover from this problem.
   - Try using a 'while' loop for this exercise (see Matlab documentation for help).
   - You may need to make some temporary variables for `a` and `b` in order to perform the algorithm.

In this problem we are looking for integer solutions $(m, n) \in \mathcal{Z}$ to the equations $\mathbf{ma} + \mathbf{nb} = \mathbf{gcd}(\mathbf{a}, \mathbf{b})$ and $\mathbf{ma} + \mathbf{nb} = \mathbf{0}$ given positive integers $(a, b) \in \mathcal{Z}^+$. Note that this requires that either m or n be negative. These solutions may be found using the Euclidean algorithm only if (a, b) are coprime ($a \perp b$). Note that integer (whole number) polynomial relations such as these are known as 'Diophantine equations.' The above equations are *linear Diophantine equations*, possibly the simplest form of such relations.

Matrix approach: It can be difficult to keep track of the a's and b's when the algorithm has many steps. Here is an alternative way to run the Euclidean algorithm, using matrix algebra. Matrix methods provide a more transparent approach to the operations on (a, b). Thus the Euclidean algorithm can be classified in terms of standard matrix operations (discussed in Lec. 5 pp. 73-75):
Division method:
Define

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a_i/b_i \rfloor \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix}$$

Example: `gcd(2,3)=1`): For $(a, b) = (2, 3)$, the result is as follows:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}}_{m \quad n} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Thus from the above equation we find the solution (m, n) to the integer equation

$$2m + 3n = \mathbf{gcd}(2, 3) = 1,$$

namely $(m, n) = (-1, 1)$ (i.e., $-2 + 3 = 1$). There is also a second solution $(3, -2)$, (i.e., $3 \cdot 2 - 2 \cdot 3 = 0$), which represents the terminating condition. Thus these two solutions are a pair and the solution only exists if (a, b) are coprime ($a \perp b$). Subtraction method: This method is more complicated than the division algorithm, because at each stage we must check if $a < b$.
Define

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where Q sets $a_{i+1} = a_i - b_i$ and $b_{i+1} = b_i$ assuming $a_i > b_i$, and S is a 'swap-matrix' which swaps a_i and b_i if $a_i < b_i$. Using these matrices, the algorithm is implemented by assigning

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = Q \begin{bmatrix} a_i \\ b_i \end{bmatrix} \text{ for } a_i > b_i, \qquad \begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = QS \begin{bmatrix} a_i \\ b_i \end{bmatrix} \text{ for } a_i < b_i.$$

The result of this method is a cascade of Q and S matrices. For $(a, b) = (2, 3)$, the result is as follows:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{S} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{m \quad n} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus we find two solutions (m, n) to the integer equation $2m + 3n = \gcd(2, 3) = 1$.

1. By inspection, find at least one integer pair (m, n) that satisfies $12m + 15n = 3$.

2. Using matrix methods for the Euclidean algorithm, find integer pairs (m, n) that satisfy $12m + 15n = 3$ and $12m + 15n = 0$. Show your work!!!

3. Does the equation $12m + 15n = 1$ have integer solutions for n and m? Why, or why not?

Here we explore the continued fraction algorithm (CFA), as discussed in Lec. 6 (Chapters 1 and 2). In its simplest form the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \hat{\mathbf{x}}$, as an example, because its CFA representation will be infinitely long. We can represent the CFA coefficents α as a vector of integers n_k, $k = 1, 2 \ldots \infty$

$$\begin{aligned} \alpha &= [n_1., n_2, n_3, n_4, \ldots] \\ &= n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \ldots}}} \end{aligned}$$

As discussed in Section 1.2.5, the CFA is recursive, with three steps per iteration:

For $\alpha_1 = \pi$, $n_1 = 3$, $r_1 = \pi - 3$ and $\alpha_2 \equiv 1/r_1$.

$$\begin{aligned} \alpha_2 &= 1/0.1416 = 7.0625 \ldots \\ \alpha_1 = n_1 + \frac{1}{\alpha_2} &= n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \ldots \end{aligned}$$

In Matlab (Octave) script

```
alpha0 = pi;
K=10;
n=zeros(1,K); alpha=zeros(1,K);
alpha(1)=alpha0;

for k=2:K  %k=1 to K
n(k)=round(alpha(k-1));
%n(k)=fix(alpha(k-1));
alpha(k)= 1/(alpha(k-1)-n(k));
%disp([fix(k), round(n(k)), alpha(k)]); pause(1)
end
disp([n; alpha]);
%Now compair this to matlab's rat() function
rat(alpha0,1e-20)
```

1. By hand (you may use Matlab as a calculator), find the first 3 values of n_k for $\alpha = e^{\pi}$.
2. For part (1), what is the error (remainder) when you truncate the continued fraction after $n_1, ..., n_3$? Give the absolute value of the error, and the percentage error relative to the original α.
3. Use the Matlab program provided to find the first 10 values of n_k for $\alpha = e^{\pi}$, and verify your result using the Matlab command `rat()`.
4. Discuss the similarities and differences between the Euclidean algorithm (EA) and CFA.

EXERCISES NS-3

Topic of this homework: Pythagorean triples, Pell's equation, Fibonacci sequence
Deliverable: Answers to problems
Euclid's formula for the Pythagorean triples a, b, c is: $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.

1. What condition(s) must hold for p and q such that a, b, and c are always positive and nonzero?

2. Solve for p and q in terms of a, b and c.

3. The ancient Babylonians (c2000BEC) cryptically recorded (a,c) pairs of numbers on a clay tablet, archeologically denoted *Plimpton-322*.

   To Do: Find p and q for the first five pairs of a and c from the tablet entries:

Table 1: First five (a,c) pairs of Plimpton-322.

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

4. Based on Euclid's formula, show that $c > (a, b)$.

5. What happens when $c = a$?

6. Is $b + c$ a perfect square? Discuss.

:Pell's equation is one of the most historic (i.e., important) equations of Greek number theory, because it was used to show that $\sqrt{2} \in \hat{\mathbf{x}}$. We seek integer solutions

$$x^2 - Ny^2 = 1.$$

As shown in Lec 8 of the lecture notes, the solutions x_n, y_n for the case of $N = 2$ are given by the 2x2 matrix recursion of the form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \jmath^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Diagonalization of a matrix ("eigenvalue/eigenvector decomposition"): As derived in Appendix C of the lecture notes, the most efficient way to compute A^n is to *diagonalize* the matrix A, by finding its *eigenvalues* and *eigenvectors*.

To do: *Hint: Use Matlab's function* `[E,Lambda] = eig(A)` *to check your results!*

1. Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.

2. Find the first 3 values of $[x_n, y_n]^T$ by hand and show that they satisfy Pell's equation for N=2.

3. By hand, find the eigenvalues $\lambda_\pm$ of the 2×2 Pell's equation matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

4. By hand, show that the matrix of eigenvectors, E, is

$$E = \begin{bmatrix} \vec{e}_+ & \vec{e}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

5. Using the eigenvalues and eigenvectors you found for A, verify that

$$E^{-1} A E = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

6. Now that you have diagonalized A (Equation **??**), use your results for E and Λ to solve for the $n = 10$ solution $[x_{10}, y_{10}]^T$ to Pell's equation with $N = 2$.

The Fibonacci sequence is famous in mathematics, and has been observed to play a role in the mathematics of genetics. Let x_n represent the Fibonacci sequence,

$$x_n = x_{n-1} + x_{n-2}, \tag{1.191}$$

where the current output sample, x_n, is equal to the sum of the previous two inputs. This is a 'discrete time' *recurrence relation*. To solve for x_n, we require some *initial conditions*. In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ for $n = 0, 1, 2, 3, \ldots$.

Here we seek the general formula for x_n. Like the Pell's equation, Eq. 1.191 has a recursive, eigen decomposition solution. To find it we must recast x_n as a 2x2 matrix relation, and then proceed as we did for the Pell case.

1. Show that Eq. 1.191 is equivalent to the 2×2 matrix equation

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}. \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.192}$$

   and that the Fibonacci sequence x_n as described above may be generated by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   What is the relationship between y_n and x_n?

2. Write a Matlab program to compute x_n using the matrix equation above (you don't need to turn in your code). Test your code using the first few values of the sequence. Using your program, what is x_{40}?
   Note: to make your program run faster, consider using the eigen decomposition of A, described by Eq. **??** *from the Pell's equation problem.*

3. Using the eigen decomposition of the matrix A (and a lot of algebra), it is possible to obtain the general formula for the Fibonacci sequence,

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{1.193}$$

   What are the eigenvalues $\lambda_\pm$ of the matrix A? How is the formula for x_n related to these eigenvalues?

4. Consider Eq. 1.193 in the limit as $n \to \infty$...

   (a) What happens to each of the two terms $[(1 \pm \sqrt{5})/2]^{n+1}$?

   (b) What happens to the ratio x_{n+1}/x_n?

5. Prove that[116]

$$\sum_1^N f_n^2 = f_N f_{N+1}.$$

6. Replace the Fibonacci sequence with

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

   such that the value x_n is the average of the previous two values in the sequence.

   (a) What matrix A is used to calculate this sequence?

   (b) Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?

   (c) What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$? *Hint: you can expect the closed-form expression for x_n to be similar to Eq. 1.193.*

7. Now consider

$$x_n = \frac{x_{n-1} + 1.01x_{n-2}}{2}.$$

   (a) What matrix A is used to calculate this sequence?

   (b) Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?

   (c) What are the eigenvalues of your new A? How do they relate to the behavior of x_n as $n \to \infty$? *Hint: you can expect the closed-form expression for x_n to be similar to Eq. 1.193.*

[116] I found this problem on a workseet for Math 213 midterm (213practice.pdf).

Appendix A

Notation

A.1 Number systems

The notation used in this book is defined in this appendix so that it may be quickly accessed.[1] Where the definition is sketchy, page numbers are provided where these concepts are fully explained, along with many other important and useful definitions. For example a discussion of $\mathbb{N}$ may be found on page 29. Math symbols such as $\mathbb{N}$ may be found at the top of the index, since they are difficult to alphabetize.

A.1.1 Units

Strangely, or not, classical mathematics (as taught today in schools) does not contain the concept of units. It seems units have been abstracted away. This makes mathematics distinct from physics, where almost everything has units. Presumably this makes mathematics more general (i.e., abstract). But for the engineering mind, this is not ideal, as it necessarily means that important physical meaning has been surgically removed, by design. We shall stick to SI units when ever possible. Spatial coordinates are quoted in meters [m], and time in seconds [s]. Angles in degrees have no units, whereas radians have units of inverse-seconds [s^{-1}.

A.1.2 Symbols and functions

We use $\ln$ as the log function base e, $\log$ as base 2, and π_k to indicate the kth prime (e.g., $\pi_1 = 2, \pi_2 = 3$).

When working with Fourier $\mathcal{FT}$ and Laplace $\mathcal{LT}$ transforms, lower case symbols are in the time domain while upper case indicates the frequency domain, as $f(t) \leftrightarrow F(\omega)$. An important exception are Maxwell's equations, because they are so widely used as upper case bold letters (e.g., $\boldsymbol{E}(\boldsymbol{x}, \omega)$). It seems logical to change this to conform to lower case, with $\boldsymbol{e}(\boldsymbol{x}, t) \leftrightarrow \boldsymbol{E}(\boldsymbol{x}, \omega)$ as the preferred notation.

A.1.3 Special symbols common to mathematical:

There are many pre-defined symbols in mathematics, too many to summarize here. We shall only use a small subset, defined here.

[1] https://en.wikipedia.org/wiki/List_of_mathematical_symbols_by_subject#Definition_symbols

- A set is a collection of objects that have a common property. A set is defined using braces. For example, if set $P = \{a, b, c\}$ such that $a^2 + b^2 = c^2$, then members of P obey the Pythagorean theorem. Thus we could say that $\{1, 1, \sqrt{2}\} \in P$.

- Number sets: $\mathbb{N}, \mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \hat{\mathbf{x}}, \mathbb{R}, \mathbb{C}$ are briefly discussed below, and in more detail in Section 1.2.1 on p. 29.

- One can define sets of sets and subsets of sets, and this is prone (in my experience) to error. For example, what is the difference between the number 0 and the null set $\varnothing = \{0\}$? Is $0 \in \varnothing$? Ask a mathematician. It is not a useful construction in the world of engineering.

- They symbol $\perp$ is used in many different ways to indicate two things are perpendicular, orthogonal, or in disjoint sets. In set theory $A \perp B$ is equivalent to $A \cap B = \varnothing$. If two vectors $\boldsymbol{E}, \boldsymbol{H}$ are perpendicular $\boldsymbol{E} \perp \boldsymbol{H}$, then their inner product $\boldsymbol{E} \cdot \boldsymbol{H} = 0$ is zero. One must infer the meaning of $\perp$ from its usage (the context).

A.1.4 Greek letters

The Greek letters used in this text include (at least) $\alpha, \beta, \gamma, \delta, \epsilon, \kappa, \rho, \xi, \omega, \sigma, \phi, \psi, \zeta$, and upper-case $\Gamma, \Xi, \Phi, \Psi, \Delta, \Omega$. Unfortunately some common Greek letters have no upper-case symbol, such as α, ζ.

Many of these are pre-associated in engineering and physics with a specific physical meaning. For example, ω [rad] is the radian frequency $2\pi f$, ρ [kgm/m^3] is commonly the density. ϕ, ψ are commonly used to indicate angles of a triangle, and $\zeta(s)$ is the Riemann zeta function. Many of these are so well established it makes no sense to define new terms, so we will adopt these common terms (and define them).

Likely you do not know all of these Greek letters, commonly used in mathematics. Some of them are pronounced in strange ways. The symbol ξ is pronounced "see," ζ is "zeta," β is "beta," and χ is "kie" (rhymes with pie and sky). I will assume you know how to pronounce the others, which are more phonetic in English. One advantage of learning LaTeX is that all of these math symbols are built in, and thus more easily learned, once you have adopted this powerful open-source math-oriented word-processing system (e.g., used to write this book).

Table A.1: Double-bold notation for the types of numbers. (#) is a page number. Symbol with an exponent denote the dimensionality. Thus $\mathbb{R}^2$ represents the real plane. An exponent of 0 denotes point, e.g., $\jmath \in \mathbb{C}^0$.

Symbol (p. #)	Genus	Examples	Counter Examples
$\mathbb{N}$ (29)	Counting	1,2,17,3, 10^{20}	0, -10, 5j
$\mathbb{P}$ (29)	Prime	2,17,3, 10^{20}	0, 1, 4, 3^2, 12, -5
$\mathbb{Z}$ (30)	Integer	-1, 0, 17, 5j, -10^{20}	1/2,π, $\sqrt{5}$
$\mathbb{Q}$ (30)	Rational	2/1, 3/2, 1.5, 1.14	$\sqrt{2}$, $3^{-1/3}$, π
$\mathbb{F}$ (30)	Fractional	1/2, 7/22	2/1, $1/\sqrt{2}$
$\hat{\mathbf{x}}$ (30)	Irrational	$\sqrt{2}$, $3^{-1/3}$, π, e	Vectors
$\mathbb{R}$ (31)	Reals	$\sqrt{2}$, $3^{-1/3}$, π	$2\pi j$
$\mathbb{C}$ (31)	Complex	$1, \sqrt{2}j$, $3^{-j/3}$, π^j	Vectors

A.1.5 Double-Bold notation

Table A.1 indicates the symbol followed by a page number indication where it is discussed, and the Genus (class) of the number type. For example, $\mathbb{N} > 0$ indicates the infinite set of *counting numbers* $\{1, 2, 3, \cdots\}$, not including zero. Starting from any counting number, you get the next one by adding 1. Counting numbers are also know as the *Cardinal numbers*.

We say that a number is in the set with the notation $3 \in \mathbb{N} \subset \mathbb{R}$, which is read as "3 is in the set of counting numbers, which in turn in the set of real numbers," or in vernacular language "3 is a real counting number."

Prime numbers ($\mathbb{P} \subset \mathbb{N}$) are taken from the counting numbers, but do not include 1.

The signed integers $\mathbb{Z}$ include 0 and negative integers. Rational numbers $\mathbb{Q}$ are historically defined to include $\mathbb{Z}$, a somewhat inconvenient definition, since the more interesting class are the *fractionals* $\mathbb{F}$, a subset of rationals $\mathbb{F} \in \mathbb{Q}$ that exclude the integers (i.e., $\mathbb{F} \perp \mathbb{Z}$). This is a useful definition because the rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are formed from the union of integers and fractionals.

The rationals may be define, using set notation (a very sloppy language, with incomprehensible syntax) as

$$\mathbb{Q} = \{p/q : q \neq 0 \ \& \ p, q \in \mathbb{Z}\}$$

which may be read as "the set '$\{\cdots\}$' of all p/q such that ':' $q \neq 0$, 'and' $p, q \subset \mathbb{Z}$. The translation of the symbols is in single ('$\cdots$') quotes.

Irrational numbers $\hat{\mathbf{x}}$ are very special: They are formed by taking a limit of fractionals, as the numerator and denominator $\to \infty$, and approach a limit point. It follows that irrational numbers must be approximated by fractionals.

The reals ($\mathbb{R}$) include complex numbers ($\mathbb{C}$) having a zero imaginary part (i.e., $\mathbb{R} \subset \mathbb{C}$).

The *size* of a set is denoted by taking the absolute value (e.g., $|\mathbb{N}|$). Normally in mathematics this symbol indicates the cardinality, so we are defining it differently from the standard notation.

Classification of numbers: From the above definitions there exists a natural heretical structure of numbers:

$$\mathbb{P} \in \mathbb{N}, \quad \mathbb{Z} : \{\mathbb{N}, 0, -\mathbb{N}\}, \quad \mathbb{F} \perp \mathbb{Z}, \quad \mathbb{Q} : \mathbb{Z} \cup \mathbb{F}, \quad \mathbb{R} : \mathbb{Q} \cup \hat{\mathbf{x}} \subset \mathbb{C}$$

1. The primes are a subset of the counting numbers: $\mathbb{P} \subset \mathbb{N}$.

2. The signed integers $\mathbb{Z}$ are composed of $\pm\mathbb{N}$ and 0, thus $\mathbb{N} \subset \mathbb{Z}$.

3. The fractionals $\mathbb{F}$ do not include of the signed integers $\mathbb{Z}$.

4. The rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are the union of the signed integers and fractionals

5. Irrational numbers $\hat{\mathbf{x}}$ have the special properties $\hat{\mathbf{x}} \perp \mathbb{Q}$.

6. The reals $\mathbb{R} : \mathbb{Q}, \hat{\mathbf{x}}$ are the union of rationals and *irrationals* $\hat{\mathbf{x}}$

7. Reals $\mathbb{R}$ may be defined as a subset of those complex numbers $\mathbb{C}$ having zero imaginary part.

A.2 Periodic functions

Fourier series tells us that periodic functions are discrete in frequency, with frequencies given by nT_x, where T_s is the sample period ($T_s = 1/2F_{max}$ and $F_{min} = F_{max}/\text{NFT}$).

This concept is captured by the *Fourier series*, which is a frequency expansion of a periodic function. This concept is quite general. Periodic in frequency implies discrete in time. Periodic and discrete in time requires periodic and discrete in frequency (the case of the DFT). The modulo function $mmodn$ is periodic with period n.

A periodic function may be conveniently indicated using double-parentheses notation. This is sometimes known as modular arithmetic. For example

$$f((t))_T = f(t) = f(t \pm kT),$$

is periodic on $t, T \in \mathbb{R}$ with a period of T and $k \in \mathbb{Z}$. This notation is useful when dealing with Fourier series of periodic functions.

When a discrete valued (e.g., time) sequence is periodic we use square brackets

$$f[[n]]_N = f[n] = f[n \pm kN],$$

with $n, k, N \in \mathbb{Z}$ and period N. This notation will be used with discrete-time signals that are periodic, such as the case of the DFT.

A.3 Differential equations vs. Polynomials

A polynomial has *degree* N defined by the largest power. A quadratic equation is degree 2, and a cubic has degree 3. We shall indicate a polynomial by the notation

$$P_N(z) = z^N + a_{N-1}z^{N-1} \cdots a_0.$$

It is a good practice to normalize the polynomial so that $a_N = 1$. This will not change the roots, defined by Eq. 1.27 (p. 59). The coefficient on z^{N-1} is always the sum of the roots z_n ($a_{N-1} = \sum_n^N z_n$), and a_0 is always their product ($a_0 = \prod_n^N z_n$).

Differential equations have *order* (polynomials have degree). If a second order differential equation is Laplace transformed (Lec. 1.3.14, p. 98), one is left with a degree 2 polynomial.

Example:

$$\frac{d^2}{dt^2}y(t) + b\frac{d}{dt}y(t) + cy(t) = \alpha\left(\frac{d}{dt}x(t) + \beta x(t)\right) \leftrightarrow \tag{A.1}$$

$$(s^2 + bs + c)Y(s) = \alpha(s + \beta)X(s). \tag{A.2}$$

$$\frac{Y(s)}{X(s)} = \alpha\frac{s + \beta}{s^2 + bs + c} \equiv H(s) \leftrightarrow h(t). \tag{A.3}$$

Using the same argument as for polynomials, the lead coefficient must always be 1. The coefficient $\alpha \in \mathbb{R}$ is called the *gain*. The complex variable s is the *Laplace frequency*. Discuss half-derivatives, etc.

The ratio of the output $Y(s)$ over the input $X(s)$ is called the system *transfer function* $H(s)$. When $H(s)$ is the ratio of two polynomials in s, the transfer function is said to be *bilinear*, since it is linear in both the input and output. The roots of the numerator are called the *zeros* and those of the denominator, the *poles*. The inverse Laplace transform of the transfer function is called the system *impulse response*, which describes the system's output signal $y(t)$ for any given input signal $x(t)$, via convolution (i.e., $y(t) = h(t) \star x(t)$).

Appendix B

Matrix algebra: Systems

B.1 Vectors

Vectors as columns of ordered sets of scalars $\in \mathbb{C}$. When we write then out in text, we typically use row notation, with the *transpose* symbol:

$$[a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

This is strictly to save space on the page. The notation for *conjugate transpose is* †, for example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^\dagger = \begin{bmatrix} a^* & b^* & c^* \end{bmatrix}.$$

The above example is said to be a 3 *dimensional* vector, because it has three components.

Row vs. column vectors: With rare exceptions, vectors are columns, denoted *column-major*.[1] To avoid confusion, it is a good rule to make your mental default column-major, in keeping with most signal processing (vectorized) software.[2] Column vectors are the unstated default of Matlab/Octave, only revealed when matrix operations are performed. The need for the column (or row) major is revealed as a consequence of efficiency when accessing long sequences of numbers from computer memory. For example, when forming the sum of many numbers using the Matlab/Octave command `sum(A)`, where `A` is a matrix, by default Matlab/Octave operates on the columns, returning a row vector, of column sums. Specifically

$$\texttt{sum} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [4, 6].$$

If the data were stored in "row-major" order, the answer would have been the column vector $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

[1] `https://en.wikipedia.org/wiki/Row-_and_column-major_order`

[2] In contrast, reading words in English is 'row-major.'

Dot products: A vector *dot product* is defined to weight vector elements before summing them, resulting in a scalar. The transpose of a vector (a *row-vector*), typically used as *weights* on the elements of a vector. Dot products play an important role in vector algebra and calculus. For example

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}^T \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix} \begin{bmatrix}1\\2\\3\end{bmatrix} = 1+2+3 = 6.$$

Dialects of vector notation: Physical fields are, by definition, functions of space $\boldsymbol{x}$ [m], and in the most general case, time t[s]. When Laplace transformed, the fields become functions of space and complex frequency (e.g., $\boldsymbol{E}(\boldsymbol{x},t) \leftrightarrow \boldsymbol{E}(\boldsymbol{x},s)$). As before, there are several equivalent vector notations. For example, vector $\boldsymbol{E}(\boldsymbol{x},t)$ may also written as

$$\boldsymbol{E}(\boldsymbol{x},t) = \begin{bmatrix}E_x(\boldsymbol{x},t)\\E_y(\boldsymbol{x},t)\\E_z(\boldsymbol{x},t)\end{bmatrix} = \begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix}(\boldsymbol{x},t) = \begin{bmatrix}E_x, & E_y, & E_z\end{bmatrix}^T \equiv E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}.$$

The above equation, with an equation number, is called a "displayed" equation. The in-line dialect is $\boldsymbol{E}(\boldsymbol{x},t) == \begin{bmatrix}E_x, & E_y, & E_z\end{bmatrix}^T E_x(\boldsymbol{x},t)\hat{\mathbf{x}} + E_y(\boldsymbol{x},t)\hat{\mathbf{y}} + E_z(\boldsymbol{x},t)\hat{\mathbf{z}}$. These are all shorthand notations for expressing the vector. Such usage is similar to a dialect in a language. Another dialect is column-transpose or "in-line" notation, so to place the vector on one line, saving space: $\boldsymbol{E}(\boldsymbol{x},t) = [E_x, E_y, E_z](\boldsymbol{x},t)]^T$. These several variants on vector notation are frequently used, even interchangeably.

Complex elements: When the elements are complex ($\in \mathbb{C}$), the transpose is defined as the complex conjugate of the elements. In such complex cases the transpose conjugate is denoted with a † rather than T. Vectors are also frequency written using a bold font

$$\boldsymbol{a}^T = \begin{bmatrix}2\jmath & -3\jmath & 1\end{bmatrix} \in \mathbb{C}.$$

For this case when the elements are complex, the dot product is a real number

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}^\dagger\boldsymbol{b} = \begin{bmatrix}a_1^* & a_2^* & a_3^*\end{bmatrix} \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = a_1^*b_1 + a_2^*b_2 + a_3^*b_3 \in \mathbb{R}.$$

Norm of a vector: The dot product of a vector with itself is called the *norm* of $\boldsymbol{a}$, designated as

$$||\boldsymbol{a}|| = \sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}} \geq 0$$

which is always non-negative.

Such a construction is useful when $\boldsymbol{a}$ and $\boldsymbol{b}$ are related by an impedance matrix

$$\boldsymbol{V}(s) = \boldsymbol{Z}(s)\boldsymbol{I}(s)$$

and we wish to compute the power. For example, the impedance of a mass is ms and a capacitor is $1/sC$. When given a system of equations (a mechanical or electrical circuit) one may define an impedance matrix.

Complex power: In this special case, the *complex power* $\mathcal{P}(s) \in \mathbb{R}(s)$ is defined, in the complex frequency domain (s) as

$$\mathcal{P}(s) = \boldsymbol{I}^{\dagger}(s)\boldsymbol{V}(s) = \boldsymbol{I}^{\dagger}(s)\boldsymbol{Z}(s)\boldsymbol{I}(s) \leftrightarrow p(t). \quad \text{[W]}$$

The real part of the complex power must be positive. The imaginary part corresponds to available stored energy.

GIVE MORE EXAMPLES

The case of three-dimensions is special, allowing definitions that are not defined in more dimensions. A vector in $\mathbb{R}^3$ labels the point having the coordinates of that vector.

B.1.1 Vectors in $\mathbb{R}^3$

Dot product in $\mathbb{R}^3$: The dot $\boldsymbol{B} \cdot \boldsymbol{C} = ||\boldsymbol{B}||\,||\boldsymbol{C}||\cos(\theta)$, and $\cos(\theta)$ is called the *direction cosine* between $\boldsymbol{B}$ and $\boldsymbol{C}$.

Norm of a vector in $\mathbb{R}^3$: The norm of a vector is the dot product of it with itself

$$||\boldsymbol{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$$

Euclidean distance between two points in $\mathbb{R}^3$: The dot product of the difference between two vectors $(\boldsymbol{A} - \boldsymbol{B}) \cdot (\boldsymbol{A} - \boldsymbol{B})$ is the Euclidean distance between the points they define

$$||\boldsymbol{A} - \boldsymbol{B}|| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

Cross product: and cross product $\boldsymbol{A} \times \boldsymbol{B} = ||\boldsymbol{A}||\,||\boldsymbol{B}||\sin(\theta)$ are defined between the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

The triple product: This is defined between three vectors as

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3, \\ c_1 & c_2 & c_3 \end{vmatrix}$$

also defined in Fig. 1.16. This may be indicated without the use of parentheses, since there can be no other meaningful interpretation. However for rigor, parentheses should be used. The triple product is the volume of the parallelepiped (3D-crystal shape) outlined by the three vectors, shown in Fig. 1.16, p. 80.

B.2 NxM Matrices

When working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements of the column-vector of weights multiplied from the right. For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\ a_{21} & a_{22} & a_{32} & \cdots & a_{3M} \\ & & \ddots & & \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_M \end{bmatrix} = w_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{21} \\ \cdots \\ a_{N1} \end{bmatrix} + w_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \cdots \\ a_{N2} \end{bmatrix} \ldots w_M \begin{bmatrix} a_{1M} \\ a_{2M} \\ a_{3M} \\ \cdots \\ a_{NM} \end{bmatrix},$$

where the weights are $\left[w_1, w_2, \ldots, w_M.\right]^T$ Alternatively the matrix is a set of row vectors of weights, each of which re applied to the column vector on the right ($[w_1, w_2, \cdots, W_M]^T$).

The determinant of a matrix is denoted either as $\det \boldsymbol{A}$ or simply $|\boldsymbol{A}|$, (as in the absolute value.) The inverse of a square matrix is $\boldsymbol{A}^{-1}$ or inv$\boldsymbol{A}$. If $|A| = 0$, the inverse does not exist. $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A}$.

Matlab's notional convention for a row-vector is $[a, b, c]$ and a column-vector is $[a; b; c]$. A prime on a vector takes the complex conjugate transpose. To suppress the conjugation, place a period before the prime. The : argument converts the array into a column vector, without conjugation. A tacit notation in Matlab is that *vectors* are columns and the index to a vector is a row vector. Matlab defines the notation $1:4$ as the "row-vector" $[1, 2, 3, 4]$, which is unfortunate as it leads users to assume that the default vector is a row. This can lead to serious confusion later, as Matlab's default vector is a column. I have not found the above convention explicitly stated, and it took me years to figure this out for myself.

When writing a complex number we shall adopt $1\jmath$ to indicate $\sqrt{-1}$. Matlab/Octave prefer this as well, as its explicit.

Units are SI; Angles in degrees [deg] unless otherwise noted. The units for π are always in radians [rad]. Ex: $\sin(\pi), e^{\jmath 90^\circ} e^{\jmath \pi/2}$.

B.2.1 NxN matrices and 2x2 systems

this discussion belong in ndix A?

Definitions:

1. *Scalar*: A number, e.g. $\{a, b, c, \alpha, \beta, \cdots\} \in \{\mathbb{Z}, \mathbb{Q}, \hat{\mathbf{x}}, \mathbb{R}, \mathbb{C}\}$

2. *Vector*: A quantity having direction as well as magnitude, often denoted by a bold-face letter with an arrow, $\boldsymbol{x}$. In matrix notation, this is typically represented as a single row $[x_1, x_2, x_3, \ldots]$ or single column $[x_1, x_2, x_3 \ldots]^T$ (where T indicates the transpose). In this class we will typically use column vectors. The vector may also be written out using unit vector notation to indicate direction. For example: $\boldsymbol{x}_{3\times 1} = x_1\hat{\mathbf{x}} + x_2\hat{\mathbf{y}} + x_3\hat{\mathbf{z}} = [x_1, x_2, x_3]^T$, where $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are unit vectors in the x, y, z Cartesian directions (here the vector's subscript 3×1 indicates its dimensions). The type of notation used may depend on the engineering problem you are solving.

3. *Matrix:* $A = \left[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \cdots, \boldsymbol{a}_M\right]_{N\times M} = \{a_{n,m}\}_{N\times M}$, can be a non-square matrix if the number of elements in each of the vectors (N) is not equal to the number of vectors (M). When $M = N$, the matrix is square. It may be inverted if its determinant $|A| = \prod \lambda_k \neq 0$ (where λ_k are the eigenvalues).

   We shall only work with 2×2 and 3×3 square matrices throughout this course.

4. *Linear system of equations*: $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{x}$ and $\boldsymbol{b}$ are vectors and matrix A is a square.

   (a) *Inverse:* The solution of this system of equations may be found by finding the inverse $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

   (b) *Equivalence:* If two systems of equations $A_0\boldsymbol{x} = \boldsymbol{b}_0$ and $A_1\boldsymbol{x} = \boldsymbol{b}_1$ have the same solution (i.e., $\boldsymbol{x} = A_0^{-1}\boldsymbol{b}_0 = A_1^{-1}\boldsymbol{b}_1$), they are said to be equivalent.

(c) *Augmented matrix:* The first type of augmented matrix is defined by combining the matrix with the right-hand-side. For example, given the linear system of equations $A\boldsymbol{x} = \boldsymbol{y}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

then the augmented matrix is

$$A|y = \left[\begin{array}{cc|c} a & b & y_1 \\ c & d & y_2 \end{array}\right]$$

A second type of augmented matrix may be used for finding the inverse of a matrix (rather than solving a specific instance of linear equations $Ax = b$). In this case the augmented matrix is

$$A|I = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

Performing Gaussian elimination on this matrix, until the left side becomes the identity matrix, yields A^{-1}. This is because multiplying both sides by A^{-1} gives $A^{-1}A|A^{-1}I = I|A^{-1}$.

5. *Permutation matrix* (P): A matrix that is equivalent to the identity matrix, but with scrambled rows (or columns). Such a matrix has the properties $det(P) = \pm 1$ and $P^2 = I$. For the 2x2 case, there is only one permutation matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A permutation matrix P swaps rows or columns of the matrix it operates on. For example, in the 2x2 case, pre-multiplication swaps the rows

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ a & b \end{bmatrix},$$

whereas post-multiplication swaps the *columns*

$$AP = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ \beta & \alpha \end{bmatrix}$$

For the 3x3 case there are $3 \cdot 2 = 6$ such matrices, including the original 3x3 identity matrix (swap a row with the other 2, then swap the remaining two rows).

6. *Gaussian elimination (GE) operations* G_k: There are 3 types of elementary row operations, which may be performed without fundamentally altering a system of equations (e.g. the resulting system of equations is *equivalent*). These operations are (1) swap rows (e.g. using a permutation matrix), (2) scale rows, or (3) perform addition/subtraction of two scaled rows. All such operations can be performed using matrices.

   For lack of a better term, we'll describe these as 'Gaussian elimination' or 'GE' matrices.[3] We will categorize any matrix that performs only elementary row operations (but any number of them) as a 'GE' matrix. Therefore, cascade of GE matrices is also a GE matrix.

[3]The term 'elementary matrix' may also be used to refer to a matrix that performs an elementary row operation. Typically, each elementary matrix differs from the identity matrix by one single row operation. A cascade of elementary matrices could be used to perform Gaussian elimination.

Consider the GE matrix

$$G = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

(a) This pre-multiplication scales and subtracts row (1) from (2) and returns it to row (2).

$$GA = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} a & b \\ a-\alpha & b-\beta \end{bmatrix}$$

The shorthand for this operation is Gaussian elimination operation is $(1) \leftarrow (1)$ and $(2) \leftarrow (1) - (2)$.

(b) Post-multiplication adds and scales *columns*.

$$AG = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & b \\ \alpha-\beta & \beta \end{bmatrix}$$

Here the second column is subtracted from the first, and placed in the first. The second column is untouched. **This operation is *not* a Gaussian elimination**. Therefore, to put Gaussian elimination operations in matrix form, we form a cascade of **pre-multiply** matrices.

Here $\det(G) = 1$, $G^2 = I$, which won't always be true if we scale by a number greater than 1. For instance, if $G = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ (scale and add), then we have $\det(G) = 1$, $G^n = \begin{bmatrix} 1 & 0 \\ n \cdot m & 1 \end{bmatrix}$.

Exercise: Find the solution to the following 3x3 matrix equation $Ax = b$ by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Matlab.

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

1. Show (i.e., verify) that the first GE matrix G_1, which zeros out all entries in the first column, is given by

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Identify the elementary row operations that this matrix performs. **Solution:** Operate with GE matrix on A

$$G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

The second row of G_1 scales the first row by -3 and adds it to the second row

$$(2) \Leftarrow -3(1) + (2).$$

The third row of $G1$ scales the first row by -1 and adds it to the third row $[(3) \Leftarrow -(1)+(3)]$.

2. Find a second GE matrix, G_2, to put G_1A in upper triangular form. Identify the elementary row operations that this matrix performs. **Solution:**

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

or $[(2) \Leftarrow -(2) + (3)]$. Thus we have

$$G_2G_1[A|b] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

3. Find a third GE matrix, G_3, which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs. **Solution:**

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which scales the second row by -1/2. Thus we have

$$G_3G_2G_1[A|b] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

4. Finally, find the last GE matrix, G_4, that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$). **Solution:**

$$G_4 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we have

$$G_4G_3G_2G_1[A|b] = \left[\begin{array}{ccc} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

5. Solve for $[x_1, x_2, x_3]^T$ using the augmented matrix format $G_4G_3G_2G_1[A|b]$ (where $[A|b]$ is the augmented matrix). Note that if you've performed the preceding steps correctly, $x = G_4G_3G_2G_1b$. **Solution:** From the preceding problems, we see that $[x_1, x_2, x_3]^T = [3, -1, 1]^T$.

B.3 Inverse of the 2x2 matrix

We shall now apply Gaussian elimination to find the solution $[x_1, x_2]$ for the 2x2 matrix equation $Ax = y$ (Eq. 1.59, left). We assume to know $[a, b, c, d]$ and $[y_1, y_2]$. We wish to show that the intersection (solution) is given by the equation on the right.

Here we wish to prove that the left equation (*i*) has an inverse given by the right equation (*ii*):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (i); \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (ii).$$

How to take inverse:
1) Swap the diagonal, 2) change the signs of the off-diagonal, and 3) divide by Δ.

B.3.1 Derivation of the inverse of a 2x2 matrix

1. Step 1: To derive (*ii*) starting from (*i*), normalize the first column to 1.

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 1 & \frac{d}{c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

2. Step 2: Subtract row (1) from row (2): (2) ← (2)-(1)

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{d}{c} - \frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{c} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

3. Step 3: Multiply row (2) by ca and express result in terms of the determinate $\Delta = ad - bc$.

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

4. Step 4: Solve row (2) for x_2: $x_2 = -\frac{c}{\Delta} y_1 + \frac{a}{\Delta} y_2$.

5. Step 5: Solve row (1) for x_1:

$$\begin{aligned} x_1 &= \frac{1}{a} y_1 - \frac{b}{a} x_2 \\ &= \frac{1}{a} y_1 - \frac{b}{a} \left[-\frac{c}{\Delta} y_1 + \frac{a}{\Delta} y_2 \right] \\ &= \left[\frac{1}{a} + \frac{b}{a} \frac{c}{\Delta} \right] y_1 - \frac{b}{a} \frac{a}{\Delta} y_2. \end{aligned}$$

Rewriting in matrix format, in terms of $\Delta = ad - bc$, gives:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} + \frac{bc}{a\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{\Delta + bc}{a} & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

since $d = (\Delta + bc)/a$.

Summary: This is a lot of messy algebra, that is why it is essential you memorize it:
1) Swap diagonal, 2) change off-diagonal signs, 3) normalize by Δ.

Appendix C

Eigen Analysis

In the following discussion we show how to determine $\boldsymbol{E}$ and D (i.e., Λ), given $\boldsymbol{A}$.

Calculating the eigenvalue matrix (Λ): The matrix equation for $\boldsymbol{E}$ is

$$\boldsymbol{A}\boldsymbol{E} = \boldsymbol{E}\Lambda. \tag{C.1}$$

Pre-multiplying by $\boldsymbol{E}^{-1}$ diagonalizes $\boldsymbol{A}$, given the *eigenvalue matrix* (D in Matlab)

$$\Lambda = \boldsymbol{E}^{-1}\boldsymbol{A}\boldsymbol{E}. \tag{C.2}$$

Post-multiplying by $\boldsymbol{E}^{-1}$ recovers $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{E}\Lambda\boldsymbol{E}^{-1}. \tag{C.3}$$

Matrix power formula: This last relation is the entire point of the eigenvector analysis, since it shows that any power of $\boldsymbol{A}$ may be computed from powers of the eigen values. Specifically

$$\boldsymbol{A}^n = \boldsymbol{E}\Lambda^n\boldsymbol{E}^{-1}. \tag{C.4}$$

For example, $\boldsymbol{A}^2 = \boldsymbol{A}\boldsymbol{A} = \boldsymbol{E}\Lambda\ (\boldsymbol{E}^{-1}\boldsymbol{E})\ \Lambda\boldsymbol{E}^{-1} = \boldsymbol{E}\Lambda^2\boldsymbol{E}^{-1}$.

Equations C.1, C.2 and C.3 are the key to eigenvector analysis, and you need to memorize them. You will use them repeatedly throughout this course, and possibly for a long time after it is over.

Showing that $\boldsymbol{A} - \lambda_\pm\boldsymbol{I}_2$ is singular: If we restrict Eq. C.1 to a single eigenvector (one of $\boldsymbol{e}_\pm$), along with the corresponding eigenvalue $\lambda_\pm$, we obtain a matrix equations

$$\boldsymbol{A}\boldsymbol{e}_\pm = \boldsymbol{E}_\pm\lambda_\pm = \lambda_\pm\boldsymbol{E}_\pm$$

Note the important swap in the order of $\boldsymbol{E}_\pm$ and $\lambda_\pm$. Since $\lambda_\pm$ is a scalar, this is legal (and critically important), since this allows us to remove (factored out) $\boldsymbol{E}_\pm$

$$(\boldsymbol{A} - \lambda_\pm\boldsymbol{I}_2)\boldsymbol{E}_\pm = 0. \tag{C.5}$$

This means that the matrix $\boldsymbol{A} - \lambda_\pm\boldsymbol{I}_2$ must be singular, since when it operates on $\boldsymbol{E}_\pm$, which is not zero, it gives zero. It immediately follows that its determinant is zero (i.e., $|(\boldsymbol{A} - \lambda_\pm\boldsymbol{I}_2)| = 0$). This equation is used to uniquely determine the eigenvalues $\lambda_\pm$. Note the important difference between $\lambda_\pm\boldsymbol{I}_2$ and Λ (i.e., $|(A - \Lambda)| \neq 0$).

Calculating the eigenvalues $\lambda_\pm$: The eigenvalues $\lambda_\pm$ of $\boldsymbol{A}$ may be determined from $|(\boldsymbol{A} - \lambda_\pm \boldsymbol{I}_2)| = 0$

$$\begin{vmatrix} 1-\lambda_\pm & N \\ 1 & 1-\lambda_\pm \end{vmatrix} = (1-\lambda_\pm)^2 - N^2 = 0.$$

For our case of $N = 2$, $\lambda_\pm = (1 \pm \sqrt{2})$.[1]

Calculating the eigenvectors $e_\pm$: Once the eigenvalues have been determined, they are substitute them into Eq. C.5, which determines the eigenvectors $\boldsymbol{E} = \left[\mathbf{e}_+, \mathbf{e}_-\right]$, by solving

$$(\boldsymbol{A} - \lambda_\pm)\mathbf{e}_\pm = \begin{bmatrix} 1-\lambda_\pm & 2 \\ 1 & 1-\lambda_\pm \end{bmatrix} \boldsymbol{E}_\pm = 0,$$

where $1 - \lambda_\pm = 1 - (1 \pm \sqrt{2}) = \mp\sqrt{2}$.

Recall that Eq. C.5 is singular, because we are using an eigenvalue, and each eigenvector is pointing in a unique direction (This is why it is singular). You might respectively suggest that this equation has no solution. In some sense you would be correct. When we solve for $\boldsymbol{E}_\pm$, the two equations defined by Eq. C.5 *co-linear* (the two equations describe parallel lines). This follows from the fact that there is only one eigenvector for each eigenvalue.

Expecting trouble, yet proceeding to solve for $\boldsymbol{E} = [e_1^+, e_2^+]^T$,

$$\begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1^+ \\ e_2^+ \end{bmatrix} = 0.$$

This gives two identical equations $-\sqrt{2}e_1^+ + 2e_2^+ = 0$ and $e_1^+ - \sqrt{2}e_2^+ = 0$. This is the price of an over-specified equation (the singular matrix is degenerate). The most we can determine is $\mathbf{e}_+ = c\ [-\sqrt{2}, 1]^T$, where c is a constant. We can determine eigenvector direction, but not its magnitude.

Following *exactly* the same procedure for λ_-, the equation for $\boldsymbol{e}_-$ is

$$\begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} e_1^- \\ e_2^- \end{bmatrix} = 0.$$

In this case the relation becomes $e_1^- + \sqrt{2}e_2^- = 0$, thus $\boldsymbol{E}_- = c\ [\sqrt{2}, 1]^T$ where c is a constant.

Normalization of the eigenvectors: The constant c may be determined by normalizing the eigenvectors to have unit length. Since we cannot determine the length, we set it to 1. In some sense the degeneracy is resolved by this normalization. Thus $c = 1/\sqrt{3}$, since

$$\left(\pm\sqrt{2}\right)^2 + 1^2 = 3 = 1/c^2.$$

Summary: Thus far we have shown

$$\boldsymbol{E} = [\boldsymbol{e}_+, \boldsymbol{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}.$$

[1] It is a convention to order the eigenvalues from largest to smallest.

Verify that $\Lambda = \boldsymbol{E}^{-1}\boldsymbol{A}\boldsymbol{E}$: To find the inverse of $\boldsymbol{E}$, 1) swap the diagonal values, 2) change the sign of the off diagonals, and 3) divide by the determinant $\Delta = 2\sqrt{2}/\sqrt{3}$ (see Appendix **??**)

$$\boldsymbol{E}^{-1} = \frac{\sqrt{3}}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.6124 & 0.866 \\ -0.6124 & 0.866 \end{bmatrix}.$$

By definition for any matrix $\boldsymbol{E}^{-1}\boldsymbol{E} = \boldsymbol{E}\boldsymbol{E}^{-1} = \boldsymbol{I}_2$. Taking the product gives

$$\boldsymbol{E}^{-1}\boldsymbol{E} = \frac{\cancel{\sqrt{3}}}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} \cdot \frac{1}{\cancel{\sqrt{3}}}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}_2.$$

We wish to show that $\Lambda = \boldsymbol{E}^{-1}\boldsymbol{A}\boldsymbol{E}$

$$\begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}. = \frac{\cancel{\sqrt{3}}}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\cancel{\sqrt{3}}}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix},$$

which is best verified with Matlab.

Verify that $\boldsymbol{A} = \boldsymbol{E}\Lambda\boldsymbol{E}^{-1}$: We wish to show that

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\cancel{\sqrt{3}}}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} \cdot \frac{\cancel{\sqrt{3}}}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix},$$

which is best verified with Matlab (or Octave). All the above equations have been verified both with Matlab and Octave.

Appendix D

Symbolic analysis of $\boldsymbol{TE} = \boldsymbol{E}\Lambda$

Update: Using Octave (version 4.2.2, June 21, 2018):

$$e_{1,1} = \frac{b}{\frac{a}{2} - \frac{d}{2} + \frac{1}{2}\sqrt{a^2 - 2ad + 4bc + d^2}} = \frac{2b}{(a-d) + \sqrt{(a-d)^2 + 4bc}}$$

This looks very different from what I have below!

Here is what I get with the same command, from Matlab (June 21, 2018)

$$e_{1,1} = \frac{\frac{a}{2} + \frac{d}{2} - \frac{\sqrt{a^2 - 2\,a\,d + d^2 + 4\,b\,c}}{2}}{c} - \frac{d}{c} = \frac{1}{2c}\left[(a-d) - \sqrt{(a-d)^2 + 4\,b\,c}\right]$$

Here we derive the eigen-matrix $\boldsymbol{E}$, and eigen-value matrix Λ given a 2x2 Transmission matrix

$$\boldsymbol{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix},$$

such that $\boldsymbol{TE} = \boldsymbol{E}\Lambda$, using symbolic algebra methods, given by the Matlab/Octave's script

```
syms A B C D T E L %Use symbolic Matlab/Octave
   T=[A B;C D]     %Given matrix T
  [E,L]=eig(T)     %Find eigen-vector matrix E and
                   %eigen-value matrix L
```

D.1 General case

The eigenvectors $\boldsymbol{e}_\pm$ are

$$\boldsymbol{e}_\pm = \begin{pmatrix} \frac{1}{2\mathcal{C}}\left[(\mathcal{A}-\mathcal{D}) \mp \sqrt{(\mathcal{A}-\mathcal{D})^2 + 4\mathcal{B}\mathcal{C}}\right] \\ 1 \end{pmatrix} \tag{D.1}$$

and eigenvalues are

$$\lambda_\pm = \frac{1}{2}\begin{pmatrix} (\mathcal{A}+\mathcal{D}) - \sqrt{(\mathcal{A}-\mathcal{D})^2 + 4\mathcal{B}\mathcal{C}} \\ (\boldsymbol{A}+\mathcal{D}) + \sqrt{(\mathcal{A}-\mathcal{D})^2 + 4\mathcal{B}\mathcal{C}} \end{pmatrix} \tag{D.2}$$

The term under the radical (i.e., the discriminant) may be written in terms of the determinant of $\boldsymbol{T}$

$$(\mathcal{A}-\mathcal{D})^2 + 4\mathcal{B}\mathcal{C} = \mathcal{A}^2 + \mathcal{D}^2 - 4(\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C}) = \mathcal{A}^2 + \mathcal{D}^2 - 4\Delta_{\boldsymbol{T}}.$$

This becomes especially important for the case of reciprocal systems where $\Delta_{\boldsymbol{T}} = 1$, or for anti-reciprocal systems where $\Delta_{\boldsymbol{T}} = -1$.

D.2 Special cases having symmetry

Each 2x2 matrix has four entries, each of which can be complex. This leads to 4x2=8 possible special symmetries (an eightfold way), discussed next, in quasi-order of their importance. Each symmetry is related to properties of $\boldsymbol{T}$. For example if $\boldsymbol{T} = \boldsymbol{T}^\dagger$, the matrix is said to have Hermitian symmetry. When $\boldsymbol{T} = \boldsymbol{T}^T$ the matrix is symmetric, and when $\boldsymbol{T} = -\boldsymbol{T}^T$ is is said to be *skew-symmetric*. Each of these eightfold symmetries corresponds to some sort of physical constraint, as discussed below.

D.2.1 Reversible systems

When the values of $\boldsymbol{T}$ on its diagonal are equal ($\mathcal{A} = \mathcal{D}$), the matrix is called *reversible*, and the eigenvectors and eigenvalues greatly simplify to

$$\boldsymbol{E} = \begin{bmatrix} -\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} & +\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \mathcal{A} - \sqrt{\mathcal{B}\mathcal{C}} & 0 \\ 0 & \boldsymbol{A} + \sqrt{\mathcal{B}\mathcal{C}} \end{bmatrix} \tag{D.3}$$

This is a common symmetry, thus it is very important. It is useful in describing transmission lines, as been discussed in length in Sections 1.3.9 (p. 89), H.3.2 and H.3.3 (pp 252-254).

D.2.2 Reciprocal systems

When the matrix is symmetric ($\mathcal{B} = \mathcal{C}$), the corresponding system is said to be *reciprocal*. Most physical systems are reciprocal. The determinant of the transmission matrix of a reciprocal network $\Delta_{\boldsymbol{T}} = \mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C} = 1$. For example Electrical networks, composed of inductors, capacitors and resistors are always reciprocal. It follows that the complex impedance matrix is symmetric (Van Valkenburg, 1964a).

Magnetic systems, such as dynamic loudspeakers are anti-reciprocal, and correspondingly $\Delta_{\boldsymbol{T}} = -1$ The impedance matrix of a loudspeaker is skew symmetric (Kim and Allen, 2013). All impedance matricies are either symmetric or anti-symmetric, depending on if they are reciprocal (LRC networks) or anti-reciprocal (magnetic networks). These system have complex eigen values with negative real parts, corresponding to lossy systems. In some sense, all of this follows from conservation of energy, but the precise general case is waiting for enlightenment. The impedance matrix is never Hermitian. It is easily proved that Hermitian matricies have real eigen-values, which correspond to lossless networks. Any physical system of equations that has any type of loss, cannot be Hermitian.

In summary, given a reciprocal system, the $\boldsymbol{T}$ matrix has $\Delta_{\boldsymbol{T}} = 1$, and the corresponding impedance matrix is symmetric (*not* Hermitian).

D.2.3 Impedance

As previously discussed in Section 1.3.9 (p. 89), the $\boldsymbol{T}$ matrix corresponding to an impedance matrix is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \boldsymbol{Z}(s) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_{\boldsymbol{T}} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

Reciprocal systems have skew-symmetric impedance matrices, namely $z_{12} = z_{21}$. When the system is both reversible $\mathcal{A} = \mathcal{D}$ and reciprocal, the impedance matrix simplifies to

$$\boldsymbol{Z}(s) = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & 1 \\ 1 & \mathcal{A} \end{bmatrix}$$

For such systems there are only two degrees of freedom, $\mathcal{A}$ and $\mathcal{C}$. As discussed previously Section 1.3.9, p. 89, these each have a physical meaning, $1/\mathcal{C}$ is the Thévenin source voltage and $\mathcal{A}/Cc$ is the Thévenin impedance.

D.2.4 Transmission matrices and symmetry

The transmission matrix fully characterizes a two-port network (Sect. 1.3.9, p. 89).

D.2.5 Hermitian symmetry

When a system is Hermitian its matrix is conjugate symmetric

$$Z(s) = Z^\dagger(s),$$

a stronger condition that reciprocal, but not the symmetric symmetry of a Brune impedance matrix. Impedance is not Hermitian, unless the diagonal elements are real, but it does have *symmetric* symmetry.

In the case of a Hermitian matrix, the eigenvalues are always real. To show this start from the definition of an impedance eigen-equation ($\mathbf{V}$ is a vector of voltages, $\mathbf{I}$ a current vector $\mathbf{Z}$ and an impedance matrix)

$$\mathbf{V} = \mathbf{Z}\mathbf{I} = \mathbf{I}\boldsymbol{\Lambda},$$

where $\mathbf{Z}, \mathbf{I}, \mathbf{V}, \boldsymbol{\Lambda} \in \mathbb{C}$, $A = A^\dagger$ is a square conjugate-symmetric matrix, and $\mathbf{I}, \mathbf{V}$ are vectors of the size of $\mathbf{Z}$. Here $\mathbf{Z}^\dagger$ is the complex transpose (see Appendix A, p. 188). The power $\mathcal{P}$ is the real part of the voltage times the current

$$2\mathcal{P} = \mathbf{V}^\dagger\mathbf{I} + \mathbf{V}\mathbf{I}^\dagger = (\mathbf{Z}\mathbf{I})^\dagger\mathbf{I} + \mathbf{Z}\mathbf{I}\,\mathbf{I}^\dagger = \mathbf{I}^\dagger\mathbf{Z}^\dagger\mathbf{I} + \mathbf{Z}\mathbf{I}\,\mathbf{I}^\dagger$$

??? Subtracting the two equations gives

D.2.6 Double roots

For the 2x2 case of double roots the matrix has Jordan form

$$\boldsymbol{T} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Then

$$\boldsymbol{T}^2 = \begin{bmatrix} \lambda^n & n\lambda \\ 0 & \lambda^n \end{bmatrix}.$$

This generalizes to nxn matrices having arbitrary combinations of degeneracies (multiple roots), as in symmetric (square) drums, for example,

Appendix E

Analysis of Pell equation (N=2, 3, M)

Section G.2.3 (p. 235) showed that the solution $[x_n, y_n]^T$ to Pell's equation, for $N = 2$, is given by powers of Eq. 1.10. To find an explicit formula for $[x_n, y_n]^T$, one must compute powers of

$$\boldsymbol{A} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{E.1}$$

We wish to find the solution to Pell's equation (Eq. 1.10), based on the recursive solution, Eq. 1.11 (p. 50). Thus we need is powers of A, that is A^n, which gives the a closed form expression for $[x_n, y_n]^T$. By the diagonalization of A, its powers are simply the powers of its eigenvalues. This diagonalization is called an *eigenvalue analysis*, a very general method rooted in linear algebra. This type of analysis allows us to find the solution to most of the linear the equations we encounter.

From Matlab with $N = 2$ the eigenvalues of Eq. E.1 are $\lambda_\pm \approx [2.4142\jmath, -0.4142\jmath]$ (i.e., $\lambda_\pm = 1\jmath(1 \pm \sqrt{2})$). The final solution to Eq. E.1 is given in Eq. G.11 (p. 235). The solution for $N = 3$ is provided in Appendix E.1.1 (p. 204).

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 1\jmath^n \boldsymbol{A}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\jmath^n \boldsymbol{E} \Lambda^n \boldsymbol{E}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalue matrix D is diagonal with the eigenvalues sorted, largest first. The Matlab command `[E,D]=eig(A)` is helpful to find D and E given any A. As we saw above,

$$\Lambda = 1\jmath \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 2.414\jmath & 0 \\ 0 & -0.414\jmath \end{bmatrix}.$$

E.1 Pell equation eigenvalue-eigenvector analysis

Here we show how to compute the eigenvalues and eigenvectors for the 2x2 Pell matrix for $N = 2$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The Matlab command `[E,D]=eig(A)` returns the eigenvector matrix $\boldsymbol{E}$

$$\boldsymbol{E} = [\boldsymbol{e}_+, \boldsymbol{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774. \end{bmatrix}$$

and the eigenvalue matrix Λ (Matlab's D)

$$\Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{bmatrix}.$$

The factor $\sqrt{3}$ on $\boldsymbol{E}$ normalizes each eigenvector to 1 (i.e., The Matlab command `norm([`$\sqrt{2}$`,1])` gives $\sqrt{3}$).

In the following discussion we show how to determine $\boldsymbol{E}$ and D (i.e., Λ), given $\boldsymbol{A}$.

Table E.1: *This table summarizes the solution of Pell's equation due to the Pythagoreans using matrix recursion, for the case of N=3. The integer solutions are shown in on the right. Not that $x_n/y_n \to \sqrt{3}$, in agreement with the Euclidean algorithm. The Matlab program for generating this data is `Pell3Sol.m`. It seem likely that β_0 could be absorbed in the starting solution, and then be removed from the generating function, other than as the known factor β_0^n*

Pell's Equation for $N = 3$
Case of $N = 3$ & $[x_0, y_0]^T = [1, 0]^T$, $\beta_0 = \jmath/\sqrt{2}$
Note: $x_n^2 - 3y_n^2 = 1$, $\qquad x_n/y_n \underset{\infty}{\longrightarrow} \sqrt{3}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (1\beta_0)^2 - 3(1\beta_0)^2 = 1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left(4\beta_0^2\right)^2 - 3\left(2\beta_0^2\right)^2 = 1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \left(10\beta_0^3\right)^2 - 3\left(6\beta_0^3\right)^2 = 1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 28 \\ 16 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} \qquad \left(28\beta_0^4\right)^2 - 3\left(16\beta_0^4\right)^2 = 1$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 76 \\ 44 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 28 \\ 16 \end{bmatrix} \qquad \left(76\beta_0^5\right)^2 - 3\left(44\beta_0^5\right)^2 = 1$$

E.1.1 Pell equation for $N = 3$

In Fig. E.1 for $N = 3$ is given, with $\beta_0 = \jmath/\sqrt{2}$. Perhaps try other trivial solutions such as $[-1, 0]^T$ and $[\pm\jmath, 0]^T$, to provide clues to the proper value of β_0 for cases where $N > 3$.[1]

Exercise: I suggest that you verify $\boldsymbol{E}\Lambda \neq \Lambda\boldsymbol{E}$ and $\boldsymbol{AE} = \boldsymbol{E}\Lambda$ with Matlab. Here is the Matlab program which does this:

```
A = [1 2; 1 1]; %define the matrix
[E,D] = eig(A); %compute the eigenvector and eigenvalue matrices
A*E-E*D %This should be $\approx 0$, within numerical error.
E*D-D*E %This is not zero
```

[1] My student Kehan found the genaral formula for β_o.

Appendix F

Tables of Fourier and Laplace Transforms

Properties of Fourier Transforms:

1. Both time t and frequency ω are real.
2. For the forward transform (time to frequency), the sign of the exponential is negative.
3. The limits on the integrals in both the forward and reverse FTs are $[-\infty, \infty]$.
4. When taking the inverse Fourier transform, the scale factor of $1/2\pi$ is required to cancel the 2π in the differential $d\omega$.
5. The Fourier step function may be defined by the use of superposition of 1 and $\text{sgn}(t) = t/|t|$ as

$$\widetilde{u}(t) \equiv \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & \text{if } t > 0 \\ 1/2 & t = 0 \\ 0 & \text{if } t < 0 \end{cases} .$$

   Taking the FT of a delayed step function

$$\widetilde{u}(t - T_o) \leftrightarrow \frac{1}{2} \int_{-\infty}^{\infty} [1 - \text{sgn}(t - T_o)]\, e^{-j\omega t} dt = \pi \tilde{\delta}(\omega) + \frac{e^{-j\omega T_o}}{j\omega}$$

   Thus the FT of the step function has the term $\pi\delta(\omega)$ due to the 1 in the definition of the Fourier step. This term introduces a serious flaw with the FT of the step function: While it appears to be causal, it is not.
6. The convolution $\widetilde{u}(t) \star \widetilde{u}(t)$ is not defined because both $1 \star 1$ and $\widetilde{\delta}^2(\omega)$ do not exist (and cannot be defined).
7. The inverse FT has convergence problems whenever there is a discontinuity in the time response. This we indicate with a hat over the reconstructed time response. The error between the target time function and the reconstructed is zero in the root-mean sense, but not pointwise.

   Specifically, at the discontinuity point for the Fourier step function ($t = 0$), $\widetilde{u}(t) \neq u(t)$, yet $\int |\widetilde{u}(t) - u(t)|^2 dt = 0$. At the point of the discontinuity the reconstructed function displays *Gibbs ringing* (it oscillates around the step, hence does not converge at the jump).[1] The $\mathcal{LT}$ does not exhibit Gibbs ringing.

[1] https://en.wikipedia.org/wiki/Gibbs_phenomenon

8. The FT is not always analytic in ω, as in this example of the step function. The step function cannot be expanded in a Taylor series about $\omega = 0$, because $\widetilde{\delta}(\omega)$ is not analytic in ω.

9. The Fourier δ function is denoted $\widetilde{\delta}(t)$, to differentiate it from the Laplace delta function $\delta(t)$. They differ because the step functions differ, due to the convergence problem described above.

10. One may define

$$\widetilde{u}(t) = \int_{-\infty}^{t} \widetilde{\delta}(t)dt,$$

and define the somewhat questionable notation

$$\widetilde{\delta}(t) = \frac{d}{dt}\widetilde{u}(t),$$

since the Fourier step function is not analytic.

11. The $\text{rec}(t)$ function is defined as

$$\text{rec}(t) = \frac{\widetilde{u}(t) - \widetilde{u}(t - T_o)}{T_o} = \begin{cases} 0 & \text{if } t > 0 \\ 1/T_o & 0 < t < T_o \\ 0 & \text{if } t < 0 \end{cases}.$$

It follows that $\widetilde{\delta}(t) = \lim_{T_o \to 0}$. Like $\widetilde{\delta}(t)$, the $\text{rec}(t)$ has unit area.

Table F.1: *Summary of key properties of FTs.*

FT Properties	
$\frac{d}{dt}v(t) \leftrightarrow \jmath\omega V(\omega)$	deriv
$f(t) \star g(t) \leftrightarrow F(\omega)G(\omega)$	conv
$f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega) \star G(\omega)$	conv
$f(at) \leftrightarrow \frac{1}{a}F\left(\frac{\omega}{a}\right)$	scaling

Table F.2: *A brief table of simple Fourier Transforms. Note $a > 0 \in \mathbb{R}$ has units [rad/s]. To flag this necessary condition, we use $|a|$ to assure this condition will be met. The other constant $T_o \in \mathbb{R}$ [s] has no restrictions, other than being real. Complex constants may not appear as the argument to a delta function, since complex numbers do not have the order property.*

$f(t) \leftrightarrow F(\omega)$	Name
$\widetilde{\delta}(t) \leftrightarrow 1(\omega) \equiv 1 \; \forall \; \omega$	Dirac
$1(t) \equiv 1 \, \forall \, t \leftrightarrow 2\pi\widetilde{\delta}(\omega)$	Dirac
$\text{sgn}(t) = \dfrac{t}{\lvert t \rvert} \leftrightarrow \dfrac{2}{\jmath\omega}$	
$\widetilde{u}(t) = \dfrac{1(t) + \text{sgn}(t)}{2} \leftrightarrow \pi\widetilde{\delta}(\omega) + \dfrac{1}{\jmath\omega} \equiv \widetilde{U}(\omega)$	step
$\widetilde{\delta}(t - T_o) \leftrightarrow e^{-\jmath\omega T_o}$	delay
$\widetilde{\delta}(t - T_o) \star f(t) \leftrightarrow F(\omega)e^{-\jmath\omega T_o}$	delay
$\widetilde{u}(t)e^{-\lvert a \rvert t} \leftrightarrow \dfrac{1}{\jmath\omega + \lvert a \rvert}$	exp
$\text{rec}(t) = \dfrac{1}{T_o}\left[\widetilde{u}(t) - \widetilde{u}(t - T_o)\right] \leftrightarrow \dfrac{1}{T_o}\left(1 - e^{-\jmath\omega T_o}\right)$	pulse
$\widetilde{u}(t) \star \widetilde{u}(t) \leftrightarrow \widetilde{\delta}^2(\omega)$ Not defined	NaN

Properties of the Laplace Transform

1. Time $t \in \mathbb{R}$ [s] and *Laplace frequency* [rad] are defined as $s = \sigma + \omega\jmath \in \mathbb{C}$.

2. Given a *Laplace transform* ($\mathcal{LT}$) pair $f(t) \leftrightarrow F(s)$, in the engineering literature, the *time domain* is always lower case [$f(t)$] and causal (i.e., $f(t<0)=0$) and the *frequency domain* is upper-case [e.g. $F(s)$]. Maxwell's venerable equations are the unfortunate exception to this otherwise universal rule.

3. The target time function $f(t<0)=0$ (i.e., it must be causal). The time limits are $0^- < t < \infty$. Thus the integral must start from slightly below $t=0$ to integrate over a delta functions at $t=0$. For example if $f(t)=\delta(t)$, the integral must include both sides of the impulse. If you wish to include non-causal functions such as $\delta(t+1)$ it is necessary to extend the lower time limit. In such cases simply set the lower limit of the integral to $-\infty$, and let the integrand ($f(t)$) determine the limits.

4. When taking the forward transform ($t \rightarrow s$), the sign of the exponential is negative. This is necessary to assure that the integral converges when the integrand $f(t) \rightarrow \infty$ as $t \rightarrow \infty$. For example, if $f(t) = e^t u(t)$ (i.e., without the negative σ exponent), the integral does not converge.

5. The limits on the integrals of the forward transform are $t : (0^-, \infty) \in \mathbb{R}$, and the reverse LTs are $[\sigma_o - \infty\jmath, \sigma_o + \infty\jmath] \in \mathbb{C}$. These limits will be further discussed in Section 1.4.9 (p. 123).

6. When taking the inverse Laplace transform, the normalization factor of $1/2\pi\jmath$ is required to cancel the $2\pi\jmath$ in the differential ds of the integral.

7. The frequency for the LT must be is complex, and in general $F(s)$ is complex analytic for $\sigma > \sigma_o$. It follows that the real and imaginary parts of $F(s)$ are related. Given $\Re\{F(s)\}$ it is possible to find $\Im\{F(s)\}$ (Boas, 1987). More on this in Section 1.3.14 (p. 98).

8. To take the inverse Laplace transform, we must learn how to integrate in the complex s plane. This will be explained in Sections 1.4.5-1.4.9 (p. 118-123).

9. The Laplace step function is defined as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt = \begin{cases} 1 & \text{if } t > 0 \\ \text{NaN} & t = 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

   Alternatively one could define $\delta(t) = du(t)/dt$.

10. It is easily shown that $u(t) \leftrightarrow 1/s$ by direct integration

$$F(s) = \int_0^\infty u(t)\, e^{-st} dt = -\frac{e^{-st}}{s}\bigg|_o^\infty = \frac{1}{s}.$$

    With the LT step ($u(t)$) there is no Gibbs ringing effect.

11. In many physical applications, the Laplace transform takes the form of a ratio of two polynomials. In such case the roots of the numerator polynomial are call the *zeros* while the roots of the denominator polynomial are called the *poles*. For example the LT of $u(t) \leftrightarrow 1/s$ has a pole at $s = 0$, which represents integration, since

$$u(t) \star f(t) = \int_{-\infty}^{r} f(\tau) d\tau \leftrightarrow \frac{F(s)}{s}.$$

12. The LT is quite different from the FT in terms of its analytic properties. For example, the step function $u(t) \leftrightarrow 1/s$ is complex analytic everywhere, except at $s = 0$. The FT of $1 \leftrightarrow 2\pi\widetilde{\delta}(\omega)$ is not analytic anywhere.

13. Dilated step function ($a \in \mathbb{R}$)

$$u(at) \leftrightarrow \int_{-\infty}^{\infty} u(at) e^{-st} dt = \frac{1}{a} \int_{-\infty}^{\infty} u(\tau) e^{-(s/a)\tau} d\tau = \frac{a}{|a|} \frac{1}{s} = \pm \frac{1}{s},$$

where we have made the change of variables $\tau = at$. The only effect that a has on $u(at)$ is the sign of t, since $u(t) = u(2t)$. However $u(-t) \neq u(t)$, since $u(t) \cdot u(-t) = 0$, and $u(t) + u(-t) = 1$, except at $t = 0$, where it is not defined.

Once complex integration in the complex plane has been defined (Section 1.4.2, p. 107), we can justify the definition of the inverse LT (Eq. 1.77).[2]

[2] https://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms

Table F.3: *The following table provides a brief table of Laplace Transforms of $f(t), \delta(t)$, $u(t)$, rect(t),$T_o, p, e, \in \mathbb{R}$ and $F(s), G(s)$,$s, a \in \mathbb{C}$. Given a* Laplace transform *($\mathcal{LT}$) pair $f(t) \leftrightarrow F(s)$, the frequency domain will always be upper-case [e.g. $F(s)$] and the time domain lower case [$f(t)$] and causal (i.e., $f(t<0)=0$). An extended table of transforms is given in Table F.4 on page 211.*

$f(t) \leftrightarrow F(s)$	$t \in \mathbb{R};\ s, F(s) \in \mathbb{C}$	Name
$\delta(t) \leftrightarrow 1$		Dirac
$\delta(\lvert a\rvert t) \leftrightarrow \frac{1}{\lvert a\rvert}$	$a \neq 0$	time-scaled Dirac
$\delta(t-T_o) \leftrightarrow e^{-sT_o}$		delayed Dirac
$\delta(t-T_o) \star f(t) \leftrightarrow F(s)e^{-sT_o}$		–
$\sum_{n=0}^{\infty} \delta(t-nT_o) = \frac{1}{1-\delta(t-T_o)} \leftrightarrow \frac{1}{1-e^{-sT_o}} = \sum_{n=0}^{\infty} e^{-snT_o}$		one-sided impulse train
$u(t) \leftrightarrow \frac{1}{s}$		step
$u(-t) \leftrightarrow -\frac{1}{s}$		anti-causal step
$u(at) \leftrightarrow \frac{a}{s}$	$a \neq 0 \in \mathbb{R}$	dilated or reversed step
$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$	$a > 0 \in \mathbb{R}$	damped step
$\cos(at)u(t) \leftrightarrow \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)$	$a \in \mathbb{R}$	cos
$\sin(at)u(t) \leftrightarrow \frac{1}{2\jmath}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)$	$a \in \mathbb{C}$	"damped" sin
$u(t-T_o) \leftrightarrow \frac{1}{s}e^{-sT_o}$	$T_0 > 0 \in \mathbb{R}$	time delay
$\text{rect}(t) = \frac{1}{T_o}\left[u(t)-u(t-T_o)\right] \leftrightarrow \frac{1}{T_o}\left(1-e^{-sT_o}\right)$		rect-pulse
$u(t) \star u(t) = tu(t) \leftrightarrow 1/s^2$		ramp
$u(t) \star u(t) \star u(t) = \frac{1}{2}t^2u(t) \leftrightarrow 1/s^3$		double ramp
$\frac{1}{\sqrt{t}}u(t) \leftrightarrow \sqrt{\frac{\pi}{s}}$		
$t^p u(t) \leftrightarrow \frac{\Gamma(p+1)}{s^{p+1}}$		$\Re p > -1, q \in \mathbb{C}$
$J_n(\omega_o t)\, u(t) \leftrightarrow \frac{\left(\sqrt{s^2+\omega_o^2}-s\right)^n}{\omega_o^n\sqrt{s^2+\omega_o^2}}$		

Table F.4: *Functional relationships between Laplace Transforms.*

$\mathcal{LT}$ functional properties		
$f(t) \star g(t) = \int_{t=0}^{t} f(t-\tau)g(\tau)d\tau \leftrightarrow F(s)G(s)$		convolution
$u(t) \star f(t) = \int_{0^-}^{t} f(t)dt \leftrightarrow \frac{F(s)}{s}$		convolution
$f(at)u(at) \leftrightarrow \frac{1}{a}F(\frac{s}{a})$	$a \in \mathbb{R} \neq 0$	scaling
$f(t)e^{-at}u(t) \leftrightarrow F(s+a)$		damped
$f(t-T)e^{-a(t-T)}u(t-T) \leftrightarrow e^{-sT}F(s+a)$		damped and delayed
$f(-t)u(-t) \leftrightarrow F(-s)$		reverse time
$f(-t)e^{-at}u(-t) \leftrightarrow F(a-s)$		time-reversed & damped
$\frac{d}{dt}f(t) = \delta'(t) \star f(t) \leftrightarrow sF(s)$		deriv
Additional transforms		
$\frac{\sin(t)u(t)}{t} \leftrightarrow \tan^{-1}(1/s)$		half-sync

F.1 Methods for automating the calculation of residues

In this appendix we shall set up the general problem of finding K_k given Eq. 1.50 (Gustavsen and Semlyen, 1999).

$$Z(s) = \frac{N(s)}{D(s)} = sL_o + R_o + \sum_{k=0}^{K} \frac{K_k}{s - s_k}, \tag{F.1}$$

given the roots s_k of polynomial $D(s) = \Pi_{k=1}^{K}(s - s_k) = 0$.

Needs significant work.

1. First discuss the general properties of $Z(s) = K_{-1}s + K_0 + \sum_{k=k_1}^{K} \frac{K_k}{s-s_k}$.

Exercise: The impedance may be written as

$$Z(s) = \frac{N(s)}{D(s)} = \frac{\sum_m n_m s^m}{\sum_k d_k s^k}$$

Thus the companion matrix for the numerator polynomial $N(s)$ is

$$Z_N = \begin{bmatrix} -d_{N-1} & d_{N-2} & \cdots & \cdots & -d_0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \dots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{F.2}$$

thus numerator polynimial of $Z(s)$ is $N(s) = |Z_N - sI_N|$.

The companion matrix for the denomonator polynomial $D(s)$ is

$$Y_N = \begin{bmatrix} -n_{N-1} & n_{N-2} & \cdots & \cdots & -n_0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \dots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{F.3}$$

while the denomonator polynomial of $Z(s)$ is $D(s) = |Y_N - sI_N|$.

2. Note that these equations need to include the determination of unknowns L_o and R_o, which in some cases will be zero.

3. Next transform the residue epansion into its *companion matrix* (Sect. 1.3.2, Eq. 1.36 p. 65). By definition, the eigen-values of the companion matrix are the same as the roots of the impedance matrix.

4. General method to substitute $s = s_k$ in $Z(s)$, to define a non-degenerate linear system of equations in K_k, having nonzero determinant ($\Delta_K \neq 0$) (Gustavsen and Semlyen, 1999). The method is recursive if the roots are not accurately known. It might be useful to review Newton's method along with the Vandermonde determinant, to see if this might be further optimized.

5. This method has close ties to the classic CFA, where it has been called the Cauer decomposition, named after its inventor Wilhelm Cauer (Cauer and Mathis, 1995; Cauer *et al.*, 2000; Cauer, 1958; Cauer *et al.*, 1958), who acted as the primary thesis advisor for Brune (Brune, 1931b; Van Valkenburg, 1964b).

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