1 Colorized complex mapping

The mapping from \( z = x + iy \) to \( w = u + iv \) is a 2x2 dimensional graph, which is difficult to visualize. One way to visualize it is to use color to represent the phase and hue (dark to light) to represent the magnitude. The Matlab program `zviz.m` (see lecture 20 on the class website) has been provided to do this (http://bear.beckman.illinois.edu/Public/zviz.m).

To use the program in Matlab, use the syntax `zviz <function of z>` (e.g. `zviz z^2`, for example). This should plot a colorized version of the function (e.g. `zviz z^2` shows \( w = z^2 \)). A list of functions you can plot with `zviz` is given in the comments at the top of the `zviz.m` program. You can view the colormap (how color and hue related to magnitude and phase) by running `zviz z`.

Problem: For the following functions, explain what you see (and understand) about the `zviz` plot of each function. For example, `sin(\pi(z-i))` is plotted in Figure 1 using the command `zviz sin(pi*(z-1i))`. The abscissa (x axis) is the horizontal axis and the ordinate (y axis) is the complex \( iy \) axis. The zeros of \( \sin \) are at multiples of \( \pi \) and these are the dark regions. The graph is offset along the \( y \) axis by \( i \) since it is \( z - i \) that is being displayed. Along the \( x \) (real) axis the function oscillates between blue (negative) and red (positive). Along the vertical axis the function is either a \( \cosh \) or \( \sinh \).

```
1. z (How does the color relate to the phase of z? How does the hue related to the magnitude of z?)
2. z^2
3. e^z
4. cos(\pi z)
5. cosh(\pi z)
```
2 Fundamental theorem of algebra (FTA) vs. Bézout’s theorem

1. State the fundamental theorem of algebra (FTA).

2. State Bézout’s theorem.

3. What is the relationship between these two theorems?

3 2-port network analysis

\[
\begin{align*}
Z_1 & \quad I_1 \\
C & \quad I_2 \\
Z_3 & \quad V_2
\end{align*}
\]

\[
\begin{align*}
L & = 1 \\
C & = 2 \\
V_1 & \quad I_1 \\
V_2 & \quad I_2
\end{align*}
\]

Figure 2: Left (a lowpass RC filter): The resistor values are in kilo-ohms, and the capacitor value is in micro-farads. Let \( Z_1 = Z_3 = 10 \ [k\Omega] \) and \( C = 0.01 \ [\mu F] \) (i.e., \( R = 10^4 \ [\Omega] \) and \( C = 10^{-8} \ [Fd] \)). Right: Here the circuit elements are labeled with their values, with whole integers to make the algebra simple.

Perform a simple analysis of electrical 2-port networks, shown in Figure 2. The definition of the ABCD (transmission matrix) is

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]  

(1)

The impedance matrix, where \( \Delta_T = AB - CD \), is given by

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{C}
\begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]  

(2)

1. Derive the formula for the impedance matrix (Eq. 2) given the transmission matrix definition (Eq. 1). Show your work.

2. Find the ABCD matrix for each of the circuits of Figure 2. For each circuit, show the chain of transmission matrices in terms of the complex frequency \( s = \sigma + j\omega \). Then, you may substitute \( s = 1j \) to calculate the total transmission matrix at a single frequency (otherwise the matrix multiplication gets very complicated, particularly for the second circuit).

3. Convert both transmission (ABCD) matrices to impedance matrices using Equation 2. Do this for the specific frequency \( s = 1j \), as in the previous part.

4 M"obius transforms and infinity

The bilinear transform: The bilinear \( z \) transform (aka Möbius transformation) is used in signal processing to design digital filters from analog filters. The goal of the transform is to take a function of analog frequency \( \omega_a \), where \( \omega_a \in (-\infty, \infty) \), and map it to a finite digital frequency range, \( \omega_d \in [-\pi, \pi] \). You will learn more about this if you take ECE 310.
The bilinear $z$ transform is expressed in terms of the complex Laplace frequency $s \equiv \sigma + j\omega$, where $\omega$ is the analog frequency in radians/second, and $z \equiv \rho e^{j\theta}$, where $\theta$ is the digital frequency in radians. 

The transform is given by

$$s = \alpha \frac{1 - z^{-1}}{1 + z^{-1}},$$

where $\alpha$ is a real constant. This is a specific example of a Möbius transform.

1. Substitute $s = j\omega_a$ and $z = e^{j\omega_d}$ into the above equation to determine the relationship between $\omega_a$ and $\omega_d$. Express your final result using a tangent function. 

   **Hint:** Try to form sine and cosine terms! Recall that $\sin(\omega) = (e^{j\omega} - e^{-j\omega})/2j$ and $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$.

2. By hand, draw a graph of the relationship you found the previous part, $\omega_a = f(\omega_d)$. Make sure to specify the behavior of $\omega_a$ at $\omega_d = 0, \pm \pi/2, \pm \pi$.

3. Explain how this relationship handles $\omega_a \to \pm \infty$.

4. Now consider the complex frequency planes, $s = \sigma_a + j\omega_a$ and $z = e^{\sigma_d + j\omega_d}$. To map $\omega_a = f(\omega_d)$, we set $\sigma_a, \sigma_d = 0$. Draw the $s$ and $z$ planes, showing the real parts on the horizontal axes and the imaginary parts on the vertical axes. Show (e.g. using thick lines) which sets of values are considered when $\sigma_a, \sigma_d = 0$.

5. Geometrically, what is the effect of this möbius transform? Consider your drawing in the previous part.

## 5 Fourier and Laplace Transforms

In this problem we briefly review Fourier and Laplace transforms along with key similarities and differences.

**Basic definitions:** The Dirac delta function $\delta(t)$ is not a true function, rather it is defined under an integral sign as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt.$$ 

From the fundamental theorem of calculus it follows that

$$\delta(t) \equiv \frac{d}{dt} u(t).$$

It is used to evaluate integrals where its argument is zero, as in this example:

$$3 \sin(\pi/2) = \int_{t=0}^{\pi} 3 \sin(t) \delta\left(t - \frac{\pi}{2}\right) dt.$$ 

The Heaviside step function is defined as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt = \begin{cases} 1 & \text{if } t > 0 \\ \text{Not Defined} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The Fourier step function is quite different:

$$\tilde{u}(t) \equiv \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & \text{if } t > 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$
5.1 Fourier transforms (FT)

To take the FT of a step function it must be defined in the following very special way, using linearity (FTs obey superposition)

\[ \tilde{u}(t) = \frac{1 + \text{sgn}(t)}{2}. \]

This definition allows one to define a step function because the FT of 1 and of sgn\((t)\) are both defined for all values of \(t\).

**Problem:** Use the FTs of 1 and sgn\((t)\) to find the FT of \(\tilde{u}(t)\).

5.2 Laplace transforms (LT)

Fill out the table of some basic Fourier and Laplace transforms. Here we define \(f(t) \leftrightarrow F(\omega)\) as a Fourier transform and \(f(t)u(t) \leftrightarrow F(s)\) as a Laplace transform, where \(s = \sigma + j\omega\) is the Laplace radian frequency. If the transform does not exist, write ‘DNE.’

<table>
<thead>
<tr>
<th>(f(t))</th>
<th>(\leftrightarrow F(\omega))</th>
<th>(F(s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta(t - T_0))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
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<tr>
<td>sgn((t))</td>
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</tr>
<tr>
<td>(\tilde{u}(t))</td>
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<tr>
<td>(u(t))</td>
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<tr>
<td>(tu(t) = u(t) \ast u(t))</td>
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<tr>
<td>(e^{-at}u(t))</td>
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