Integration of Analytic (and non–Analytic) functions

There are three basic definitions related to Cathy’s integral formula. They are very similar, and can be thought of as various forms of the same concept.

1. **Cauchy’s Theorem** (Stillwell, p. 319; Boas, p. 45)

   \[ \oint_{C} f(z) \, dz = 0, \]

   if, and only if, \( f(z) \) is complex-analytic inside of \( C \).

   An alternative version of this formula is

   \[ \int_{a}^{b} f(z) \, dz = F(a) - F(b) \]

   where \( F(z) \) is the anti-derivative of \( f(z) \), namely \( f(z) = dF/dz \). I recommend you look up Stokes’ Theorem and the related Kelvin–Stokes theorem (also known as the curl theorem). If you take ECE 329, you will need this more general version of Cauchy’s Integral Theorem.

2. **Cauchy’s (integral) formula**

   \[ \frac{1}{2\pi j} \oint_{C} \frac{f(z) \, dz}{z - z_{0}} = \begin{cases} f(z_{0}), & z_{0} \in C \text{ (inside)} \\ 0, & z_{0} \notin C \text{ (outside)} \end{cases} \]

   Here \( f(z) \) is analytic everywhere within (and on) the contour \( C \). This means that \( f(z) \) may be determined at any and every point \( z_{0} \) inside \( C \), given values on \( C \). Likewise if there is a pole at \( z_{0} \), the integral (sum) of all the values around the pole is \( f(z_{0}) \).

   This formula has important generalizations to 3 dimensions.

3. **Cauchy’s Residue Theorem** (Boas, p. 51)

   \[ \oint_{C} f(z) \, dz = 2\pi j \sum_{k=1}^{K} \text{Res}_{k} \]

   Where \( \text{Res}_{k} \) are the residues of all poles of \( f(z) \) enclosed by \( C \). If \( f(z) \) is analytic at \( z_{0} \) then the residue of \( f(z)/(z - z_{0}) \) is \( f(z_{0}) \).
1.1 Problems

State where the function is and is not analytic. Integrate \( w = f(z) = u(x, y) + iv(x, y) \) over the curve \( C \in z = x + iy \), as given.

\[
\oint_C f(z) \, dz
\]

for each \( f(z) \), where \( C \) is the unit circle, defined as \( z = e^{i\theta}, \, 0 \leq \theta \leq 2\pi \).

1. \( f(z) = z \)
2. \( f(z) = \sin(z) \)
3. \( f(z) = 1/z \)
4. \( f(z) = 1/(2 - z) \)
5. \( f(z) = 1/\sqrt{z} \)
6. \( f(z) = 1/\sqrt{z} \) twice around the unit circle \((0 \leq \theta \leq 4\pi)\). This function has a branch cut, can you apply the Cauchy theorem?
7. \( f(z) = 1/z^2 \) on the unit circle.
2 Inverse Laplace Transforms

This problem investigates the inverse Laplace transform \( f(t) = \mathcal{L}^{-1}F(s) \), defined as

\[
f(t) = \int_{-\infty}^{\sigma_0+j\infty} F(s)e^{st} \frac{ds}{2\pi j}.
\]

Here constant \( \sigma_0 \) determines the abcissa of the integration path. The path must be to the right of all the poles to assure that \( f(t) = 0 \) for \( t < 0 \). We may take \( \sigma_0 = 0 \) if all the poles are strictly in the left-hand plane. The parameter \( \sigma_0 \) must be set such that all the poles are left (west) of \( \sigma_0 \). If a pole lies on the \( j\omega \) axis (e.g., \( 1/s \) has a pole at \( s = 0 \)) then \( \sigma_0 > 0 \).

To guarantee that \( f(t) = 0 \) for \( t < 0 \) we must apply the Cauchy formula. Thus we must close the curve as \( |\omega| \to \infty \). For negative time we close the curve on the right, so that no poles are included within the path of integration. For positive time, we must close the curve on the left, thus including all the poles within the path of integration. We call the path of integration (i.e., \( s = \sigma_0 + j\omega \) from \( -\infty < \omega < \infty \)) the “principal path” of this closed curve. To close the curve we must connect the two end points at \( \pm\infty \) with a semicircular contour from \( \sigma_0 - j\infty \) to \( \sigma_0 + j\infty \). This portion of the integral at \( \infty \) must be zero for the formula to give the desired \( f(t) \). Thus the inverse Laplace transform is intimately related to Cauchy’s Residue Theorem.

If one knows the forward Laplace Transform (FLT) of a causal function then they immediately know the Inverse Laplace Transform (ILT), since they are a matched pair. A few basic Laplace transform pairs are

\[
\begin{align*}
\delta(t) &\leftrightarrow 1 \tag{1} \\
\delta(t - T_0) &\leftrightarrow e^{-sT_0} \tag{2} \\
u(t) &\leftrightarrow \frac{1}{s} \tag{3} \\
t u(t) &\leftrightarrow \frac{1}{s^2} \tag{4} \\
e^{at} u(t) &\leftrightarrow \frac{1}{s-a} \tag{5} \\
(e^{at} \pm e^{bt}) u(t) &\leftrightarrow \frac{1}{s-a} \pm \frac{1}{s-b} = \frac{(s-a) \mp (s-b)}{(s-a)(s-b)} \tag{6} \\
\sin(\omega_0 t) u(t) &\leftrightarrow \frac{b}{s^2 + \omega_0^2} \tag{7} \\
\frac{1}{\sqrt{\pi t}} &\leftrightarrow \frac{1}{\sqrt{s}} \tag{8} \\
\frac{1}{\sqrt{\pi t}} &\leftrightarrow \frac{1}{\sqrt{s}} \tag{9}
\end{align*}
\]

2.1 Problems

1. Find the LT of \( f(t) = \delta(t - T_0) \).

2. Find the \( \mathcal{L}^{-\infty} \) of \( F(s) = 1/s \). Since the pole is at \( s = 0 \), we must take \( \sigma_0 > 0 \). As long as the path is to the right of the pole, its exact value is not important (due to Cauchy’s Theorem).
3 Waves in a tube

Assume an acoustic transmission line (TL) of length $L$ that delays the input pressure signal $p(x, t)$, at $x = 0$, by $T = L/c$, where $c$ is the speed of sound. The TL obeys the wave equation

$$\nabla^2 p(x, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(x, t)$$

In 1747 d’Alembert published the general solution to the wave equation\(^1\)

$$p(x, t) = f(x - ct) + g(x + ct),$$

where $f(x)$ and $g(x)$ are arbitrary, differentiable, functions.

The formula for the LT of the input impedance of the tube at any point along the TL at location $x$ is the ratio of the pressure $P(x, s)$ over the volume velocity $U(x, s)$\(^2\)

$$Z(x, s) \equiv \frac{P(L, s)}{U(L, s)}.$$ 

In keeping with d’Alembert’s solution to the wave equation, the pressure $P(x, s)$ (a scaler potential) may be written as the sum of a forward $P^+$ and backward wave $P^-$,

$$P(x, s) = P^+(x, s) + P^-(x, s).$$

The volume velocity $U(x, s)$ (a vector flow) is the difference (taking direction into account) of the two flows

$$U(x, s) = U^+(x, s) - U^-(x, s).$$

One may keep track of the ratio of the two at every point in terms of the reflectance, defined as

$$\Gamma(x, s) \equiv \frac{P^-(x, s)}{P^+(x, s)} = \frac{U^-(x, s)}{U^+(x, s)}.$$ 

This decomposition leads to the following formula for the impedance:

$$Z(x, s) = \frac{P^+(x, s) + P^-(x, s)}{U^+(x, s) - U^-(x, s)} = r_0 \frac{1 + \Gamma(L, s) e^{-2\kappa(s)(L-x)}}{1 - \Gamma(L, s) e^{-2\kappa(s)(L-x)}}, \quad (10)$$

where $r_0 \equiv P^+/U^+$ is the transmission line’s characteristic resistance; $\Gamma(L, s)$ is the reflection coefficient at the output load location (defined as $x = L$) (defined as the reflectance evaluated at the boundry $x = L$); $\kappa(s)$ is the complex analytic propagation function, and $x$ is where the measurement is made. (The input was defined as $x = 0$.)

In the following exercise:

1. Take the characteristic resistance ($r_0 = 1$) and the reflection coefficient at $\Gamma(x, s) = 1$ at $x = L$.

2. When the TL is loss-less the propagation function $\kappa(s) = j\omega/c$ has a zero real part.

3. At time $t = 0$ and $x = 0$, assume that the input pressure is an ideal impulse $p(x, 0) = \delta(t)$.

4. The tube is blocked at the end ($x = L$) (the velocity $U(L, s) \propto \partial p/\partial x = 0$).

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\(^1\)https://books.google.com/books?id=lJQDAAAAMAAJ&pg=PA214
\(^2\)By Newton’s second law, $U(x, s) \propto -\nabla P(x, s)$ (https://en.wikipedia.org/wiki/Transmission_line.)
3.1 Problems

1. (a) Given d’Alembert’s result, show that the pressure, for \(0 < t < L/c\) and \(0 < x < L\), is \(p(x, t) = \delta(t - x/c)\).

(b) Find the Laplace transform (LT) (i.e., frequency response) of the input pressure \(P(x, s)\) at \(x = 0\) and \(x = L\).

(c) Once the pressure pulse reaches the end of the tube, it reflects off a blocked end, reversing the direction (but not the sign) of the pressure pulse. Determine the pressure \(p(x, t)\) between \(0 < x < L\), at time \(t = 3L/2c\)?

(d) Use the LT to express the total pressure at the input \(x = 0\) for \(0 \leq t \leq 2L/c\).

2. Derivation of the formula for the input impedance of a loss-less TL:

(a) Show that Eq. 10 may be reduced to

\[
Z(0, s) = \frac{e^{\kappa(s)L} + e^{-\kappa(s)L}}{e^{\kappa(s)L} - e^{-\kappa(s)L}} = r_0 \tanh(\kappa L).
\]  
(11)

(b) Show that the propagation function is given by the log of a Möbius transformation of the impedance:

\[
\kappa(s) = \frac{1}{2L} \ln \left( \frac{r_0 + Z(0, s)}{r_0 - Z(0, s)} \right),
\]

Hint: i) Start from Eq. 11, solve for \(e^{\kappa 2L}\). ii) Take the log of both sides.

(c) Discuss the physical significance of this relationship.
4 Riemann Sheets and Branch cuts

In a previous homework we looked at `viz.m` for polynomials $F(s)$. Now we shall study the neighborhood of the poles of $F(s)$.

When describing multi-valued functions, be sure to address the role of *branch cuts*.

1. Use `viz.m f(s)` to describe the response of the following functions

   (a) $1/s$
   (b) $1/s^2$
   (c) $f(s) = 1/\sqrt{s}$.
   (d) $1/\sqrt{1-s^2}$.
   (e) $f(s) = \text{atan}(js)$.
   (f) $s/(s+1)$

2. Describe the branch cut for $f(z) = e^z$. 