Topic of this homework: Fundamental Theorem of Complex Calculus; Inverse Laplace transforms; ROC; Integration of analytic functions; Cauchy integral formula;

1 Fundamental Theorem of Complex Calculus

To define the inverse Laplace Transform we will need to integrate $F(s)$ in the $s$ plane. Thus we must learn how to define integration and differentiation with respect to complex frequency $s$. This is best done using the Fundamental Theorem of Calculus.

According to the Fundamental Theorem of Real Calculus

$$F(x) = F(a) + \int_a^x f(x')dx'$$

This is known as an indefinite integral (since the upper limit is unspecified). It follows that

$$\frac{dF(x)}{dx} = \frac{d}{dx} \int_a^x f(x')dx' = f(x).$$

This justifies also calling the indefinite integral an anti-derivative.

Complex integration and differentiation When one replaces $x \in \mathbb{R}$ with the complex variable $s \in \mathbb{C}$, a bit of magic happens, due to the way that integration and differentiation are defined. Here we let $s = \sigma + j\omega$ be the complex frequency and $Z(s) = R(\sigma, \omega) + jX(\sigma, \omega)$ be an impedance (i.e., a complex function of a complex variable), the sum of a resistance $R(s) \equiv \Re Z(s)$ and reactance $X(s) \equiv \Im Z(s)$.

Integration of the impedance in the complex plane gives

$$H(s) = H(a) + \int_a^s Z(s')ds' = \int_a^s \left[R(\sigma', \omega') + jX(\sigma', \omega')\right] d(\sigma' + j\omega') \quad (1)$$

Taking the derivative with respect to $s$ gives

$$\frac{dH(s)}{ds} = \frac{d}{ds} \int_a^s Z(s')ds' = Z(s).$$

As for the real case, the derivative of an indefinite integral returns the integrand, justifying that the indefinite may be recognized as the anti-derivative.

It immediately follows that the integration is independent of the integration path (since the integral only depends on the end points), and, as discussed in class, the Cauchy-Riemann (CR) follow from this independence of the path of integration.1

1If one requires that the integral be an anti-derivative, then it follows that the integral must be independent of the path. The CR conditions and thus $\nabla^2 R(\sigma, \omega) = 0$ and $\nabla^2 X(\sigma, \omega) = 0$ follow.
1.1 Problems

Find \( I(s) = \int_0^s F(z)dz \) where \( F(s) = e^s \) for each value of \( c \) given below. The integration path \( C \) to be from 0 to \( s \) in the \( s \in \mathbb{C} \) plane, as specified in the problem below.

1. \( c = 1/e = 1/2.7183\ldots \) with \( C = [0, i, s] \)

2. \( c = 2 \ldots \) with \( C = [0, 1 + i, s] \)

3. \( c = i \) with \( C \) is an inward spiral \( 0.99^t e^{i2\pi t} \), with \( t \) from 0 to \( \infty \).

4. \( c = it \) with \( C \) from \( 1 - i\infty \) to \( s \to 1 + i\infty \). Namely
\[
I(st) = \int_{1-i\infty}^{1+i\infty} itdz.
\]

2 Integration of Analytic functions

There are three basic definitions related to Cauchy’s integral formula. They are very similar, and can be thought of as various forms of a similar, if not the same concept.

1. Cauchy’s Theorem (Stillwell, p. 319; Boas, p. 45)
\[
\oint_C f(z)dz = 0,
\]
if, and only if, \( f(z) \) is complex-analytic inside of \( C \).

An related version of this formula is
\[
\int_a^b f(z)dz = F(b) - F(a)
\]
where \( F(z) \) is the anti-derivative of \( f(z) \), namely \( f(z) = dF/dz \).

2. Cauchy’s integral formula (Boas, p. 51; Stillwell, p. 220)
\[
\frac{1}{2\pi j} \oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0), & z_0 \in C \text{ (inside)} \\ 0, & z_0 \not\in C \text{ (outside)} \end{cases}
\]
Here \( f(z) \) is analytic everywhere within (and on) the contour \( C \). \( f(z_0) \) is called the residue. This means that \( f(z) \) may be determined at any and every point \( z_0 \) inside \( C \), given values on \( C \). Likewise if there is a pole at \( z_0 \), the integral (sum) of all the values around the pole is \( f(z_0) \).

This formula has important generalizations to 3 dimensions.

3. The Residue Theorem (Boas, p. 72)
\[
\oint_C f(z)dz = 2\pi j \sum_{k=1}^K \text{Res}_k
\]
Where \( \text{Res}_k \) are the residues of all poles of \( f(z) \) enclosed by \( C \). If \( f(z) \) is analytic at \( z_0 \) then the residue of \( f(z)/(z-z_0) \) is \( f(z_0) \).
2.1 Problems

1. In one or two brief sentences, describe the relation between the three theorems:
   (a) (1) and (2)
   (b) (1) and (3)
   (c) (2) and (3)

3 Inverse Laplace transforms

This homework will explore the inverse Laplace transform (ILT, also denoted $\mathcal{L}^{-1}$)

$$f(t) = \oint_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} \frac{ds}{2\pi j},$$

where $s = \sigma + j\omega$ [radians] and $t$ is time [seconds]. To define the ILT we need to understand integration in the complex plane.

The domain of functions that are subject to a Laplace transform are causal functions, namely those functions that are zero for negative time. The formulae for the Laplace transform is

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt.$$ 

In class I have referred to such causal functions as system functions, namely those that describe systems (vs. signals which are described by Fourier Transforms).

**Notation:** Many loosely adhere to the convention that the frequency domain uses upper-case [e.g. $F(s)$] while the time domain uses lower case [e.g. $f(t)$]. LT pairs are commonly expressed as $f(t) \leftrightarrow F(s)$.

As long as the domain (time vs. frequency) is clear from context, order may be reversed, if desired. While radians are useful units for calculations, when providing physical insight in discussions of problem solutions, it is easier to work with Hertz, since frequency in [Hz] and time in [s] are mentally more more natural units than radians. The same is true of degrees vs radians. Boas (p. 10) recommends the use of degrees over radians. He gives the example of $3\pi/5$ [radians], which is more easily visualize as $108^\circ$.

2 Linear, time-invariant systems are described by an ordinary differential equation. For example consider the first-order linear differential equation

$$a_1 \frac{dy}{dt} + b_1 \frac{dx}{dt} + b_0 x(t).$$

Define Laplace transforms $y(t) \leftrightarrow Y(s)$ and $x(t) \leftrightarrow X(s)$, then this equation may be written in the frequency domain as

$$a_1 sY(s) + b_1 sX(s) + b_0 X(s).$$

The transfer function is defined as $H(s) = \frac{Y(s)}{X(s)}$ which for this example ($I_2 = 0$) is $b_1/a_1 + b_0/a_1 s$.

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\footnote{Of course this is a matter of personal preference, just like SI (now the standard), vs cgs or English units. I recommend you stick to SI units so your not spending your time converting form one system to another.}
Figure 1: This three element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$.

Examples of analytic functions Examples of electrical impedance are an inductor $sL$, a capacitor $1/sC$ and a resistor $R$. Figure 1 is an example of a simple RC low-pass filter, which acts as a leaky integrator having a transfer function $H(s)$ given by

$$H(s) = \frac{V_1}{V_2} = \frac{1}{1 + R_1Cs} = \sum_{k=0}^{\infty} (-R_1Cs)^k.$$ 

For example the impedance of a series circuit of an inductor, capacitor and resistor is given by

$$Z(s) = sL + R + 1/sC = \frac{LCs^2 + RCs + 1}{sC}.$$ 

In mechanical circuits the impedance may be a mass is $sM$, a spring $K/s$ and a damper $R$. Other than the physics, a mass-spring-damper would have an identical formula. This impedance has a pole at $s = 0$ and a pair of zeros at the roots of the numerator polynomial.

Problems:

1. Use the ABCD method to find the matrix representation of Fig. 1.
2. Assuming that $I_2 = 0$ find $H(s) \equiv V_2/V_1$. From the results of the ABCD matrix you determined above, show that

$$H(s) = \frac{1}{1 + R_1Cs}.$$ 

3. Find the Residue of $H(s)$.
4. Find the inverse Laplace transform of $H(s) \leftrightarrow h(t)$.
5. Assuming that $V_2 = 0$ find $Y_{12}(s) \equiv I_2/V_1$.
6. Find the input impedance $Z_{22} \equiv V_2/I_2$.
7. Compute the determinant of the ABCD matrix? (Hint, it is always 1)
8. Compute the derivative of $H(s) = V_2/V_1|_{I_2=0}$, given by problem 2 above.
Recognizing and identifying complex analytic functions

It is important to be able identify complex analytic functions from their colorized plots. This helps to develop an intuition for complex analytic functions.

Many classic complex analytic functions are periodic (or multi-valued) along one axis and monotonic along the other. A classic example is \( \sinh(iz) = i \sin(z) \), where \( z = x + iy \) (prove this using \texttt{zviz}).

Letting \( x = 0, \ iz = -y \in \mathbb{R} \)

\[
\sinh(-y) \equiv \frac{e^{-y} - e^y}{2} = i \sin(iy),
\]

which is monotonic in \( y \).

Letting \( y = 0, \ iz = ix \in \mathbb{R} \)

\[
\sinh(ix) \equiv \frac{e^{ix} - e^{-ix}}{2} = i \sin(x),
\]

which is periodic in \( x \). Verify this with \texttt{zviz} for both \( \sinh(iz) \) and \( \cosh(iz) \).

The inverses of two equivalent functions [i.e., \( i \text{asinh}(z) \equiv \text{asin}(iz) \)] must also be equivalent, and therefore have identical symmetries. This is both a useful and important concept.

Valuedness and the inverse: Functions can be single-valued [\( \sin(t), e^x \)] or multi-valued [\( \pm \sqrt{x} \)], which impacts the inverse. The a multi-valued function has branch-cuts. A single-valued function does not. But what exactly is this relationship? The terms \textit{one-to-one} and \textit{onto} are used to generalize valuenance.\(^3\) To understand the function \( w(z) = f(z) \) you must understand the complex relationship to its inverse \( z(w) = f^{-1}(w) \). I’m guessing that Riemann was the first to fully appreciate these symmetries.

Problems:

In the following, describe the relationship between periodic functions and the branch-cut of its inverse function. When describing multi-valued functions, be sure to address the role of any branch cuts.

1. Use \texttt{zviz.m} \( f(s) \) to describe the response of the following functions:

   (a) \( F(s) = \sin(\pi s/2) \). Where is the function periodic and where is it monotonic (or constant)?

   \[
f(s) = \sin(\pi s/2)
   \]

   (b) \( F(s) = \cosh(\pi s) \). Where is the function periodic and where is it monotonic (constant)?

\(^3\)Even more general (and difficult) are the terms \textit{surjective} (onto), \textit{injective} (1:1) and \textit{bijective} (1:1 & onto).
2. Inverse functions:

(a) Based on the following colorized plot of \( F(s) = e^s \), where \( s = \sigma + j\omega \), explain where \( F(s) \) is single valued or periodic.

(b) Given what you said about \( e^s \), what can you conclude about its inverse \( F(s) = \log(s) \)?

3. Discuss the output from \( \texttt{zviz \ -log((i-z)/(i+z))*i/2 \ atan(z)} \). Look at page 314 of Stillwell for help. Then relate your answer to the following plots:
\[ f(s) = \tan(z) \]

\[ f(s) = \arctan(z) \]