An Invitation to Mathematical Physics

and its History

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Abstract

An understanding of physics requires knowledge of mathematics. The contrary is not true. By definition, pure mathematics contains no physics. Yet historically, mathematics has a rich history filled with physical applications. Mathematics was developed by people with intent of making things work. In my view, as an engineer, I see these creators of early mathematics, as budding engineers. This book is an attempt to tell this story, of the development of mathematical physics, as viewed by an engineer.

The book is broken down into three topics, called streams, presented as five chapters: 1) Introduction, 2) Number systems, 3) Algebra Equations, 4) Scalar Calculus, and 5) Vector Calculus. The material is delivered as 40 “Lectures” spread out over a semester of 15 weeks, three lectures per week, with a 3 lecture time-out for administrative duties. Problems are provided for each week’s assignment. These problems are written out in LATEX, with built in solutions, that may be expressed by un-commenting one line. Once the home-works are turned in, each student is given the solution. The students rated these Assignments as the most important part of the course. There is a built in interplay between these assignments and the lectures. On many occasions I solved the homework in class, as motivation for coming to class.

There were four exams, one at the end of each of the three sections, and a final. One of the exams was in class, three others and the final were evening exams. Each of the exams and assignments are LATEX files, with solutions encoded with a one line software switch. When the exam was returned by the student, the full solution was provided, while the exam was fresh in the students mind, providing a teaching moment. The Exams are largely based on the Assignments. It is my philosophy that, in principle, the students could see the exam in advance of taking it. In a sense they do, since each exam was based on the homework.

Author’s Personal Statement

An expert is someone who has made all possible mistakes in a small field. I don’t know if I would be called and expert, but I certainly have made my share of mistakes. I openly state that “I love making mistakes, because I learn so much from them.” One might call that the “expert’s corollary.”

This book has been written out of both my love for the topic of mathematical physics, and a frustration for wanting to share many key concepts, and even new ideas on these basic concepts. Over the years I have developed a certain physical sense of math, along with a related mathematical sense of physics. While doing my research,¹ I have come across what I feel are certain conceptual holes that need filling, and sense many deep relationships between math and physics, that remain unidentified. While what we presently teach is not wrong, it is missing these relationships. What is lacking is an intuition for how math “works.” A good scientist “listens” to their data. In the same way we need to start listening to the language of mathematics. We need to let mathematics guide us toward our engineering goals.

It is my suspicion that over the centuries many others have had similar insights, and like me, have been unable to convey this slight misdirection. I hope these views can be useful to open young minds.

As summarized in Fig. 1, this marriage of math, physics and engineering will help us make progress in understanding the physical world. We must turn to mathematics and physics when trying to understand the universe. I have arrived in my views following a lifelong attempt to understand human

¹https://auditorymodels.org/index.php/Main/Publications
communication, i.e., the perception and decoding of human speech sounds. This research arose from my 32 years at Bell Labs in the Acoustics Research Department. There such lifelong pursuits were not only possible, they were openly encouraged. The idea was that if you were successful at something, take it as far as you can. But on the other side, don’t do something well, that’s not worth doing. People got fired for the latter. I should have left for University after a mere 20 years, but the job was just too cushy.

In this text it is my goal to clarify conceptual errors while telling the story of physics and mathematics. My views have been inspired by classic works, as documented in the bibliography. This book was inspired by my reading of Stillwell (2002), through Chapter 21 (Fig. 2) (p. 10). Somewhere in Chapter 22 I switched to the third edition (Stillwell, 2010), at which point I realized I had much more to master. It became clear that by teaching this material to first year engineers I could absorb the advanced material at a reasonable pace. This book soon followed.

**Back Cover Summary**

This is foremost a math book, but not the typical math book. First, this book is for the engineering minded, for those who need to understand math to do engineering, to learn how things work. In that sense it is more about physics and engineering. Math skills are essential for to making progress in building things, be it pyramids or computers, as clearly shown by the many great civilizations of the Chinese, Egyptians, Arabs (people of Mesopotamia), and Greeks and Romans.

Second, this is a book about the math that developed to explain physics, to allow people to engineer complex things. To sail around the world one needs to know how to navigate. This requires a model of the planets and stars. You can only know where you are on earth once you understand where earth is relative to the heavens. The answer to such a deep questions will depend on who you ask. The utility and accuracy of that answer depends critically on the depth of understanding of how the worlds

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I started around December 1970, fresh out of Graduate school, and retired in December 2002.
and heavens work. Who is qualified to answer such question? It is best answered by those who study mathematics applied to the physical world.

The English astronomer Edmond Halley (1656–1742) asked Newton (1643–1727) for the equation that describes the orbit of the planets. Halley was obviously interested in comets. Newton immediately answered “an ellipse.” It is said that Halley was stunned by the response (Stillwell, 2010, p. 176), as this was what had been experimentally observed by Kepler (c1619), and thus he knew Newton must have some deeper insight.

When Halley asked Newton to explain how he knew, Newton responded “I calculated it.” But when challenged to show the calculation, Newton was unable to reproduce it. This open challenge eventually led to Newton’s grand treatise, Philosophiae Naturalis Principia Mathematica (July 5, 1687). It had a humble beginning, as a letter to Halley, explaining how to calculate the orbits of the planets. To do this Newton needed mathematics, a tool he had mastered. It is widely accepted that Isaac Newton and Gottfried Leibniz invented calculus. But the early record shows that perhaps Bhāskara II (1114–1185 AD) had mastered this art well before Newton.\(^3\)

Third, the main goal of this book is to teach motivated engineers mathematics, in a way that it can be understood, mastered and remembered. How can this impossible goal be achieved? The answer is to fill in the gaps with “who did what, and when.” An historical story may be easily mastered.

To be an expert in a field, one must know its history. This includes who the people were, what they did, and the credibility of their story. Do you believe the Pope or Galileo on the roles of the sun and the earth? The observables provided by science are clearly on Galileo’s side. Who were those first engineers? They are names we all know: Archimedes, Pythagoras, Leonardo da Vinci, Galileo, Newton, . . . . All of these individuals had mastered mathematics. This book teaches the tools taught to every engineer. Do not memorize complex formulas, rather make the equations “obvious” by learning the simplicity of the underlying concepts.

Credits

Besides thanking my parents, I would like to credit John Stillwell for his constructive, historical summary of mathematics. My close friend and colleague Steve Levinson somehow drew me into this project, without my even knowing it. My brilliant graduate student Sarah Robinson was constantly at my side, grading home-works and exams, and tutoring the students. Without her, I would not have survived the first semester the material was taught. Her proof-reading skills are amazing. Thank you Sarah for your infinite help. Finally I would like to thank John D’Angelo for putting up with my many silly questions. When it comes to the heavy hitting, John was always there to provide a brilliant explanation that I could easily translate into Engineer’ese (Matheering?) (i.e., Engineer language).

To write this book I had to master the language of mathematics (John’s language). I had already mastered the language of engineering, and a good part of physics. One of my secondary goals is to bring down this scientific Tower of Babble by unifying the terminology and removing the jargon.

Finally I would like to thank my wife (Sheau Feng Jeng aka Patricia Allen) for her unbelievable support and love. She delivered constant piece of mind, without which this project could never have been started, much less finish.

There are many others who played important roles, but they must remain anonymous, out of my fear of offending someone I forgot to mention.


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\(^3\)http://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch8_5.html
Mathematics and its History *(Stillwell, 2002)*

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*Figure 2: Table of contents of Stillwell (2002)*
Preface

It is widely acknowledged that interdisciplinary science is the backbone of modern scientific investigation. This is embodied in the STEM (Science, Technology, Engineering, and Mathematics) programs. Contemporary research is about connecting different areas of knowledge, thus it requires understanding of cross-disciplines. However, while STEM is being taught, interdisciplinary science is not, due to its inherent complexity and breadth. Furthermore there are few people to teach it. Mathematics, Engineering and Physics (MEP) are at the core of such studies.

**STEM vs. MEP**

Mathematics is about rigor. Mathematicians specifically attend to the definitions of increasingly general concepts. Thus mathematics advances slowly, as these complex definitions must be collectively agreed upon. Since it embraces rigor, mathematics shuns controversy, the opposite of uncertainty. Physics explores the fringes of uncertainty. Physicists love controversy. Engineering addresses the advancement the technology. Engineers, much like mathematicians, are uncomfortable with uncertainty, but are trained to deal with it.

To create such an interdisciplinary STEM program, a unified MEP curriculum is needed. In my view this unification could (should) take place based on a core mathematical training, from a historical perspective, starting with Euclid or before (i.e., Chinese mathematics), up to modern information theory and logic. As a bare minimum, the **fundamental theorems of mathematics** (arithmetic, algebra, calculus, vector calculus, etc.) need to be appreciated by every MEP student. The core of this curriculum is outlined in Table 1.1 (p. 21).

If first year students are taught a common MEP methodology and vocabulary, presented in terms of the history of mathematics, they will be equipped to

1. Exercise interdisciplinary science (STEM)
2. Communicate with other MEP trained (STEM) students and professors.

The goal is a comprehensive understanding of the fundamental concepts of mathematics, defined as those required for engineering. We assume that students with this deep understanding will end up being in the top 0.1% of Engineering. Time will tell if and when this assumption is proved to be correct.

The key tool is methodology. The traditional approach is a five to eight course sequence: Calc I, II, III, Linear Algebra IV, DiffEq V, Real analysis VI, Complex analysis VII and Number theory VIII, over a time frame of three or more years (six semesters). This was the way I learned math. Following this regime I felt I had not fully mastered the material, so I started over. I now consider myself to be self-taught. We need a more effective teaching method. I am not suggesting we replace the standard 6 semester math curriculum, rather I am suggesting replacing Calc I, II with a mathematical physics course, based on the historical thread, designed for those students who have demonstrated advanced ability. One needs more than a high school education to succeed in college engineering courses.

By teaching mathematics in the context of history, the student can fully appreciate the underlying principles. Including the mathematical history provides a uniform terminology for understanding the fundamentals of mathematics. The present teaching method, using abstract proofs, with no (or few)
figures or physical principles, by design, removes intuition and the motivation that was available to the creators of these early theories. This present six semester approach does not function for many students, leaving them with poor, or even no intuition.
Chapter 1

Introduction

Much of early mathematics centered around the love of art and music, due to our sensations of light and sound. Exploring our physiological senses required a scientific understanding of vision and hearing, as first explored by Newton (1687) and Helmholtz (1863a)\(^1\) (Stillwell, 2010, p. 261). Our sense of color and musical pitch are determined by the frequencies of light and sound. The Chinese and Pythagoreans are well known for their early contributions to music theory.

Pythagoras strongly believed that “all is integer,” meaning that every number, and every mathematical and physical concept, could be explained by integral relationships. It is likely that this belief was based on Chinese mathematics from thousands of years earlier. It is also known that his ideas about the importance of integers were based on what was known about music theory. For example it was known that the relationships between the musical notes (pitches) obey natural integral relationships.

Modern applications of number theory include:

- Public-private key encryption: which require computationally intensive prime factoring of large integers
- IEEE Floating point\(^2\)

As acknowledged by Stillwell (2010, p. 16), the Pythagorean view is relevant today.

*With the digital computer, digital audio, and digital video coding everything, at least approximately into sequences of whole numbers, we are closer than ever to a world in which “all is number.”*

Mersenne (1588-1647) contributed to our understanding of the relationship between the wavelength and the length of musical instruments. These results were extended by Galileo’s father, and then by Galileo himself (1564-1642). Many of these musical contributions resulted in new mathematics, such as the discovery of the wave equation by Newton (c1687), followed by its one-dimensional general solution by d’Alembert (c1747).

By the time of Newton there was a basic understanding that sound and light traveled at very different speeds (thus why not the velocities of different falling weights?).

Ole Römer first demonstrated in 1676 that light travels at a finite speed (as opposed to instantaneously) by studying the apparent motion of Jupiter’s moon Io. In 1865, James Clerk Maxwell proposed that light was an electromagnetic wave, and therefore traveled at the speed \(c\) appearing in his theory of electromagnetism.\(^3\)

In 1589 Galileo famously conceptualized an experiment where he suggested dropping two different weights from the Leaning Tower of Pisa, and showed that they must take the same time to hit the

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CHAPTER 1. INTRODUCTION

ground. Conceptually this is an important experiment, driven by a mathematical argument in which
he considered the two weights to be connected by an elastic cord. This resulted in the concept of
conservation of energy, one of the cornerstones of physical theory, since that time.

\[ \frac{c_p}{c_v} = \sqrt{1.4} = 1.183 \]

While Newton may be best known for his studies on light, he was the first to predict the speed of
sound. However his theory was in error\(^4\) of \(\sqrt{\frac{c_p}{c_v}} = \sqrt{1.4} = 1.183\). This famous error would not be
resolved for over two hundred years, awaiting the formulation of thermodynamics and the equi-partition
theorem, by Maxwell and Boltzmann, and others. What was needed was the concept of constant-heat,
or adiabatic process. For audio frequencies (0.02-20 [kHz]), the small temperature gradients cannot
diffuse the distance of a wavelength in one cycle (Pierce, 1981; Boyer and Merzbach, 2011), “trapping”
the heat energy in the wave. There were several other physical enigmas, such as the observation that
sound disappears in a vacuum and that a vacuum cannot draw water up a column by more than 34
feet.

There are other outstanding examples where physiology impacted mathematics. Leonardo da Vinci
is well known for his studies of the human body. Helmholtz’s theories of music and the percep-
tion of sound are excellent examples of under-appreciated fundamental mathematical contributions
(Helmholtz, 1863a). Lord Kelvin (aka William Thompson),\(^5\) was one of the first true engineer-scientists,
equally acknowledged as a mathematical physicist, well known for his interdisciplinary research, and
knighted by Queen Victoria in 1866. Lord Kelvin coined the term thermodynamics, a science more fully
developed by Maxwell (the same Maxwell of electrodynamics). Thus the interdisciplinary nature of
science has played many key roles in the development of thermodynamics.\(^6\) Lord Rayleigh’s (William
Strutt) book on the theory of sound (Rayleigh, 1896) is a classic text, read even today by those who
study acoustics.

It seems that we have somewhat retracted from this venerable interdisciplinary view of science, by
splitting the disciplines into into smaller parts, whenever we perceived a natural educational boundary.
Reforging these natural connections into the curriculum, is essential for the proper training of STEM
students (scientists, mathematicians and engineers).\(^7\)

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\(^4\)The square root of the ratio of the specific heat capacity at constant pressure \(c_p\) to that at constant volume \(c_v\).

\(^5\)Lord Kelvin was one of half a dozen interdisciplinary mathematical physicists, all working about the same time, that
made a fundamental change in our scientific understanding. Others include Helmholtz, Stokes, Green, Heaviside, Rayleigh
and Maxwell.

\(^6\)Thermodynamics is another example of a course that needs reworking along historical lines (Kuhn, 1978).

\(^7\)Perhaps its time to put the MEP Humpty Dumpty back together.
1.1 Early Science and Mathematics

The first 5,000 years is not well documented, but the basic record is clear, as outlined in Fig. 1.2. Thanks to Euclid and later Diophantus (c250 CE), we have some vague understanding Chinese thought. For example, Euclid’s formula (Eq. 1.7, p. 38; Sec. 2.2.1, Fig. 2.3, p. 134) provides a method for computing Pythagorean triplets, a formula known long before Euclid (Stillwell, 2010, pp. 4-9). Chinese Bells and stringed musical instruments were exquisitely developed in their tonal quality, as documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was so rich that one must question why the Chinese failed to initiate the industrial revolution. Specifically, why did Europe eventually dominate with its innovation when it was the Chinese who did the extensive early invention?

According to Lin (1995) this is known as the Needham question:

“Why did modern science, the mathematization of hypotheses about Nature, with all its implications for advanced technology, take its meteoric rise only in the West at the time of Galileo[, but] had not developed in Chinese civilization or Indian civilization?”

Needham cites the many developments in China:8

“Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon considered as the three most important inventions facilitating the West’s transformation from the Dark Ages to the modern world, were invented in China.” (Lin, 1995)

“Needham’s works attribute significant weight to the impact of Confucianism and Taoism on the pace of Chinese scientific discovery, and emphasizes what it describes as the ‘diffusionist’ approach of Chinese science as opposed to a perceived independent inventiveness in the western world. Needham held that the notion that the Chinese script had inhibited scientific thought was ‘grossly overrated’ ” (Grosswiler, 2004).

Lin was focused on military applications, missing the importance of non-military contributions. A large fraction of mathematics was developed to better understand the solar system, acoustics, musical instruments and the theory of sound and light. Eventually the universe became a popular topic, as it still is today.

1.1.1 Lec 1 The Pythagorean theorem

While early Asian mathematics is not fully documented, it clearly defined the course for at least several thousand years. The first recorded mathematics was that of the Chinese (5000-1200 BCE) and the Egyptians (3,300 BCE). Some of the best early record were left by the people of Mesopotamia (Iraq, 1800 BCE). Thanks to Euclid’s Elements (c323 BCE) we have an historical record, tracing the progress in geometry, as defined by the Pythagorean theorem for any right triangle

\[ c^2 = a^2 + b^2, \]  

having sides of lengths \((a, b, c)\) that are either positive real numbers, or more interesting, integers, such that \(c > |a, b|\) and \(a + b > c\). Integer solutions were likely found by trial and error rather than by Euclid’s formula.

If \(a, b, c\) are lengths, then \(a^2, b^2, c^2\) are each the area of a square. Equation 1.1 says that the square

---

area $a^2$ plus the square area $b^2$ equals the square area $c^2$. Today a simple way to prove this is to compute the magnitude of the complex number $c = a + bj$, which forces the right angle

$$|c|^2 = (a + bj)(a - bj) = a^2 + b^2.$$ 

However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers and algebra had yet to be invented.

Almost 700 years after Euclid’s Elements, the Library of Alexandria was destroyed by fire (391 CE), taking with it much of the accumulated Greek knowledge. Thus one of the best technical records may be Euclid’s Elements, along with some sparse mathematics due to Archimedes (c300 BCE) on geometrical series, computing the volume of a sphere, and the area of the parabola, and elementary hydrostatics. Additionally, a copy of a book by Diophantus Arithmetic was discovered by Bombelli (c1572) in the Vatican library (Stillwell, 2010, p. 51). This book became an inspirational resource for Fermat.

**Chronological history pre 16th century**

200th BCE Chinese (Primes; quadratic equation; Euclidean algorithm (GCD))
180th BCE Babylonian (Mesopotamia/Iraq) (quadratic equation)
6th BCE Pythagoras (Thales) and the Pythagorean “tribe”
4th BCE Euclid (quadratic equation) 300BCE; Archimedes

3rd BCE Diophantus c250CE;
4th CE Alexandria Library destroyed 391CE;
6th CE Brahmagupta (negative numbers; quadratic equation) 598-670CE
10th CE al-Khwârîmî (algebra) 830CE; Hasan Ibn al-Haytham (Alhazen) 965-1040CE
12th CE Bhaskara (calculus) 1114-1183
15th Leonard & Copernicus 1473-1543
16th Tartaglia (cubic eqs); Bombelli 1526-1572; Galileo Galilei 1564-1642

**Time Line**

![Mathematical time-line between 1500 BCE and 1650 CE](fig:TimeLineBCE)

1.1.2 Pythagorean Triplets

Well before Pythagoras, the Babylonians had tables of *Pythagorean triplets* (PTs), integers $[a, b, c]$ that obey Eq. 1.1. An example is $[3, 4, 5]$. A stone tablet (Plimpton-322) dating back to 1800 BCE (Fig. 1.9, p. 39) was found with integers for $[a, c]$. Given such sets of two numbers, which determined a third positive integer $b$ such that $b = \sqrt{c^2 - a^2}$, this table is more than convincing that the Babylonians were well aware of PTs, but less convincing that they had access to Euclid’s formula (Eq. 1.7 p. 38).

It seems likely that Euclid’s Elements was largely the source of the fruitful 6th century era due to the Greek mathematician Diophantus (Fig. 1.2), who developed the concept of *discrete mathematics*, now known as *Diophantine analysis*. The work of Diophantus was followed by a rich mathematical era, with the discovery of 1) early calculus (Brahmagupta, 7th CE), 2) algebra (al-Khwârizmî, 9th CE), and 3) complex arithmetic (Bombelli, 15th CE). This period overlapped with the European middle (aka, dark) ages. Presumably European
1.1. EARLY SCIENCE AND MATHEMATICS

intellectuals did not stop thinking during these many centuries, but what happened in Europe is presently unclear given the available records.\(^9\)

1.1.3 What is mathematics?

Mathematics is a language, not so different from other languages. Today’s mathematics is a written language with an emphasis on symbols and glyphs, biased toward Greek letters. The etymology of these symbols would be interesting to study. Each symbol is dynamically assigned a meaning, appropriate for the problem being described. These symbols are then assembled to make sentences. It is similar to Chinese in that the spoken and written version are different across dialects. In fact, like Chinese, the sentences may be read out loud in the language (dialect) of your choice, while the mathematical sentence (like Chinese) is universal.

Math (i.e., the syntax) is a language: It seems strange when people complain that they “can’t learn math,”\(^10\) but they claim to be good at languages. Math is a language, with the symbols taken from various languages, with a bias toward Greek, due to the popularity of Euclid’s Elements. Learning languages is an advanced social skill. However the social outcomes are very different between learning a romance language and math. Learning a new language is fun because it opens doors to other cultures.

Math is different due to the rigor of the rules of the language, along with the way it is taught (e.g., not as a language). A third difference between math and the romance languages is that math evolved from physics, with important technical applications. This was the concept behind the Pythagorean school, a band of followers called the Pythagoreans.

A further problem is that pre-high-school students confuse arithmetic with math. The two topics are very different, and students need to understand this. One does not need to be good at arithmetic to be good at math (but it doesn’t hurt).

There are many rules that must be mastered. These rules are defined by algebra. For example the sentence \(a = b\) means that the number \(a\) has the same value as the number \(b\). The sentence is spoken as “\(a\) equals \(b\).” The numbers are nouns and the equal sign says they are equivalent, playing the role of a verb, or action symbol. Following the rules of algebra, this sentence may be rewritten as \(a - b = 0\). Here the symbols for minus and equal indicate two types of actions.

Sentences can become arbitrarily complex, such as the definition of the integral of a function, or a differential equation. But in each case, the mathematical sentence is written down, may be read out loud, has a well defined meaning, and may be manipulated into equivalent forms following the rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, known as theorems.

The writer of an equation should always translate (explicitly summarize the meaning of the expression), so the reader will not miss the main point. This is simply a matter of clear writing.

Language may be thought of as mathematics (turning this idea on its head). To properly write correct English it is necessary to understand the construction of the sentence. It is important to identify the subject, verb, object, and various types of modifying phrases. If you wish to read about this, look up the distinction between the words that and which, which make a nice example of this concept. Most of us work directly with what we think “sounds right,” but if you’re learning English as a second language, it is very helpful to understand the rules, which are arguably easier to master than the foreign phones (i.e., speech sounds).

1.1.4 Early physics as mathematics

The role of mathematics is to summarize algorithms (i.e., sets of rules). As outlined in Fig. 1.2, it was clear to Pythagoras (and many others before him), that there was an important relationship between

\(^9\)It might be interesting to search the archives of the monasteries, where the records were kept, to figure out what happened during this strange time.

\(^{10}\)“It looks like Greek to me.”
Figure 1.3: Above: Jakob (1655-1705) and Johann (1667-1748) Bernoulli; Below: Leonhard Euler (1707) and Jean le Rond d’Alembert (1717-1783). Euler was blind in his right eye, hence the left portrait view. The figure numbers are from Stillwell (2010).

Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Figure 13.11: Johann Bernoulli

Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Figure 13.11: Johann Bernoulli

Figure 10.4: Leonhard Euler
1.1. EARLY SCIENCE AND MATHEMATICS

mathematics and the physical world. Pythagoras may have been one of the first to capitalize on this relationship, using science and mathematics to design and make things. This was the beginnings of technology as we know it, coming from the relationship between physics and math, impacting map making, tools, implements of war (the wheel, gunpowder), art (music), sound, water transport, sanitation, secure communication, food, . . . , etc.

Why is Eq. 1.1 called a theorem, and what exactly needs to be proved? We do not need to prove that \((a, b, c)\) obey this relationship, since this is a condition that is observed. We do not need to prove that \(a^2\) is the area of a square, as this is the definition of the area of a square. What needs to be proved is that this relation only holds if the angle between the two shorter sides is \(90^\circ\), when the sum of the areas \(a^2\) and \(b^2\) equals area \(c^2\). It follows that lengths \(a\) and \(b\) are less than length \(c\), which is less than \(a + b\) (i.e., \(0 < a \leq b < c < a + b\)).

To appreciate the significance of this development it is helpful to trace the record back to before the time of the Greeks. The Pythagorean theorem (Eq. 1.1) did not begin with Euclid or Pythagoras. Rather Euclid and Pythagoras appreciated the importance of these ideas and documented them.

In the end the Pythagoreans were destroyed by fear. This may be the danger of mixing technology and politics:

"Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Mesopotamia in 497 BCE."

–Stillwell (2010, p. 16)

Chronological history 17–18th centuries

![Figure 1.4: Exploded time-line of the two centuries between the 16th and 18th CE. Given the time line, it seems likely that Bombelli’s discovery of Diophantus’ book on “Arithmetic” in the Vatican library, triggered many of the ideas presented by Descartes and Fermat, followed by others (i.e., Newton). Bombelli’s discovery might be considered as a magic moment in mathematics. The vertical red lines connect mentor-student relationships. Figure 1.2 (p. 16) gives the time-line from 1500BCE to 1650CE. Figure 1.11 (p. 45) give the 17–20 CE (Newton–Einstein) view from 1640–1950, and Fig. 1.17 (p. 76) outlines the full (Bombelli–Einstein, 1525–1925) period.](fig:TimeLine16_18)

1.1.5 The birth of modern mathematics

Modern mathematics (what we know today) was born in the 15–16th century, in the hands of Leonardo da Vinci, Bombelli, Galileo, Descartes, Fermat, and many others (Stillwell, 2010). Many of these early master were, like the Pythagoreans, secretive to the extreme about how they solved problems. This soon changed due to Mersenne, Descartes and Newton, causing mathematics to blossom.

The amazing Bernoulli family The first individual that seems to have openly recognized the importance of mathematics, to actually teach it, was Jacob Bernoulli (Fig. 1.3). Jacob worked on what is now viewed as the standard package of analytic “circular” (i.e., periodic) functions: \(\sin(x), \cos(x),\)

\footnote{It seems likely that the Chinese and Egyptians also did this, but that is more difficult to document.}
exp(x), \log(x). \footnote{Eventually the full details were developed (for real variables) by Euler (Sections 1.3.1 p. 49 and 3.1.1, p. 140).}

From Fig. 1.17 we see that Jacob was contemporary to Merenne, Descartes, Fermat, Huygens, Newton, and Euler. Thus it seems likely that he was strongly influenced by Newton, who in turn was influenced by Descartes, \footnote{It seems clear that Descartes was also a teacher.} Vi`te and Wallis (Stillwell, 2010, p. 175). With the closure of Cambridge University due to the plague of 1665, Newton returned home to Woolsthorpe-by-Colsterworth (95 [mi] north of London), to work by himself, for over a year.

Jacob Bernoulli, like all successful mathematicians of the day, was largely self taught. Yet Jacob was in a new category of mathematicians, because he was an effective teacher. Jacob taught his sibling Johann, who then taught his sibling Daniel. But most importantly, Johann taught Leonhard Euler (Figs. 1.3, 1.17), the most prolific (thus influential) of all mathematicians. This resulted in an explosion of new ideas and understanding. It is most significant that all four mathematicians published their methods and findings. Much later, Jacob studied with students of Descartes\footnote{There are at least three useful exponential scales: Factors of 2, factors of $e \approx 2.7$, and factors of 10. The decibel uses factors of 2 (6 [dB]) and factors of 10 (20 [dB]). Information theory uses powers of 2 (1 [bit]), 4 (2 [bits]). Circuit theory uses in factors of $e \approx 3.7$ (11.4 [dB]).} (Stillwell, 2010, p. 175).

Euler went far beyond all the Bernoulli family, Jacob, Johann and Daniel, (Stillwell, 2010, p. 315). A special strength of Euler was the degree to which he published. First he would master a topic, and then he would publish. His papers continued to appear long after his death (Calinger, 2015).

Another individual of that time of special note, who also published extensively, was d’Alembert (Figs. 1.3, 1.17). Some of the most important tools were first proposed by d’Alembert. Unfortunately, and perhaps somewhat unfairly, his rigor was criticized by Euler, and later by Gauss (Stillwell, 2010).

Once the tools were being openly published, mathematics grew exponentially.\footnote{Lincoln traveled through Mahomet IL (where I live) on his way to the Urbana Court house.} It was one of the most creative times in mathematics. Figure 1.17 shows the list of the many famous names, and their relative time-line. To aid in understand the time line, note that Leonhard Euler was a contemporary of Benjamin Franklin, and James Clerk Maxwell of Abraham Lincoln.\footnote{https://en.wikipedia.org/wiki/Early_life_of_Isaac_Newton}

### 1.1.6 Three Streams from the Pythagorean theorem

From the outset of his presentation, Stillwell (2010, p. 1) defines “three great streams of mathematical thought: Numbers, Geometry and Infinity,” that flow from the Pythagorean theorem, as summarized in Table 1.1. The Pythagorean theorem is the spring from which flow the three streams of all mathematics. This is a useful concept, based on reasoning not as obvious as one might think. Many factors are in play here. One of these was the strongly held opinion of Pythagoras that all mathematics should be based on integers. The rest are tied up in the long, necessarily complex history of mathematics, as best summarized by the fundamental theorems (Table 1.2, p. 29), which are each discussed in detail in a relevant chapter.

Stillwell’s concept of three streams following from the Pythagorean theorem is the organizing principle behind the this book, organized by chapter:

1. **Introduction** (Chapter 1) A detailed overview of the fundamentals and the three streams are presented in Sections
   1.2 Number systems
   1.3 Algebraic equations
   1.4 Scalar Calculus
   1.5 Vector Calculus

\footnote{The log and tan functions are related by $\log(z) = -\frac{1}{2} \ln(\frac{1-z}{1+z})$.}
Chapter 1 is intended to be self contained. Chapter 2-5 are dedicated to advanced topics. If you're a novice, stick to Chapter 1.

As of Version 0.82.03, Chapters 2-5 are not finished.

2. **Number Systems** (Chapter 2: Stream 1) Fundamentals of number systems, starting with prime numbers, through complex numbers, vectors and matrices.

3. **Algebraic Equations** (Chapter 3: Stream 2) Algebra and its development, as we know it today. The theory of real and complex equations and functions of real and complex variables. Complex impedance \( Z(s) \) of complex frequency \( s = \sigma + \omega j \) is covered with some care, given its importance for engineering mathematics.

4. **Scalar Calculus** (Chapter 4: Stream 3a) Ordinary differential equations. Integral theorems. Acoustics.


<table>
<thead>
<tr>
<th>Table 1.1: Three streams followed from Pythagorean theorem: Number Systems (Stream 1), Geometry (Stream 2) and Infinity (Stream 3).</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The Pythagorean Theorem is the mathematical spring which bore the three streams.</td>
</tr>
<tr>
<td>• ≈ Several centuries per stream:</td>
</tr>
<tr>
<td>1) Numbers:</td>
</tr>
<tr>
<td>6(^{th}) BCE ( \mathbb{N} ) counting numbers, ( \mathbb{Q} ) (Rationals), ( \mathbb{P} ) Primes</td>
</tr>
<tr>
<td>5(^{th}) BCE ( \mathbb{Z} ) Common Integers, ( \mathbb{I} ) Irrationals</td>
</tr>
<tr>
<td>7(^{th}) CE ( \mathbb{Z} ) zero ( \in \mathbb{Z} )</td>
</tr>
<tr>
<td>2) Geometry: (e.g., lines, circles, spheres, toroids, ...)</td>
</tr>
<tr>
<td>17(^{th}) CE Composition of polynomials (Descartes, Fermat)</td>
</tr>
<tr>
<td>Euclid’s Geometry + algebra ( \Rightarrow ) Analytic Geometry</td>
</tr>
<tr>
<td>18(^{th}) CE Fundamental Theorem of Algebra</td>
</tr>
<tr>
<td>3) Infinity: (( \infty \rightarrow \text{Sets} ))</td>
</tr>
<tr>
<td>17-18(^{th}) CE Taylor series, Functions, Calculus (Newton)</td>
</tr>
<tr>
<td>19(^{th}) CE ( \mathbb{R} ) Real, ( \mathbb{C} ) Complex 1851</td>
</tr>
<tr>
<td>20(^{th}) CE Set theory</td>
</tr>
</tbody>
</table>

### 1.2 Stream 1: Number Systems

Number theory (discrete, i.e., integer mathematics) was a starting point for many key ideas. For example, in Euclid’s geometrical constructions the Pythagorean theorem for \( [a, b, c] \in \mathbb{R} \) was accepted as true, but the emphasis in the early analysis was on integer constructions (i.e., \([a, b, c] \in \mathbb{N}\)), such as Euclid’s formula for Pythagorean triplets (Eq. 1.7, Fig. 1.8, p. 38). As we shall see, the Pythagorean theorem is a rich source of mathematical constructions, such as composition of polynomials, and solutions of Pell’s equation by eigenvector and recursive analysis methods. Recursive difference equation solutions predate calculus, at least going back to the Chinese (c2000 BCE). These are early (pre-limit) forms of differential equations, best analyzed using an eigenvector expansion, a powerful geometrical concept from linear algebra, of an expansion in terms of an orthogonal set of normalized (unit-length) vectors (Appendix C, p. 191),
The first use of zero and $\infty$:

It is hard to imagine that one would not appreciate the concept of zero and negative numbers when using an abacus. If five beads are moved up, and one is moved down, then four are left. Then if four more are move down, that leaves zero. Taking away (subtracting) is the opposite of giving (adding). Thus subtraction, to obtain zero beads, is no different than subtraction from zero, giving negative four beads. On an abacus, subtraction is obviously the inverse of addition.

To assume the Romans or the Chinese did not understand negative numbers, seems a bit absurd.

However, understanding the concept of zero and negative numbers is not the same as having a symbolic notation. The Roman number system has no such symbols. The first recorded use of a symbol for zero is said to be by Brahmagupta in 628 CE.\(^{17}\) It does not take much imagination to go from counting numbers $\mathbb{N}$ to the set of all integers $\mathbb{Z}$, including zero, but apparently it takes 600 years to functionally develop a terminology, that represents these ideas. Likely this is more about politics and government rule, than mathematics. Defining the rules of subtraction required the creation of algebra (c830 CE, Fig. 1.2).

The concept that caused much more difficulty was $\infty$. Until Riemann’s thesis in 1851 it was not clear if $\infty$ was a number, many numbers, or even definable.

1.2.1 Lec 2: The Taxonomy of Numbers: $\mathbb{N}, \mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{I}, \mathbb{R}, \mathbb{C}$

Once symbols for zero and negative numbers were accepted, progress was made. In a similar manner, to fully understand numbers, a transparent notation is required. First one must differentiate between the different classes (genus) of numbers, providing a notation that defines each of these classes, along with their relationships. It is logical to start with the most basic counting numbers, which we indicate with the double-bold symbol $\mathbb{N}$. All the double-bold symbols and set-theory symbols, such as $\{\cdot\}, \cup, \cap, \in, \mathcal{E}, \perp$ etc. are summarized in Appendix A on mathematical notation, for easy access.

Counting numbers $\mathbb{N}$: These are known as the “natural numbers” $\mathbb{N} = 1, 2, 3, \cdots, \infty$, denoted by the double-bold symbol $\mathbb{N}$. For increased clarity we shall refer to the natural numbers as counting numbers, to clarify that natural means integer. The mathematical sentence “$2 \in \mathbb{N}$” is read as $2$ is a member of the set of counting numbers. The word set means the sharing of a specific property.

Primes $\mathbb{P}$: A prime number $\mathbb{P}$ is an integer that may not be factored by other than 1 and itself. Since $1 = 1 \cdot 1$, $1 \notin \mathbb{P}$, as it is seen to violate this basic definition of a prime. Prime numbers $\mathbb{P}$ are a subset of the counting numbers ($\mathbb{P} \subset \mathbb{N}$). A somewhat amazing fact, well known from the beginning of mathematics, is that every integer may be written as a unique product of primes. A second key idea is that the density of primes is inversely proportional to the log of the number, an observation first observed by Gauss (Goldstein, 1973). In fact, there is a prime between every integer $N$ and $2N$.

We shall use the convenient notation $\pi_n$ for the prime numbers, indexed by $n \in \mathbb{N}$. The first 12 primes ($n = 1, \ldots, 12$) are $\pi_n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37$. Since, $4 = 2^2$ and $6 = 2 \cdot 3$ may be factored, $4, 6 \notin \mathbb{P}$ (read as: 4 and 6 are not in the set of primes). Given this definition, multiples of a prime, i.e., $[2, 3, 4, 5, \ldots] \cdot \pi_k$ of any prime $\pi_k$, cannot be prime. It follows that all primes except 2 must be odd and every integer $N$ is unique in its factorization.

Coprimes are numbers with no common factors. For example, $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$ are not coprime, as they have 2 as a common factor, whereas $21 = 3 \cdot 7$ and $10 = 2 \cdot 5$ are. By definition all distinct primes are coprime. The notation $m \perp n$ indicates that $m, n$ are coprime. The ratio of two coprimes is reduced, as it has no factors to cancel. The ratio of two numbers that are not coprime, may always be reduced to a smaller ratio, once the common factors are canceled. This is called the reduced form, or irreducible fraction. When doing numerical work, it is always beneficial to work with the reduced form. This is especially important when working with the ratio of polynomials, to remove common roots, typically by long division or deconvolution.

\(^{17}\)The fall of the Roman Empire has been established as Sept. 4, 476 CE.
1.2. STREAM 1: NUMBER SYSTEMS (10 LECTURES)

The fundamental theorem of arithmetic states that all integers may be uniquely expressed as a product of primes. The Prime Number Theorem estimates the mean density of primes over \( \mathbb{N} \).

**Integers** \( \mathbb{Z} \): These include positive and negative counting numbers and zero. Notionally we might indicate this using set notation as \( \mathbb{Z} : \{-\mathbb{N}, 0, \mathbb{N}\} \). Read this as The integers are in the set composed of the negative of the natural numbers \(-\mathbb{N}\), zero, and \( \mathbb{N} \). Note that \( \mathbb{N} \subseteq \mathbb{Z} \).

**Rational numbers** \( \mathbb{Q} \): These are defined as numbers formed from the ratio of two integers. Since the integers \( \mathbb{Z} \) include 1, it follows that integers are a subset of rational numbers \( (\mathbb{Z} \subseteq \mathbb{Q}) \). For example, the rational number \( \frac{3}{1} \in \mathbb{Z} \). The main utility of rational numbers is that they can efficiently approximate any number on the real line, to any precision. For example the rational approximation \( \pi \approx \frac{22}{7} \), has a relative error of \( \approx 0.04\% \).

**Fractional number** \( \mathbb{F} \): The fractionals \( \mathbb{F} \) are defined as the ratio of two coprimes. If \( a \perp b \), then \( c = a/b \in \mathbb{F} \). Given this definition, \( \mathbb{F} \subseteq \mathbb{Q} \), \( \mathbb{F} \subset \mathbb{Q} = \mathbb{Z} \cup \mathbb{F} \). Because of the powerful approximating power of rational numbers, the fractional set \( \mathbb{F} \) represent the most important (and the largest) portion of the rational numbers, dwarfing the size of the integers, another good reason for defining the two distinct subsets. For example \( \pi \approx \frac{22}{7} \), with 22 \( \perp \) 7, and \( \frac{9}{6} = \frac{3}{2} = 1 + \frac{1}{2} \) with 3 \( \perp \) 2.

**Irrational numbers** \( \mathbb{I} \): Every real number that is not rational is irrational \( (\mathbb{Q} \perp \mathbb{I}) \). Irrational numbers include \( \pi, e \) and the square roots of the primes \( (\in \sqrt{\mathbb{F}}) \). These are decimal numbers that never repeat, thus requiring infinite precision in their representation. Such numbers can not, and never will be represented on any computer, as they would require an infinite number of bits (precision).

The rationals \( \mathbb{Q} \) and irrationals \( \mathbb{I} \) split the reals \( (\mathbb{R} : \mathbb{Q} \cup \mathbb{I}, \mathbb{Q} \perp \mathbb{I}) \), thus each is a subset of the reals \( (\mathbb{Q} \subseteq \mathbb{R}, \mathbb{I} \subseteq \mathbb{R}) \). This is analogous to the integers \( \mathbb{Z} \) and fractionals \( \mathbb{F} \), which split the rationals \( (\mathbb{Q} : \mathbb{Z} \cup \mathbb{F}, \mathbb{Z} \perp \mathbb{F}) \) (thus each is a subset of the rationals \( (\mathbb{Z} \subseteq \mathbb{Q}, \mathbb{F} \subseteq \mathbb{Q}) \)).

Irrational numbers \( (\mathbb{I}) \) were famously problematic for the Pythagoreans, who incorrectly theorized that all numbers were rational. Like \( \infty \), irrational numbers require a new and difficult concept before they may even be defined: It was essential to understand the factorization of counting numbers into primes (i.e., the fundamental theorem of arithmetic) before the concept of irrationals could be sorted out.

As discussed in Sect. Lec. 1.2.5 (p. 36), fractionals can approximate any irrational number with arbitrary accuracy. Integers are also important, but for a very different reason. All numerical computing today is done with \( \mathbb{Q} = \mathbb{F} \cup \mathbb{Z} \). Indexing uses integers \( \mathbb{Z} \), while the rest of computing (flow dynamics, differential equations, etc.) is done with fractionals \( \mathbb{F} \) (i.e., IEEE-754). Computer scientists are trained on these topics, and computer engineers need to be at least conversant with them.

**Real numbers** \( \mathbb{R} \): Reals are the union of rational and irrational numbers, namely \( \mathbb{R} = \mathbb{I} \cup \mathbb{Q} \). The lengths in Euclidean geometry are reals. Many people assume that IEEE 754 floating point numbers (c1985) are real (i.e., \( \in \mathbb{R} \)). In fact they are rational \( (\mathbb{Q} : \{\mathbb{F} \cup \mathbb{Z}\}) \) approximations to real numbers, designed to have a very large dynamic range. There can be no machine realization of irrational numbers, since such a number would require infinite precision (\( \infty \) bits). The hallmark of fractional numbers \( (\mathbb{F}) \) is their power in making highly accurate approximations of any real number.

Using Euclid’s compass and ruler methods, one can make line length proportionally shorter or longer, or (approximately) the same. A line may be made be twice as long, an angle bisected. However, the concept of an integer length in Euclid’s geometry was not defined.\(^\text{18}\) Nor can one construct an imaginary or complex line as all lines are assumed to be real.

\(^{18}\)As best I know.
Real numbers were first fully accepted only after set theory was developed by Cantor (1874) (Stillwell, 2010, pp. 461, 525, ...). This seems amazing, given how widely accepted real numbers are today. In some sense they were accepted by the Greeks, as lengths of real lines.

**The size of \( \mathbb{R} \):** The real line may be divided into a hierarchical groups, sorted by size: \( |\mathbb{R}| > |\mathbb{I}| \gg |\mathbb{Q}| > |\mathbb{F}| \gg |\mathbb{Z}| > |\mathbb{N}| > |\mathbb{P}| \). The reals may be split into irrationals and rationals, which may be further split into the integers and the fractionals. Thus, all is not integer. If a triangle has two integer sides, then the hypotenuse can be irrational (\( \sqrt{2} = \sqrt{1^2 + 1^2} \)). This leads us to a fundamental question: “Are there integer solutions to Eq. 1.1?” We need not look further than the simple example \{3, 4, 5\}. In fact the example generalizes: the formula for generating an infinite number of integer solutions is called *Euclid’s Formula* (Section 1.2.5, p. 36).

The more important point is that the size of the irrationals is much larger than any set other than the reals (i.e., complex numbers). Thus when we use computers to model physical systems, we are constantly needing to compute with irrational numbers. But this is impossible since every irrational numbers would require an infinite number of bits to represent it. Thus we must compute with rational approximations to the irrationals. This means we need to use the fractionals. In the end, we must work with the IEEE 754 floating point numbers,\(^{19}\) which are rationals.

**Complex numbers \( \mathbb{C} \):** Complex numbers are best defined as *ordered pairs of real numbers*.\(^{20}\) The word “complex,” as used here, does not mean that the number are complicated or difficult. The word “imaginary” does not mean the numbers disappear. Complex numbers are quite special in engineering mathematics, as roots of polynomials. The most obvious example is the quadratic formula for the roots of a \( 2 \)\(^{nd} \) degree polynomial, having either rational, real or complex coefficients.

Today the common way to write a complex number is using the notation \( z = a + bj \in \mathbb{C} \), where \( a, b \in \mathbb{R} \). Here \( 1j = \sqrt{-1} \). We also define \( 1t = -1j \) to account for the two possible signs of the square root. Accordingly \( 1j^2 = 1t^2 = -1 \).

Multiplication of complex numbers follows the basic rules of real algebra, for example, the rules of multiplying two polynomials. Multiplication of two first degree polynomials gives

\[
(a + bx)(c + dx) = ac + (ad + bc)x + bdx^2
\]

If we substitute \( 1j \) for \( x \), and use the definition \( 1j^2 = -1 \), we obtain the product of the two complex numbers

\[
(a + bj)(c + dj) = ac - bd + (ad + bc)j.
\]

Thus multiplication of complex numbers obey the accepted rules of algebra.

**Polar representation:** A alternative for complex multiplication is to work with polar coordinates. The polar form of complex number \( z = a + bj \) is written in terms of its magnitude \( \rho = \sqrt{a^2 + b^2} \) and angle \( \theta = \angle z = \tan^{-1}(z) = \arctan z \), as \( z = \rho e^{j\theta} \). From the definition of the complex natural log function

\[
\ln \rho e^{j\theta} = \ln \rho + \theta j,
\]

which is useful in engineering calculations.\(^{21}\) Note that the angle \( \angle \theta = \Im \{ \log z \} \). Clearly this is an improvement over \( \angle \theta = \arctan(z) \).

The polar representation make clear the utility of a complex number: Its magnitude scales while its angle \( \Theta \) rotates. This propriety of scaling and rotating is what makes complex numbers useful in engineering calculations. This is especially obvious when dealing with impedances, which have complex roots with very special properties, as discussed in Sect. 1.3.3 (p. 54).

\(^{19}\)IEEE 754: http://www.h-schmidt.net/FloatConverter/IEEE754.html.

\(^{20}\)A polynomial \( a + bx \) and a 2-vector \( [a, b]^T = \begin{bmatrix} a \\ b \end{bmatrix} \) are also examples of ordered pairs.

\(^{21}\)Chapter 2 discusses the definition of the phase, i.e., how it computed (i.e., arctan(\( e^{j\theta} \)), arctan2(\( x, y \)), and the importance of the unwrapped phase, as in the example (Table 1.5, p. 71) \( \delta(t - \tau) \leftrightarrow e^{-j\tau} \).
Matrix representation: An alternative way to represent complex numbers is in terms of 2x2 matrices. This relationship is defined by the mapping from a complex number to a 2x2 matrix:

\[
\begin{align*}
a + bj & \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \\
2 & \leftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\
-j & \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
e^{\theta j} & \leftrightarrow \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}
\end{align*}
\] (1.2)

By taking the inverse of the 2x2 matrix one can define the ratio of one complex number by another, until you try out this representation, it may not seem obvious, or even that it could work.

Exercise: Verify that

\[
\begin{align*}
a + bj + c + dj &= \frac{ab + bd + (bc - ad)j}{c^2 + d^2} \\
&\leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2}.
\end{align*}
\]

This representation proves that \(1j\) is not necessary when defining a complex number. What \(1j\) can do is to conceptually simplify the algebra. It is worth your time to become familiar with the matrix representation, to clarify any possible confusions you might have about multiplication and division of complex numbers. This matrix representation can save you time, heartache and messy algebra. Once you have learned how to multiply two matrices, it’s a lot simpler than doing the complex algebra. In many cases we will leave the results of our analysis in matrix form, to avoid the algebra altogether. Thus both representations are important. More on this topic may be found in Chapter 2.

Real versus complex numbers: As discussed in Appendix A (p. 181), all numbers may be viewed as complex. Namely every real number is complex if we take the imaginary part to be zero (Boas, 1987). For example, \(2 \in \mathbb{P} \subset \mathbb{C}\). Likewise every purely imaginary number (e.g., \(0 + 1j\)) is complex with zero real part. It follows that \(2j = 0 + \pi 2j \in \mathbb{P}\). Integers are a subset of reals, which are a subset of complex numbers. Gaussian integers are complex integers (\(\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}\)). From the above discussion it should be clear that each of these different classes of number are nested in a hierarchy, with the following embedding:

\[\pi_k \in \mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{F} \subset \mathbb{R} \subset \mathbb{C}\].

The roots of real polynomials \(x_k\) are complex (\(x_k \in \mathbb{C}\), independent of the genus of the coefficients (e.g., real integer coefficients give rise to complex roots). Each genus plays an important role in algebra, with prime numbers at the bottom (root of the tree) and complex numbers at the top. We shall explore this further in Chaps. 2 and 3.

Finally, note that complex numbers \(\mathbb{C}\) do not have “order,” meaning one complex number cannot be larger or smaller than another. It makes no sense to say that \(j > 1\) or \(j = 1\) (Boas, 1987). The real and imaginary parts and the magnitude and phase, have order. If time \(t\) were complex, there could be no yesterday and tomorrow.

Applications of integers

The most relevant question at this point is “Why are integers important?” First we count with them, so we can keep track of “how much.” But there is much more to numbers than counting: We use integers for any application where absolute accuracy is essential, such as banking transactions (making change), the precise computing of dates (Stillwell, 2010, p. 70) and location (“I’ll meet you at 34 and Vine at noon on Jan. 1, 2034.”), building roads or buildings out of bricks (objects built from a unit size).

\[\text{Sometimes we let the computer do the final algebra, numerically, as 2x2 matrix multiplications.}\]

\[\text{The plural complexs (a double /s/) seems an unacceptable word in English.}\]

\[\text{It follows that integers are a subset of Gaussian integers (the imaginary or real part of the Gaussian integer may be zero).}\]

\[\text{One can define } \xi = x + 1j \text{ ct to be complex } (x, t \in \mathbb{R}, \text{ with } x \text{ in meters [m] and } t \text{ is in seconds [s], with the speed of light } c \text{ [m/s].}\]

\[\text{Boas, 1987.}\]
To navigate we need to know how to predict the tides, the location of the moon and sun, etc. Integers are important because they are precise: Once a month there is a full moon, easily recognizable. The next day its slightly less than full. If one could represent our position as integers in time and space, we would know exactly where we are at all times.

The Pythagoreans claimed that all was integer. From a practical point of view, it seems they were right. Today all computers compute floating point numbers as fractionals. However, in theory they were wrong. The difference is a matter of precision.

**History of complex numbers:** It is notable how long it took for complex numbers to be accepted (1851), relative to when they were first introduced by Bombelli (16th century CE). One might have thought that the solution of the quadratic, known to the Chinese, would have settled this question. It seems that complex integers (aka, Gaussian integers) were accepted before non-integral complex numbers. Apparently real numbers (ℝ) were not accepted (i.e., proved to exist, thus mathematically defined) until even later. It took the development of set theory in the late 19th century to sort out a proper definition of the real number, due to the existence of irrational numbers. It makes one wonder “How many angels can stand on the point of a pin?”

**Numerical Representations of ℤ, ℝ, ℂ:** When doing numerical work, one must consider how we may compute within the family of reals (i.e., irrational). There can be no irrational number representation on a computers. IEEE floating point numbers, which are the international standard of computation, are actually rational approximations. The mantissa and the exponent are each integers, having sign and magnitude. The size of each integer depends on the precision of the number being represented. An IEEE floating-point number is rational because it has a binary (integer) mantissa, multiplied by 2 raised to the power of a binary (integer) exponent. For example π ≈ a26 with a, b ∈ ℤ. In summary, IEEE floating-point numbers are not, and cannot, be irrational, since to do that would require an infinite number of bits.

True floating point numbers contain irrational numbers, which must be approximate by rational numbers. This leads to the concept of fractional representation, which requires the definition of the mantissa, base and exponent, where both the mantissa and the exponent are signed. Numerical results must not depend on the base. One could dramatically improve the resolution of the numerical representation by the use of the Fundamental Theorem of Arithmetic (Section 1.2.2, page 30). For example one could factor the exponent into its primes and then represent the number as a26π·2ⁿ⁺ᵐ⁻¹, etc. Such a representation would improve the resolution of the representation. But even so, the irrational numbers would be approximate. For example, base ten 26 is natural using this representation since 10ⁿ = 2ⁿ5ⁿ. Thus

\[
\pi \cdot 10^5 \approx 314159.27 \ldots = 3 \cdot 2^55^1 + 1 \cdot 2^45^4 + 4 \cdot 2^35^3 + \cdots + 9 \cdot 2^05^0 + 2 \cdot 2^015^{-1} \ldots
\]

If we approximate π by 22/7, then according to the Matlab/Octave `dec2bin()` routine, the binary representation is

\[
\pi \cdot 2^{17} \approx 131072_{10} \cdot 22/7 = 110,0100,1001,0010,01012,
\]

where 1 and 0 are multipliers of powers of 2, which are then added together as follows

\[
2^{18} + 2^{17} + 2^{14} + 2^{11} + 2^8 + 2^5 + 2^2 + 2^0.
\]

In base 16 (i.e, hexadecimal) 2^{17} · 22/7 = 2^{18} · 8_{16}/7_{16}.

Computers keep track of the decimal point using the exponent, which in this case is the factor 2^{17} = 131072_{10}. The concept of the decimal point is replaced by an integer, having the desired precision, and a scale factor of any base (radix). This scale factor may be thought of as moving the decimal point to the right (larger number) or left (smaller number). The mantissa “fine-tunes” the value about a scale factor (the exponent).

---

26 Base 10 is the natural world-wide standard simply because we have 10 fingers.
Here is $x = 2^{17} \times 22/7$ using IEEE 754 double precision, as computed by an IEEE-754 floating point converter.\textsuperscript{27} \[ x = 411,940.5625_{10} = 2^{54} \times 1198372 = 0, 10010, 00, 110010, 010010, 0100102 = 0 \times 48 \times 924924_{16} \]. The commas in the binary $(0,1)$ string are to help visualize the quasi-periodic nature of the bit-stream. The mantissa is $4,793,490_{10}$ and the exponent is $218$. The numbers are stored in a 32 bit format, with 1 bit for sign, 8 bits for the exponent and 23 bits for the mantissa. Perhaps a more instructive number is $x = 4,793,490.0 = 0, 100, 1010, 100, 100, 100, 100, 100, 100, 100, 100_2 = 0 \times 40 \times 924924_{16}$, which has a repeating binary bit pattern of $(0(100))_3$, broken by the scale factor $0 \times 4a$. Another with even higher symmetry is $x = 6,344,131,191,146,9 \times 10^{-17} = 0 \times 24,924,924_{16} = 00, 100, 100, 100, 100, 100, 100, 100, 100, 100_2$. In this example the repeating pattern is clear in the Hex representation as a repeating $((942))_3$, as represented by the double brackets, with the subscript indicating the period, in this case, 3 digits. As before, the commas are to help with readability, and have no other meaning.

There are other important types of representations. As pairs of reals, complex numbers have similar approximate representations. An important representations of complex numbers is $e^z = \cos(z) + j \sin(z)$ with $z \in \mathbb{C}$, which includes the famous formula of Euler ($\theta \in \mathbb{R}$)

$$e^{i\theta} = \cos \theta + j \sin \theta \leftrightarrow \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$ 

Some of these concepts can be generalized to include vectors, matrices and polynomials.

**Integers and the Pythagoreans** The integer is the corner stone of the Pythagorean doctrine, so much so that it caused a fracture within the Pythagoreans when it was discovered that not all numbers are rational. The famous example is the isosceles triangle $1,1,\sqrt{2}$, which lead to the next triangle $[1,2,\sqrt{3}]$, etc. This is known as the Spiral of Theodorus: the short side is 1 and the long side is recursively incremented by one, using a simple compass-ruler construction.

There are right-triangles with integral lengths, the best known being $[3,4,5]$. Such triplets of integers $[a,b,c]$ that satisfy the Pythagorean formula (Eq. 1.1) are denoted Pythagorean triplets, which may be verified using Euclid’s formula (Eq. 1.7, p. 134).

To form triangles with perfect $90^\circ$ angles, the lengths need to satisfy Eq. 1.1. Such triangles are useful in constructing buildings or roads made from of bricks having a uniform size.

**Public-private key Security:** An important application of prime numbers is public-private key (RSA) encryption, essential for internet security applications (e.g., online banking). To send secure messages the security (i.e., utility) of the internet is dependent on key encryption. Most people assume this is done by a personal login and passwords. Passwords are simply not secure, for many reasons. A proper method depends on factoring integers formed from products of primes having thousands of bits.\textsuperscript{28} The security is based on the relative ease in multiplying large primes, but the virtual impossibility of factoring them.

When a computation is easy in one direction, but its inverse is impossible, it is called a trap-door function. We shall explore the reasons for this in Chapter 2. If everyone were to switch from passwords to public key encryption, the internet would be much more secure.

A third application of integers are imaginative problems that use integers. An example is the classic Chinese Four stone problem: “Find the weight of four stones that can be used with a scale to weigh any object (e.g., salt, gold) between 0, 1, 2, \ldots, 40 [gm].” As with the other problems, the answer is

\begin{footnotesize}
\begin{itemize}
\item \textsuperscript{27}http://www.h-schmidt.net/FloatConverter/IEEE754.html
\item \textsuperscript{28}It would seem that public key encryption could work by having two numbers with a common prime, and then by using Euclidean algorithm, that GCD could be worked out. One of the integers could be the public key and the second could be the private key. Given the difficulty of factoring the numbers into their primes, and ease of finding the GCD using Euclidean algorithm, a practical scheme may be possible. One problem with this idea is that one may store all the known primes.
\end{itemize}
\end{footnotesize}
CHAPTER 1. INTRODUCTION

not as interesting as the method, since the problem may be easily recast into a related one. This type of problem can be found in airline magazines as entertain on a long flight. The solution to this problem is best cast as a linear algebra problem, with integer solutions. Again, once you know the trick, it is “easy.”

1.2.2 Lec 3: The role of physics in mathematics

Bells, chimes and Eigenmodes: Integers naturally arose in art, music and science. Examples include the relations between musical notes, the natural eigenmodes (tones) of strings and other musical instruments. These relations were so common and well studied, it appeared that to understand the physical world (aka, the Universe), one needed to understand integers. This was a seductive view, but not actually correct. As will be discussed in Sections 1.3.1 (p. 43) and 3.1.1 (p. 139), it is best to view the relationship between acoustics, music and mathematics as historical, since these topics played such an important role in the development of mathematics. Also interesting is the role that integers play in quantum mechanics, also based on eigenmodes, but in this case, those of atoms. Eigenmodes follow from solutions of the wave equation, which has natural delay due to d’Alembert’s solution, along with reflecting boundary conditions (Eq. 1.121, p. 106), resulting in roots of the characteristic equation.

Engineers are so accustomed to working with real (or complex) numbers, the distinction between real (i.e., irrational) and fractional numbers are rarely acknowledged. Integers on the other hand arise in many contexts. One cannot master programming computers without understanding integer, hexadecimal, octal, and binary representations, since all numbers in a computer are represented in numerical computations in terms of rationals ($\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$).

As discussed in Section 1.2.1 (p. 22), the primary reason integers are so important is their absolute precision. Every integer $n \in \mathbb{Z}$ is unique, and has the indexing property, which is essential for making lists that are ordered, so that one can quickly look things up. The alphabet also has this property (e.g., a book’s index). Other than for hexadecimal numbers, which for notional reasons use the alphabet, other than the base, letters (base 26), are equivalent to integers (base 10) (i.e., they are just as precise). A sequence of letters is equivalent to a sequence of integers.

Because of the integer’s absolute precision, the digital computer overtook the analog computer, once it was practical to make logic circuits that were fast. The first digital computer was thought to be the Eniac at the University of Pennsylvania, but it turned out that the code-breaking effort in Bletchley Park, England, under the guidance of Alan Turing, created the first digital computer (The Colossus) to break the WWII German “Enigma” code. Due to the high secrecy of this war effort, the credit was only acknowledged in the 1970s when the project was finally declassified.

There is zero possibility of analog computing displacing digital computing, due to the importance of precision (and speed). But even with binary representation, there is a non-zero probability of error, for example on a hard drive, due to physical noise. To deal with this, error correcting codes have been developed, to reduce the error by several orders of magnitude. Today this is a science, and billions of dollars are invested to increase the density of bits per area, to increasingly larger factors. A few years ago the terabyte drive was unheard of; today it is the standard. In a few years petabyte drives will certainly become available. It is hard to comprehend how these will be used by individuals, but they are essential for on-line (cloud) computing.

The role of mathematics in physics

Modern mathematics is built on a hierarchical construct of fundamental theorems, as summarized in Table 1.2. The importance of such theorems cannot be overemphasized. Gauss’ and Stokes’ Laws play a major role in understanding and manipulating Maxwell’s equations. Every engineering student needs to fully appreciate the significance of these key theorems. If necessary, memorize them. But

29When ever someone tells you something is “easy,” you should immediately appreciate that it is very hard, but there is a concept, that once you learn, the difficulty evaporates.

30Check out the history of $1729 = 1^3 + 12^3 = 9^3 + 10^3$. 
Table 1.2: The fundamental theorems of mathematics

1. Fundamental theorems of:

   (a) **Number systems: Stream 1**
   - arithmetic
   - prime number

   (b) **Geometry: Stream 2**
   - algebra
   - Bézout

   (c) **Calculus: Stream 3**
   - Leibniz $\mathbb{R}^1$
   - complex $\mathbb{C} \subset \mathbb{R}^2$
   - vectors $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$
     - Gauss’ Law (Divergence theorem)
     - Stokes’ Law (Curl theorem, or Green’s theorem)
     - Vector calculus (Helmholtz’s theorem)

2. Other key concepts:

   - Complex analytic functions (complex roots are finally accepted!)
     - Complex Taylor series (complex analytic functions)
     - Region of convergence (RoC) of complex analytic series
     - Laplace transform, and its inverse
     - Causal time $\implies$ complex frequency $s$
     - Cauchy Integral Theorem
     - Residue integration (i.e., Green’s Thm in $\mathbb{R}^2$)
   - Riemann mapping theorem (Gray, 1994; Walsh, 1973)
   - Complex Impedance (Ohm’s Law) (Kennelly, 1893)

---

that will not do over the long run, as each and every theorem must be fully understood. Fortunately most students already know several of these theorems, but perhaps not by name. In such cases, it is a matter of mastering the vocabulary.

The theorems are naturally organized and may be thought of in terms of Stillwell’s three streams. For Stream 1 there is the Fundamental Theorem of Arithmetic and the Prime Number Theorem. For Stream 2 there is the Fundamental Theorem of Algebra and Bézout’s theorem, while for Stream 3 there are a host of theorems on calculus, ordered by their dimensionality. Some of these theorems verge on trivial (e.g., the Fundamental Theorem of Arithmetic). Others are more challenging, such as the Fundamental Theorem of Vector Calculus and Green’s theorem.

Complexity should not be confused with importance. Each of these theorems is, as stated, fundamental. Taken as a whole, they are a powerful way of summarizing mathematics.

Stream 1: Prime Number theorems:

There are two fundamental theorems about primes,

1. The fundamental theorem of arithmetic: This states that every integer \( n \in \mathbb{Z} \) may be uniquely factored into prime numbers. This raises the question of the meaning of factor (split into a product). The product of two integers \( m, n \in \mathbb{Z} \) is \( mn = \sum m n = \sum m m \).

2. The Prime Number Theorem: One would like to know how many primes there are. That is easy: \( |\mathbb{P}| = \infty \) (The size of the set of primes, is infinite). One way of asking this questions is What is the average density of primes, in the limit as \( n \to \infty \)? This question was answered, for all practical purposes, by Gauss, who in his pastime, computed the first three million primes by hand. He discovered that, to a good approximation, the primes are equally likely, on a log scale. This is nicely summarized by the jingle, attributed to the mathematician Pafnuty Chebyshev (Stillwell, 2010, p. 585)

Chebyshev said, and I say it again: There is always a prime between \( n \) and \( 2n \).

When the ratio (interval) of two frequencies (pitch) is 2, the relationship is called an octave. Thus we might say, with a slight stretch of terminology, there is at least one prime per octave. Thus one might wonder about the maximum number of primes per octave. In modern music the octave is further divided into 12 intervals called semitones (factors), equal to the \( \sqrt[12]{2} \). The product of 12 semitones is an octave. Thus one might ask how many primes there are per semitone? In the end, it is a question of the density of primes on a log (i.e., ratio) scale.

Stream 2: Fundamental theorem of algebra

This theorem states that every polynomial in \( x \) of degree \( N \)

\[
P_n(x) = \sum_{k=0}^{N} a_k x^k
\]

has at least one root. When the root is removed, the degree of the polynomial is reduced by 1. It follows that applied recursively, a polynomial of degree \( N \) has \( N \) roots. Note there are \( N+1 \) coefficients (i.e., \( a_k \)).

Besides the fundamental theorem of algebra, a second important theorem is Bézout’s theorem, which is a generalization of the fundamental theorem of algebra. It says\(^{31} \) that the composition of two polynomials has degree equal to the product of the degrees of each polynomial. For example, if \( P_3(x) = x^3 \) and \( P_5(x) = x^5 \), then \( P_3(P_5)(x) = (x^5)^3 = x^{15} \). It further states that when counting the

\(^{31}\text{Statements of the theorem speak of intersections and constructions of curves, rather than compositions. I find this somewhat confusing. For example, how does intersection differ from elimination, or construction from composition (Stillwell, 2010, p. 119)?}
1.2. STREAM 1: NUMBER SYSTEMS (10 LECTURES)

\( N \) roots of a polynomial of degree \( N \), one must include the imaginary roots, double roots and roots at infinity, some of which may difficult to identify.

One must wonder what happens when the degree is fractional, or worse, irrational? Are these cases covered by Bézout?

**Stream 3: Fundamental theorems of calculus**

In Sections 1.5.5 and 1.5.6 we will deal with each of the theorems for Stream 3, where we consider the several fundamental theorems of integration, starting with Leibniz’s formula for integration on the real line (\( \mathbb{R} \)), then progressing to complex integration in the complex plane (\( \mathbb{C} \)) (Cauchy’s theorem), which is required for computing the inverse Laplace transform. Gauss’ and Stokes’ Laws for \( \mathbb{R}^2 \) require closed and open surfaces, respectively. One cannot manipulate Maxwell’s equations, fluid flow, or acoustics without understanding these theorems. Any problem that deals with the wave equation in more than one dimension, requires an understanding of these theorems, thus are the basis of the derivation of the Kirchoff voltage and current laws. The \( \nabla \) symbol is pronounced as “del” (preferred) or “nabla.”

Finally we define the four basic vector operations based on the \( \nabla \) “operator:” the gradient \( \nabla() \), divergence \( \nabla \cdot () \), curl \( \nabla \times () \) and the Laplacian \( \nabla \cdot \nabla() = \nabla^2() \). The first three operations are defined in terms of integral operations on a surface in 1, 2 or 3 dimensions, by then taking the limit as that surface, and the volume contained within, goes to zero. These three differential operators are necessary if you wish to understand Maxwell’s Equations, the crown jewel of mathematical physics. Hence mathematics plays a key role in physics, as does physics in math.

**Other key concepts**

Besides the widely recognized fundamental theorems for the three streams, there are a number of equally important theorems that have not yet been labeled as “fundamental.”

The widely recognized Cauchy Integral Theorem is an excellent example, since it is a stepping stone to Green’s theorem and the fundamental theorem of complex calculus. In Section 1.5.3 (p. 100) we clarify the contributions of each of these special theorems.

Once these fundamental theorems of integration (Stream 3) have been mastered, the student is ready for the complex frequency domain, which takes us back to Stream 2 and the complex frequency plane (\( s = \sigma + \omega \jmath \in \mathbb{C} \)). While the Fourier and Laplace transforms are taught in mathematics courses, typically few physical connections are made, accordingly the concept of complex frequency is rarely mentioned. The complex frequency domain and causality are fundamentally related (Sects. 1.4.6–1.4.8, p. 86–87), and critical for the analysis of signals and systems, and especially the concept of impedance (Sect. 1.4.3, p. 79).

Without the concept of time and frequency, one cannot develop an intuition for the Fourier and Laplace transform relationships, especially within the context of engineering and mathematical physics.

**WEEK 2**

1.2.3 **Lec 4: Prime numbers**

If someone came up to you and asked for a theory of counting numbers, I suspect you would look them in the eye with a blank stare, and start counting. It sounds like either a stupid question or a bad joke. Yet integers are rich topic, so the question is not even slightly dumb. It is somewhat amazing that even birds and bees can count. While I doubt birds and bees can recognize primes, cicadas and other insects only crawl out of the ground in multiples of prime years, (e.g., 13 or 17 year cycles). If you have ever witnessed such an event (I have), you will never forget it. Somehow they know. Finally, there is

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32It is not clear what it takes to reach this more official sounding category.
an analytic function, first introduced by Euler, based on his analysis of the sieve, now known as the \textit{Riemann zeta function} \( \zeta(s) \), which is complex analytic, with its poles at the logs of the prime numbers.

The exact relationship between the primes and the poles will be discussed in Sections ?? (p. ??). The properties of this function are truly amazing, even fun.\footnote{The Riemann zeta function is known as the million dollar equation as there is a cash reward for a proof of the Riemann Hypothesis.} It follows that primes are a subset of the counting numbers (Section 1.2.1), that the theory of numbers (and primes) is an important topic of study. Many of the questions, and some of the answers, go back to at least the time of the Chinese (Stillwell, 2010).

\textbf{The importance of prime numbers}

Likely the first insight into the counting numbers started with the \textit{sieve}, shown in Fig. 1.5. A sieve answers the question “How can one list the prime numbers?” The answer comes from looking for irregular patterns in the counting numbers, by playing the counting numbers against themselves.

A prime is that subset of positive integers \( \mathbb{P} \subset \mathbb{N} \) that cannot be factored. The number 1 is not a prime, for some non-obvious reasons, but there is no pattern in it since it is always a (useless) factor of every counting number.

\textbf{Sieves}

A recursive sieve method for finding primes was first devised by the Greek Eratosthenes,\footnote{https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes\#Euler.27s_Sieve} and summarized in Fig. 1.5.

\begin{enumerate}
\item Write \( N - 1 \) counting number from 2 to \( N \) (List)
\item Define loop index \( k = 1 \) and a multiplier \( n \in \mathbb{N} \) denoted \( n := \{2, \cdots, N\} \).
\item The next number on the list is prime \( \pi_k \in \mathbb{P} \)
\item Remove (Cross out) all multiples \( n \cdot \pi_n \) of \( \pi_k \)
\item \( k = k + 1 \): return to step 3.
\end{enumerate}

Starting from the first prime \( \pi_1 = 2 \), one successively strikes out all the multiples of that prime. For example, starting from \( \pi_1 = 2 \) one strikes out \( 2 \cdot 2, 2 \cdot 3, 2 \cdot 4, 2 \cdot 5, \cdots, 2 \cdot (N/2) \). By definition the multiples are products of the target prime (2 in our example) and every another integer \( (n \geq 2) \). In this way all the even numbers are removed in this first iteration. The next remaining integer (3 in our example) is identified as the next (second) prime \( \pi_2 \). Then all the \( (N - 2)/2 \) multiples of \( \pi_2 = 3 \) are removed. The next remaining number is \( \pi_3 = 5 \), so all multiples of \( \pi_3 = 5 \) are removed (i.e., \( 2 \cdot 5, 3 \cdot 5, 5 \cdot 5 \) etc., \( \cdots \)). This process is repeated until all the numbers of the list have been either canceled, or identified as prime.

As the word sieve implies, this sifting process takes a heavy toll on the integers, rapidly pruning the non-primes. In four iterations of the sieve algorithm, all the primes below \( N = 50 \) are identified in \textbf{red}. The final set of primes are displayed in the caption of Fig. 1.5.

Once a prime greater than \( N/2 \) has been identified, we may stop, since twice that prime is greater than \( N \), the maximum number under consideration. Once you have reached \( \sqrt{N} \) all the primes have been identified (this follows from the fact that the next prime \( \pi_n \) is multiplied by an integer \( n = 1, \ldots, N \)). Once this number \( n \pi_n > N \) the list has been exhausted, which must be \( n < \sqrt{N} \).

When using a computer, memory efficiency and speed are the main considerations. There are various schemes for making the sieve more efficient. For example the recursion \( n \pi_k = (n - 1)\pi_k + \pi_k \) will speed up the process, by replacing the multiply with an addition of \( \pi_k \).
1. Write $N$ integers from 2 to $N - 1$. Let $k = 1$. The first element $\pi_1 = 2$ is a prime. Cross out $n \cdot \pi_n$: (e.g., $n \cdot \pi_1 = 4, 8, 16, 32, \cdots$).

$\begin{array}{cccccccc}
2 & 3 & A & 5 & A & 7 & A & 11 \\
11 & A^2 & 13 & A & 15 & A & 17 & A \\
31 & A^2 & 33 & A & 35 & A & 37 & A \\
41 & A^2 & 43 & A & 45 & A & 47 & A \\
\end{array}$

2. Let $k = 2, \pi_2 = 3$. Cross out $n\pi_k$ (6, 9, 12, 15, …):

$\begin{array}{cccccccc}
2 & 3 & A & 5 & A & 7 & A & 11 \\
11 & A^2 & 13 & A & 15 & A & 17 & A \\
31 & A^2 & 33 & A & 35 & A & 37 & A \\
41 & A^2 & 43 & A & 45 & A & 47 & A \\
\end{array}$

3. Let $k = 3, \pi_3 = 5$. Cross out $n\pi_3$. (Cross out 25, 35).

$\begin{array}{cccccccc}
2 & 3 & A & 5 & A & 7 & A & 11 \\
11 & A^2 & 13 & A & 15 & A & 17 & A \\
31 & A^2 & 33 & A & 35 & A & 37 & A \\
41 & A^2 & 43 & A & 45 & A & 47 & A \\
\end{array}$

4. Finally let $k = 4, \pi_4 = 7$. Remove $n\pi_4$: (Cross out 49). Thus there are 15 primes less than $N = 50$: $\pi_k = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sieve_of_eratosthenes.png}
\caption{Sieve of Eratosthenes for the case of $N = 50$.}
\end{figure}

**Two fundamental theorems of primes:** Early theories of numbers revealed two fundamental theorems (there are many more than two), as discussed in Section 1.2.2 and 2.1.1 (p. 127). The first of these is the fundamental theorem of arithmetic, which says that every integer greater than 1 may be uniquely factored into a product of primes

\[ n = \prod_{k=1}^{K} \pi_k^{\beta_k}, \]  

(1.4)

where $k = 1, \ldots, K$ indexes the integer’s $K$ prime factors $\pi_k \in \mathbb{P}$. Typically prime factors appear more than once, for example 25 = $5^2$. To make the notation compact we define the multiplicity $\beta_k$ of each prime factor $\pi_k$. For example 2312 = $2^3 \cdot 17^2 = \pi_1^3 \pi_7^2$ (i.e., $\pi_1 = 2, \beta_1 = 3; \pi_7 = 17, \beta_7 = 2$) and 2313 = $3^2 \cdot 257 = \pi_3^2 \pi_{55}$ (i.e., $\pi_3 = 3, \beta_3 = 2; \pi_{55} = 257, \beta_{55} = 1$). Our demonstration of this is empirical, using the Matlab/Octave \texttt{factor(N)} routine, which factors $N$.\textsuperscript{35}

What seems amazing is the unique nature of this theorem. Each counting number is uniquely represented as a product of primes. There cannot be two integers with the same factorization. Once you multiply the factors out, the result is unique ($N$). Note that it’s easy to multiply integers (e.g., primes), but expensive to factor them. And factoring the product of three primes is significantly more difficult than factoring two.

Factoring is much more expensive that division. This is not due to the difference in the cost of division over multiplication. The division of one number by another is only slightly more expensive

\textsuperscript{35}If you wish to be a Mathematician, you need to learn how to prove theorems. If you’re an Engineer, you are happy that someone else has already proved them, so that you can use the result.
than multiplying two numbers, maybe by a factor of $2$. Dividing the product of two primes, given one, is trivial, slightly more expensive that multiplying. Factoring the product of two primes is nearly impossible, as one needs to know what to divide by. This is an explosion in probability, measured by Shannon Entropy. Factoring means dividing by some integer and obtaining another integer with remainder zero. Thus one could factor a product of primes $N = \pi_k \pi_l$ by doing $M$ divisions, where $M$ is the number of primes less than $N$. This assumes the list of primes less than $N$ are known.

But the utility has to do with the density of primes (the prime number theorem, i.e., Gauss’ hypothesis). If we were simply looking up a few numbers from a short list of primes, it would be easy, but the density of primes among the integers, is logarithmic ($>1$ per octave, Section 2.1.1, p. 128).

This take us to the **Prime Number Theorem** (PNT). The security problem is the reason why these two theorems are so important: 1) Every integer has a unique representation as a product of primes, and 2) the number of primes is very dense (their are a very large number of them, the density is proportional to the log of their number). Security reduces to the “needle in the haystack problem,” the cost of a search. A more formal way to measure the density is known as Shannon entropy, couched in terms of the expected value of the log-probability of events “What is the probability of finding a prime between $N$ and $2N$?”

### 1.2.4 Lec 5: Greatest common divisor (Euclidean algorithm)

The *Euclidean algorithm* is a method to find the greatest common divisor (GCD) $k$ between two integers $n, m$, denoted $k = \text{gcd}(n, m)$, where $n, m, k \in \mathbb{N}$. For example $15 = \text{gcd}(30, 105)$ since when factored 

$$(30, 105) = (2 \cdot 3 \cdot 5, 7 \cdot 3 \cdot 5) = 3 \cdot 5 \cdot (2, 7) = 15 \cdot (2, 7).$$

The Euclidean algorithm was known to the Chinese (i.e., not discovered by Euclid) (Stillwell, 2010, p. 41).

**Why is the GCD important?** Computing the GCD is simple, whereas a full factoring is extremely expensive. The GCD is important, precisely because of the fundamental difficulty of factoring large integers into their primes. This utility surfaces when the two numbers are composed of very large primes. When two integers have no common factors they are said to be coprime, thus their GCD is 1. The ratio of two integers which are coprime is automatically in reduced form (they have no common factors).

For example $4/2 \in \mathbb{Q}$ is not reduced since $2 = \text{gcd}(4, 2)$. Canceling out the common factor 2, gives the reduced form $2/1 = 2 \in \mathbb{N}$. Thus if we wish to form the ratio of two integers, first compute the gcd, and remove it from the two numbers, to form the ratio. This assures the rational number is in its reduced form. If the GCD were $10^3$ digits it is obvious that the common factor must be removed to greatly simplify further computation.

An example: Take the two integers $[873, 582]$. In factored form these are $[\pi_{25} \cdot 3^2, \pi_{25} \cdot 3 \cdot 2]$. Given the factors, we see that the largest common factor is $\pi_{25} \cdot 3 = 291 (\pi_{25} = 97)$. When we take the ratio of the two numbers this common factor cancels

$$\frac{873}{582} = \frac{\pi_{25} \cdot 3}{\pi_{25} \cdot 2} = \frac{3}{2} = 1.5.$$  

Of course if we divide 582 into 873 this we will numerically obtain the answer $1.5 \in \mathbb{F}$.

If the common factor is large ($\pi_{25}$ in this example), a floating point number in $\mathbb{F}$ is returned, since all floating point numbers are in $\mathbb{F}$. But due to rounding errors, it may not be $3/2$. As an example, in Matlab/Octave $\text{rat}(3/2) = 2 + 1/(-2)$, due to IEEE floating point rounding. One would expect $\text{rat}(3/2) = 1 + 1/2$. To obtain the precise answer in $\mathbb{F}$, we need to remove the the GCD. Removing large common factors, without actually factoring the two numbers, has obvious practical utility. However this breaks down due to numerical issues. For example $\text{rat}(582, 97) = 582$ (the wrong answer) yet $\text{factor}(582) = 2 \cdot 3 \cdot 97$ and $\text{factor}(873) = 3 \cdot 3 \cdot 97$ (the correct answers). If we remove factors 2 or

---

36https://streamcomputing.eu/blog/2012-07-16/how-expensive-is-an-operation-on-a-cpu/
37When I understand this better, I’ll do a better job of explaining it.
3, we get the correct answer: \( \text{rat}(582/3,97) = 194=2*97, \text{rat}(582/2,97) = 291 = 3*97 \). This is the nature of rounding error in the IEEE processor.

### Greatest Common Divisor: \( k=gcd(m,n) \)

- **Examples** \((m, n, k, Z)\):
  - \( gcd(13*5,11*5) = 5 \) (The common 5 is the gcd)
  - \( gcd(13*10,11*10) = 10 \) (The gcd\((130,110) = 10 = 2*5, \) is not prime)
  - \( gcd(1234,1024) = 2 \) \((1234=2*617, 1024=2^{10})\)
  - \( gcd(\pi_k \pi_m, \pi_k \pi_n) = \pi_k \)
  - \( k=gcd(m,n) \) is the part that cancels in the fraction \( m/n \in F \)
  - \( m/gcd(m,n) \in \mathbb{Z} \)
- **Coprimes** \((m \perp n)\) are numbers with no common factors: i.e., \( gcd(m,n)=1 \)
  - The gcd of two primes is always 1: \( gcd(13,11) = 1, gcd(\pi_m, \pi_n)=1 \) \((m \neq n)\)
  - \( -m = 7 \cdot 13, n = 5 \cdot 19 \Rightarrow (7 \cdot 13) \perp (5 \cdot 19) \)
  - If \( m \perp n \) then \( gcd(m,n) = 1 \)
  - If \( gcd(m,n) = 1 \) then \( m \perp n \)
- **The GCD may be extended to polynomials:** e.g., \( gcd(ax^2+bx+c, \alpha x^2+\beta x+\gamma) \)
  - \( gcd((x-3)(x-4), (x-3)(x-5))= (x-3) \)
  - \( gcd(x^2-7x+12, 3(x^2-8x+15))= 3(x-3) \)
  - \( gcd(x^2-7x+12, (3x^2-24x+45)=3(x-3) \)
  - \( gcd((x-2\pi)(x-4), (x-2\pi)(x-5) )=(x-2\pi) \) \( \) (Needs long division)

**Figure 1.6:** The Euclidean algorithm for finding the GCD of two numbers is one of the oldest algorithms in mathematics, and is highly relevant today. It is both powerful and simple. It was used by the Chinese during the Han dynasty (Stillwell, 2010, p. 70) for reducing fractions. It may be used to find pairs of integers that are coprime (their gcd must be 1), and it may be used to identify factors of polynomials by long division. It has an important sister algorithm called the continued fraction algorithm (CFA), that is so similar in concept that Gauss referred to the Euclidean algorithm as the “continued fraction algorithm” (Stillwell, 2010, p. 48).

**Euclidean algorithm:** The algorithm is best explained by a trivial example: Let the two numbers be 6,9. At each step the smaller number (6) is subtracted from the larger (9) and the difference (the remainder) and the smaller numbers are saved. This process continues until the two resulting numbers are equal, at which point the GCD equals that final number. If we were to take one more step, the final numbers would be the gcd and zero. For our example step 1 gives 9-6=3, leaving 6 and 3. Step 2 gives 6-3=3 and 3. Since the two numbers are the same, the GCD=3. If we take one more difference we obtain (3,0). We can easily verify this result since this example is easily factored (e.g., \( 3 \cdot 3, 3 \cdot 2 \)) = \( 3(3,2)\). It may be numerically verified using the Matlab/Octave GCD command \( gcd(6,9) \), which returns 3.

**Matrix method:** The GCD may be written a matrix recursion given the starting vector \( (m_0,n_0)^T \). The recursion is then

\[
\begin{bmatrix}
m_{k+1} \\
n_{k+1}
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
m_{k} \\
n_{k}
\end{bmatrix}
\]

This recursion continues until \( m_{k+1} < n_{k+1} \), at which point \( m \) and \( n \) must be swapped. Because the operation depends on the output, this is a nonlinear recursion (Postulate P1 (Linear/nonlinear) of Section 3.5.1, p. 160).

The direct method is inefficient because in recursively subtract \( n \) many times until the resulting \( m \) is less than \( n \), as shown in Fig. 2.1 (p. 131). It also must test for \( m < n \) at each iteration, and then swap \( m \) and \( n \) once that condition is met. This recursion is repeated until \( m_{k+1} = 0 \). At that stage the
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GCD is \( k+1 \). Figure 2.1, along with the above matrix relation, give the best insight into the Euclidean Algorithm, but at the cost of low efficiency.

Below is a Matlab/Octave code to find \( k = \gcd(m, n) \) based on the strict definition of the EA as described by Stillwell (2010):

```matlab
function k = gcd(m,n)
    while m ~= 0
        A=m; B=n;
        m=max(A,B); n=min(A,B); //m>n
        m=m-n;
    end
    k = m;
end
```

This program keeps looping until \( m = 0 \). An much more efficient method is described in Section 2.1.3, p. 130, using the `floor()` function, which we call division with rounding.

1.2.5 Lec 6: Continued fraction algorithm

In its simplest form, the Continued fraction algorithm (CFA) starts from a single real decimal number \( x \in \mathbb{R} \), and recursively expands it as a fraction \( x \in \mathbb{F} \). Thus the CFA It is used in finding rational approximations to any real number. For example, \( \pi \approx \frac{22}{7} \). The GCD (the Euclidean algorithm) on the other hand operates on a pair of integers \( m, n \in \mathbb{N} \) and finds their greatest common divisor \( k \in \mathbb{N} \). Thus \( m/k, n/k \in \mathbb{N} \), reducing the ratio to its irreducible form, since \( m/k \perp n/k \).

Despite this seeming large difference in the two algorithms, apparently the CFA is closely related to the Euclidean algorithm (the GCD), so closely in fact, that Gauss referred to the Euclidean algorithm as the Continued fraction algorithm (Stillwell, 2010, P. 48). At first glance it is not clear why Gauss would call the CFA the Euclidean algorithm. One must assume that Gauss had some deeper insight into this relationship. If so, that insight would be valuable to understand. 38

In the following we refine the description of the CFA and give examples that go beyond the simple cases of expanding numbers. The CFA of any positive number, say \( x_o \in \mathbb{R}^+ \), is defined as follows:

1. Start with \( n = 0 \) and a positive input target \( x_o \in \mathbb{R}^+ \).
2. Define \( a_n = \text{round}(x_n) \), which rounds to the nearest integer.
3. Define \( r_n = x_n - a_n \), thus \(-0.5 \leq r_n \leq 0.5\). If \( r_n = 0 \), the recursion terminates.
4. Define \( x_{n+1} = 1/r_n \) and return to step 2, with \( n = n + 1 \).

The recursion may continue to any desired accuracy, since convergence is guaranteed.

An example: Let \( x_o \equiv \pi \approx 3.14159 \ldots \) Thus \( a_0 = 3, r_o = 0.14159, x_1 = 7.065 \approx 1/r_o, \) and \( a_1 = 7 \).

If we were to stop here we would have

\[
\tilde{\pi}_2 \approx 3 + \frac{1}{7 + 0.0625 \ldots} \approx 3 + \frac{1}{7} = \frac{22}{7},
\]

(1.5)

This approximation of \( \pi \approx \frac{22}{7} \) has a relative error of 0.04% \( \frac{22/7 - \pi}{\pi} = 4 \times 10^{-4} \).

For the next approximation we continue by reciprocating the remainder \( 1/0.0625 \approx 15.9966 \) which rounds to 16 giving a negative remainder of \( \approx -1/300 \), resulting in the second approximation

\[
\tilde{\pi}_3 \approx 3 + 1/(7 + 1/16) = 3 + 16/(7 \cdot 16 + 1) = 3 + 16/113 = 355/113.
\]

Note that if we had truncated 15.9966 to 15, the remainder would have been much larger, but positive, resulting in a much less accurate rational approximation. It follows that there can be a dramatic difference, depending on the rounding scheme, which must be specified.

\[38\]The resolution of this interrelationship is still unresolved.
**Rational approximation examples**

\[
\hat{\pi}_2 = \frac{22}{7} = [3,.7] \approx \hat{\pi}_2 + O(1.3 \times 10^{-3})
\]

\[
\hat{\pi}_3 = \frac{355}{113} = [3,.7,16] \approx \hat{\pi}_3 + O(2.7 \times 10^{-7})
\]

\[
\hat{\pi}_4 = \frac{10^4 348}{33215} = [3,.7,16,249] \approx \hat{\pi}_4 + O(3.3 \times 10^{-10})
\]

Figure 1.7: The expansion of π to various orders, using the CFA, along with the order of the error of each rational approximation. For example \(\hat{\pi}_2 = 22/7\) has an absolute error \(||22/7 - \pi||\) of about 0.13%.

**Notation:** Writing out all the fractions can become tedious. For example, expanding \(e = 2.7183\) using the Matlab/Octave command `rat(exp(1))` gives the approximation

\[
\hat{\pi}_6 = 3 + 1/(−4 + 1/(2 + 1/(5 + 1/(−2 + 1/(−7)))))) - O \left(1.75 \times 10^{-6}\right).
\]

A compact notation for this these coefficients of the CFA is \(\hat{\pi}_6 = [3.,-4,2,5,-2,-7]\).\(^{39}\) Note that the leading integer part may be noted by an optional decimal point.\(^{40}\) If the process is carried further, the values of \(a_n \in \mathbb{N}\) give increasingly more accurate rational approximations.

**Rounding schemes:** In Matlab/Octave there are four different rounding function: `round()`, `fix()`, `floor()`, `ceil()`. If the rounding-down (`floor`) is used \(\hat{\pi}_{12} = [3.,7,15,1,292,1,1,1,2,1,3,1]\) whereas true rounding to the nearest integer (`round`) gives \(\hat{\pi}_8 = [3.,7,16,-294,3,-4,5,-15]\). Thus `round` introduces negative coefficients each time a number rounds up to the nearest integer.

**Exercise:** Based on several examples, which rounding scheme is the most accurate? Explain why.\(^{41}\)

**Sol:** Rounding will give a smaller remainder at each iteration, resulting in a smaller net error.

When the CFA is applied and the expansion terminates \((r_n = 0)\), the target is rational. When the expansion does not terminate (which is not always easy to determine), the number is irrational. Thus the CFA has important theoretical applications regarding irrational numbers. You may try this yourself using Matlab’s `rats(pi)` command.

One of the useful things about the procedure, besides its being so simple, are its generalizations, one of which will be discussed in Section 2.1.3 (p. 130).

**Symmetry:** A **continued fraction expansion** can have a high degree of symmetry. For example, the CFA of

\[
R_1 \equiv \frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1+\cdots}} = 1.618033988749895\cdots,
\]

Here \(a_n\) in the CFA is always 1 \((R_1 \equiv [1.,1,1,\cdots])\), thus the sequence will not terminate, proving that \(\sqrt{5} \in \mathbb{R}\). A related example is \(R_2 \equiv \text{rat}(1+\text{sqrt}(2))\), which gives \(R_2 = [2.,2,2,\cdots]\).

When expanding a target irrational number \((x_o \in \mathbb{R})\), and the CFA is truncated, the resulting rational fraction approximates the irrational target. For the example above, if we truncate at three coefficients \(([1;1,1])\) we obtain

\[
1 + \frac{1}{1 + \frac{1}{1+o}} = 1 + 1/2 = 3/2 = 1.5 = \frac{1 + \sqrt{5}}{2} + 0.118\cdots.
\]

\(^{39}\)\(\pi - 3 = [-4,2,5,\cdots]\).

\(^{40}\)Unfortunately Matlab/Octave does not support the bracket notation.
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Truncation after six steps gives

\[ [1, 1, 1, 1, 1, 1] = 13/8 \approx 1.6250 = \frac{1 + \sqrt{5}}{2} + 0.0070 \cdots \]

Because all the coefficients are 1, this example converges very slowly. When the coefficients are large (i.e., remainder small), the convergence will be faster. The expansion of \( \pi \) is an example of faster convergence.

**In summary:** Every rational number \( m/n \in \mathbb{F} \), with \( m > n > 1 \), may be uniquely expanded as a continued fraction, with coefficients \( a_k \) determined using the CFA. When the target number is irrational \( (x_0 \in \mathbb{Q}) \), the CFA does not terminate, thus each step produces a more accurate rational approximation, converging in the limit as \( n \to \infty \).

Thus the CFA expansion is an algorithm that can, in theory, determine when the target is rational, but with an important caveat: one must determine if the expansion terminates. This may not be obvious. The fraction \( 1/3 = 0.33333 \cdots \) is an example of such a target, where the CFA terminates yet the fraction repeats. It must be that

\[ 1/3 = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \cdots . \]

Here \( 3*3=9 \). As a second example

\[ 41/7 = 0.142857142857142857142857 \cdots = 142857 \times 10^{-6} + 142857 \times 10^{-12} + \cdots \]

Note that 142857 * 7 = 999999. This also works for 1/11 = 0.090909 \cdots and 11 * 0.090909 = 999999. We might conclude that when the sequence of digits repeats, the sequence is predictable, thus it must be rational.

**WEEK 3**

1.2.6 **Lec 7: Pythagorean triplets (Euclid’s formula)**

**Euclid’s formula** is a method for finding three integer lengths \( [a, b, c] \in \mathbb{N} \), that satisfy Eq. 1.1. It is important to ask “Which set are the lengths \( [a,b,c] \) drawn from?” There is a huge difference, both practical and theoretical, if they are from the real numbers \( \mathbb{R} \), or the counting numbers \( \mathbb{N} \). Given \( p, q \in \mathbb{N} \) with \( p > q \), the three lengths \( [a, b, c] \in \mathbb{N} \) of Eq. 1.1 are given by

\[ a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2. \quad (1.7) \]

This result may be directly verified, since

\[ [p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2 \]

or

\[ p^4 + q^4 + 2p^2q^2 = p^4 + q^4 - 2p^2q^2 + 4p^2q^2. \]

Thus, Eq. 1.7 is easily proven, once given. Deriving Euclid’s formula is obviously more difficult.

A well known example is the right triangle depicted in Fig. 1.8, defined by the integers \( [3, 4, 5] \in \mathbb{N} \), having angles \( [0.54, 0.65, \pi/2] \) [rad], which satisfies Eq. 1.1 (p. 15). As quantified by Euclid’s formula (Eq. 1.7), there are an infinite number of Pythagorean triplets (PTs). Furthermore the seemingly simple triangle, having angles of \( [30, 60, 90] \in \mathbb{N} \) [deg] (i.e., \( [\pi/6, \pi/3, \pi/2] \in \mathbb{I} \) [rad]), has one irrational (\( I \)) length \( ([1, \sqrt{3}, 2]) \).

\(^{41}\)Taking the Fourier transform of the target number, represented as a sequence, could help to identify an underlying periodic component. The number \( 1/7 \leftrightarrow [[1, 4, 2, 8, 5, 7]], \) has a 50 [dB] notch at 0.8\( \pi \) [rad] due to its 6 digit periodicity, carried to 15 digits (Matlab/Octave maximum precision), Hamming windowed, and zero padded to 1024 samples.
1.2. STREAM 1: NUMBER SYSTEMS (10 LECTURES)

Figure 1.8: Beads on a string form perfect right triangles when number of beads on each side satisfy Eq. 1.1. For \( p = 2, q = 1 \) the sides are \([3, 4, 5]\).

<table>
<thead>
<tr>
<th>( a )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
<tr>
<td>65</td>
<td>97</td>
</tr>
<tr>
<td>319</td>
<td>481</td>
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<tr>
<td>2291</td>
<td>3541</td>
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<td>799</td>
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<td>481</td>
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<td>4961</td>
<td>8161</td>
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<td>45</td>
<td>75</td>
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<tr>
<td>1679</td>
<td>2929</td>
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<tr>
<td>161</td>
<td>289</td>
</tr>
<tr>
<td>1771</td>
<td>3229</td>
</tr>
<tr>
<td>56</td>
<td>106</td>
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</tbody>
</table>

Figure 1.9: "Plimpton-322" is a stone tablet from 1800 [BCE], displaying \( a \) and \( c \) values of the Pythagorean triplets \([a, b, c]\), with the property \( b = \sqrt{c^2 - a^2} \in \mathbb{N} \). Several of the \( c \) values are primes, but not the \( a \) values. The stone is item 322 (item 3 from 1922), from the collection of George A. Plimpton. –Stillwell (2010)

The technique for proving Euclid's formula for PTs \([a, b, c] \in \mathbb{Q}\), derived in Fig. 2.3 (p. 134) of Section 2.1.4, is much more interesting than the PTs themselves.

The set from which the lengths \([a, b, c]\) are drawn was not missed by the Indians, Chinese, Egyptians, Mesopotamians, Greeks, etc. Any equation whose solution is based on integers is called a Diophantine equation, named after the Greek mathematician Diophantus of Alexandria (c250 CE).

A stone tablet having the numbers engraved on it, as shown in Table 1.9, was discovered in Mesopotamia, from the 19th century [BCE], and cataloged in 1922 by George Plimpton.\(^\text{42}\) These numbers are \( a \) and \( c \) pairs from PTs \([a, b, c]\). Given this discovery, it is clear that the Pythagoreans were following those, long before them. Recently a second similar stone, dating between 350 and 50 [BCE] has been reported, that indicates early calculus on the orbit of Jupiter’s moons.\(^\text{43}\)

1.2.7 Lec 8: Pell’s Equation

Pell’s equation

\[ x^2 - Ny^2 = 1, \tag{1.8} \]


with non-square $N \in \mathbb{N}$ specified and $a, b \in \mathbb{N}$ unknown, is related to the Euclidean algorithm (Stillwell, 2010, 48). For example, with $N = 2$, one solution is $a = 17, b = 12$ ($17^2 - 2 \cdot 12^2 = 1$). This equation has a long history (Stillwell, 2010).

A 2x2 matrix recursion algorithm was used by the Pythagoreans to investigate the $\sqrt{2}$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}. \quad (1.9)$$

Starting with $[x_o, y_o]^T = [1, 0]^T$, results in solutions of Pell’s equations (Stillwell, 2010, p. 44). Their approach was likely motivated by the Euclidean algorithm (GCD, p. 34), since $y_n/x_n \rightarrow \sqrt{2}$ (Stillwell, 2010, p. 37, 55). Note that this is a composition method, of 2x2 matrices, since the output of one matrix multiply is the input to the next.

**Asian solutions:** The first solution of Pell’s equation was published by Brahmagupta (c628), who independently discovered the equation (Stillwell, 2010, p. 46). Brahmagupta’s novel solution introduced a different composition method (Stillwell, 2010, p. 69), and like the Greek result, these solutions were incomplete.

Then in 1150CE, Bhâskara II obtained solutions using Eq. 1.9 (Stillwell, 2010, p.69). This is the solution method we shall explore here, as summarized in Fig. 1.10.

The best way to see how this recursion results in solutions to Pell’s equation, is by example. Initializing the recursion with the trivial solution $[x_o, y_o]^T = [1, 0]^T$, gives

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 1^2 - 2 \cdot 1^2 = -1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad 3^2 - 2 \cdot 2^2 = 1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \quad (7)^2 - 2 \cdot (5)^2 = -1$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 7 \end{bmatrix}, \quad 17^2 - 2 \cdot 12^2 = 1$$

Thus the recursion results in a modified version of Pell’s equation

$$x_n^2 - 2y_n^2 = (-1)^n,$$

where only even values of $n$ are solutions. This sign change had no effect on the Pythagorean goal, since they only cared about $y_n/x_n \rightarrow \sqrt{2}$.

**Solution to Pell’s equation:** By multiplying the matrix by $1j$, all the solutions to Pell’s equation are determined. This solution is shown in Fig. 1.10 (p. 41) for the case of $N = 2$, and again for Eq. D.1 (p. 195), for $N = 3$. The math is straightforward and is easily verified using Matlab/Octave. From Fig. 1.10 we can see that every output this slightly modified matrix recursion gives solutions to Pell’s equation (Eq. 1.8). The $1j$ factor corrects the alternation in sign, so every iteration yields a solution. For $n = 0$ (the initial solution) $[x_o, y_o] = [1, 0]$, $[x_1, y_1] = j[1, 1]$, and $[x_2, y_2] = -[3, 2]$. These are easily computed by this recursion, and easily checked on a hand calculator (or using Matlab/Octave).

At each iteration, the ratio $x_n/y_n$ approaches $\sqrt{2}$ with increasing accuracy, coupling it to the CFA. The value of $41/29 \approx \sqrt{2}$, with a relative error of $<0.03\%$. The solution for $N = 3$ is discussed in Eq. D.1 (p. 195).
1.2. STREAM 1: NUMBER SYSTEMS (10 LECTURES)

- Case of $N = 2$ & $[x_0, y_0]^T = [1, 0]^T$
  
  Note: $x_n^2 - 2y_n^2 = 1$, $x_n/y_n \to \sqrt{2}$

\[
\begin{bmatrix}
x_1 \\
y_1
\end{bmatrix} = j \begin{bmatrix} 1 \\ 1 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow j^2 - 2 \cdot j^2 = 1
\]

\[
\begin{bmatrix}
x_2 \\
y_2
\end{bmatrix} = j^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow 3^2 - 2 \cdot 2^2 = 1
\]

\[
\begin{bmatrix}
x_3 \\
y_3
\end{bmatrix} = j^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} j^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} (7j)^2 - 2 \cdot (5j)^2 = 1
\]

\[
\begin{bmatrix}
x_4 \\
y_4
\end{bmatrix} = \begin{bmatrix} 17 \\ 12 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} j^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} 17^2 - 2 \cdot 12^2 = 1
\]

\[
\begin{bmatrix}
x_5 \\
y_5
\end{bmatrix} = j \begin{bmatrix} 41 \\ 29 \end{bmatrix} = j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} (41j)^2 - 2 \cdot (29j)^2 = 1
\]

Figure 1.10: This summarizes the solution of Pell’s equation for $N = 2$ using a slightly modiﬁed matrix recursion. Note that $x_n/y_n \to \sqrt{2}$ as $n \to \infty$, which was what the Pythagoreans were pursuing.

Relations to digital signal processing: Today we recognize Eq. 1.9 as a difference equation, which is a pre-limit (pre Stream 3) form of differential equation. The Greek 2x2 form is an early precursor to 17th and 18th century developments in linear algebra. Thus the Greek’s recursive solution for the $\sqrt{2}$ and Bhaskara’s (1030 CE) solution of Pell’s equation, is an early precursor to discrete-time processing, as well as to calculus.

There are similarities between Pell’s Equation and the Pythagorean theorem. As we shall see in Chapter 2, Pell’s equation is related to the geometry of a hyperbola, just as the Pythagorean equation is related to the geometry of a circle. One might wonder if there is a Euclidean formula for the solutions of Pell’s Equations, since these are all conic sections with closely related geometry.

WEEK 4

1.2.8 Lec 9: Fibonacci sequence

Another classic problem, formulated by the Chinese, was the Fibonacci sequence, generated by the relation

\[ f_{n+1} = f_n + f_{n-1}. \]  (1.10)

Here the next number $f_{n+1}$ is the sum of the previous two. If we start from $[0, 1]$, this difference equation leads to the Fibonacci sequence $f_n = [0, 1, 2, 3, 5, 8, 13, \ldots]$. Alternatively, if we define $y_{n+1} = x_n$, then Eq. 1.10 may be represented as the recursion of a 2x2 matrix equation

\[
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}. \]  (1.11)

The correspondence is easily verified. Starting with $[x_n, y_n]^T = [0, 1]^T$ we obtain for the first few steps

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix}
2 \\
1
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix}
3 \\
2
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \ldots
\]
CHAPTER 1. INTRODUCTION

From the above \( x_n = [0, 1, 1, 2, 3, 5, \ldots] \) is the Fibonacci sequence since the next \( x_n \) is the sum of the previous two and the next \( y_n \) is \( x_n \).

The mean-Fibonacci sequence: Suppose that the Fibonacci sequence recursion is replace by the mean of the last two values, namely let

\[
f_{n+1} = \frac{f_n + f_{n-1}}{2}.
\] (1.12)

This seems like a small change. But how does the solution differ? Two answer this question it is helpful to look at the corresponding 2x2 matrix.

Exercise: Find the 2x2 matrix corresponding to Eq. 1.12. Sol: In this case we find

\[
\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.
\] (1.13)

The eigenvalues of this matrix are the roots of the binomial equation \(-\lambda(1 - \lambda/2) - 1/2 = 0\) which has eigenvalues \([1, -1/2]\). Thus \([x_n, y_n] = [1, 1]^T\) and \([x_n, y_n] = [1, 1]^T\) are each solutions

Exercise: Starting from \([x_n, y_n]^T = [1, 0]^T\) compute the first 5 values of \([x_n, y_n]^T\).

Exercise: Show that the solution to Eq. 1.12 is bounded, unlike that of the Fibonacci sequence, which diverges. Explain what is going on. Sol: By studying the eigen values of Eq. 1.13 one finds that the steady state solution approaches 1. Namely \(f_n \to 1 = (f_{n-1} + f_{n-2})/2\) is the solution, as \(n \to \infty\). Namely the average of the last two values must approach 1 for large \(n\). Sol: Use the formula for the generalized diagonalization of a matrix to find the general solution of the man-Fibonacci sequence.

Summary: The GCD (Euclidean algorithm), Pell’s equation and the Fibonacci sequence may all be written as compositions of 2x2 matrices. Thus Pell’s equation and the Fibonacci sequence are special cases of 2x2 matrix composition

\[
\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.
\]

This is an important and common thread of these early mathematical findings. It will turn out that this form plays a special role in physics, mathematics and engineering. The first several thousands of years of mathematical trial and error research, set the stage for this breakthrough, but it took a long time to fully appreciate the unity of these several contributions.

1.2.9 Lec 10: Exam I (In class)

1.3 Algebraic Equations: Stream 2

The era from 1600-1850 (Fig. 1.17, p. 76) produced a stream of fundamental theorems. A few of the individuals who played a notable role in this development, in chronological (birth) order, include Galileo, Mersenne, Newton, d’Alembert, Fermat, Huygens, Descartes and Helmholtz. These individuals were some of the first to develop the basic ideas, in various forms, that were then later reworked into the proofs, that today we recognize as The fundamental theorems of mathematics.
1.3.1 Lec 11 Algebra and geometry as physics

Following Stillwell's history of mathematics, Stream 2 is geometry, which led to the merging of Euclid’s geometrical methods and the 9th century development of algebra by al-Khwarizmi (830 CE). This integration of ideas lead Descartes and Fermat to develop analytic geometry. While not entirely a unique and novel idea, it was late in coming, given what was known at that time.

The mathematics up to the time of the Greeks, documented and formalized by Euclid, served students of mathematics for more than two thousand years. Algebra and geometry were, at first, independent lines of thought. When merged, the focus returned to the Pythagorean theorem, generalized as analytic conic sections rather than as geometry in Euclid’s Elements. With the introduction of Algebra, numbers, rather than lines, could be used to represent a geometrical length. Thus the appreciation for geometry grew, given the addition of the rigorous analysis using numbers.

Physics inspires algebraic mathematics: The Chinese used music, art, and navigation to drive mathematics. Unfortunately much of our knowledge has been handed down either as artifacts, such as musical bells and tools, or mathematical relationships documented, but not created by, scholars such as Euclid, Archimedes, Diophantus, and perhaps Brahmagupta. With the invention of algebra by al-Khwārizmī (830CE) this paradigm did not shift, rather it expanded. During the 16th and 17th century, it had becoming clear that differential equations (DEs), such as the wave equation, can characterize a law of nature, at a single point in space and time. This principle was not obvious. A desire to understand motions of objects and planets precipitated many new discoveries. This period is organized in Fig. 1.4. Galileo investigated gravity and invented the telescope. For example the law of gravity (first formulated by Galileo to explain the dropping to two objects of different masses) must obey conservation of energy. Kepler investigated the motion of the planets. While Kepler was the first to appreciate that the planets were described by ellipses, it seems he under-appreciate the significance of this finding, and continued with his epicycle model of the planets. Newton (c1687) went on to show that there must be a gravitational potential between two masses \(m_1, m_2\), of the form

\[
\phi(r) \propto \frac{m_1 m_2}{r},
\]

where \(r = |x_1 - x_2|\) is the Euclidean distance between the two point masses at locations \(x_1\) and \(x_2\). Note that this a power series, but with exponent of \(-1\), which is a pole with an RoC of zero. This is the first appearance of the Laurent series, via Newton’s formula for gravitational potential. Using algebra and his calculus, Newton formalized the equation of gravity, forces and motion (Newton’s three laws) and showed that Kepler’s discovery of planetary elliptical motion naturally follows from these laws. With the discovery of Uranus (1781) “Kepler’s theory was ruined.” (Stillwell, 2010, p. 23).

Once Newton proposed the basic laws of gravity, he proceed to calculate, for the first time, the speed of sound. This required some form of the wave equation

\[
\frac{\partial^2}{\partial x^2} p(x, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(x, t),
\]

a key equation in mathematical physics. The speed of sound is \(c = 343 \text{[m/s]}\), which is a function of the density \(\rho_0 = 1.12 \text{[kg/m}^3]\) and the dynamic stiffness \(\eta P_o\) of air.\(^{44}\)

If we substitute for the pressure

\[
p(x, t) = P_o(s) e^{2\pi(f t \pm k x)},
\]

where \(t\) is time and \(x\) is position, we find that \(k = 2\pi/\lambda = 2\pi f/c\), because \(k = 2\pi/\lambda\).

This classic relation \(\lambda f = c\) is deceivingly simple, thus confusing, because \(k = 2\pi/\lambda\) becomes complex (has both real and imaginary parts) in dispersive media (e.g., acoustic waves in tubes when

\[^{44}\]c = \sqrt{\eta P_o/\rho_o}, \eta = c_p/c_v = 1.4\] is the ratio of two thermodynamic constants, and \(P_o = 10^5 \text{[Pa]}\) is the barometric pressure of air.
losses are considered (Kirchhoff, 1868); electron waves in silicon crystals (Brillouin, 1953). In these more general cases, \( k(f) = 2\pi f/c \) must be replaced with a complex analytic function \( \kappa(s) \) of \( s \), i.e.,

\[
p(x, t) = P_0(s)e^{st} e^{\pm \kappa(s)x},
\]

such that the wave number \( \kappa(s) \) is a complex analytic function of the Laplace frequency \( s \). This is because electron “waves” in a dispersive medium (e.g., a semi-conductor, such as silicon), are “filtered” in their magnitude and phase. This filter acts as a transmission line, or a complex “wave-filter,” forcing the wavelength to be a complex function of frequency. This view is elegantly explained by Brillouin (1953, Chap. 1), in an historical context. Acoustics provides a rich source of examples (Morse, 1948; Beranek, 1954; Beranek and Mellow, 2012).

Newton’s *Principia* was finally published in 1687, and the general solution to Newton’s wave equation [i.e., \( p(x, t) = G(t \pm x/c) \)], where \( G \) is any function, was first published 60 years later by d’Alembert (c1747), which showed that for sounds of a single frequency, the wavelength \( \lambda \) and frequency \( f \) were related by

\[
f \lambda = c.
\]

Today d’Alembert’s analytic wave solution can be written as Eq. 1.16 with a real wave number \( k = 1/\lambda \) [m\(^{-1}\)]. This formulation led to the frequency domain concept of Fourier analysis, based on the linearity (i.e., superposition) property of the wave equation (Postulate P2: Lec. 1.3.11, p. 73).

While Newton’s value for the speed of sound in air \( c_0 \) was incorrect by the thermodynamic constant \( \sqrt{\gamma} \), a problem that would take more than two hundred years to resolve, his success was important because it quantified the physics behind the speed of sound, and demonstrated that momentum \( mv \), not mass \( m \), was transported by the wave. His concept was correct, and his formulation using algebra and calculus represented a milestone in science. If the speed of sound in-vacuo is zero. In air, assuming no visco-elastic losses, it is constant (i.e., \( \kappa = \sqrt{\gamma\rho_0/c^2} \)). When including losses the wave number becomes a complex function of frequency, leading to Eq. 1.17. In periodic structures, again the wave number becomes complex due to diffraction, as commonly observed in optics (e.g., diffraction gratings) and acoustics. Thus Eq. 1.18 only holds for the most simple cases, but in general it must be considered as a complex analytic function of \( s \), as \( \kappa(s) \) in Eq. 1.17.

The corresponding discovery for the formula for the speed of light was made 174 years after Principia, by Maxwell (c1861). Maxwell’s formulation also required great ingenuity, as it was necessary to hypothesize an experimentally unmeasured term in his equations, to get the mathematics to correctly predict the speed of light.

**The first Algebra:** Prior to the invention of algebra, people worked out problems as sentences using an obtuse description of the problem (Stillwell, 2010, p. 93). Algebra solved this problem, resulting in a compact language of mathematics, where numbers are represented as abstract symbols (e.g., \( x \) and \( \alpha \)). The problem to be solved could be formulated in terms of sums of powers of smaller terms, the most common being powers of some independent variable (i.e., time or frequency). Today we call such an expression a **polynomial** of degree \( n \)

\[
P_n(z) \equiv z^n + a_{n-1} z^{n-1} + \cdots + a_0 z^0 = \sum_{k=0}^{n} a_k z^k = \prod_{k=0}^{n} (z - z_k).
\]

Here we have let \( a_n = 1 \). The coefficient \( a_n \) cannot be zero, or the polynomial would not be of degree \( n \). A solution is to force \( a_n = 1 \), since this simplifies the expression, and does not change the roots.

The key question is “What values of \( z = z_k \) result in \( P_n(z_k) = 0 \)” In other words, what are the roots \( z_k \) of the polynomial? The answer to this question consumed thousands of years, with intense efforts by many aspiring mathematicians. In the earliest attempts, it was a competition to evaluate mathematical acumen. Results were held as a secret to the death bed. It would be fair to view this

\[ A \text{ polynomial has zeros at its roots. An infinite series has poles at its roots.} \]
Chronological history post 17th century

17th Newton 1642-1727, Bernoulli, Johann 1667-1748
18th Bernoulli, Daniel, Cauchy 1789-1857, Euler 1707-83, d’Alembert 1717-83, Gauss 1777-1855
19th Kirchhoff 1824-87, Helmholtz 1821-1894, Riemann 1826-1866, Maxwell 1831-1879, Rayleigh 1842-1919, Heaviside 1850-1925, Poincare 1854-1912,

Figure 1.11: Time-line of the three centuries from the 18th to 20th CE. This was one of the most productive of all times, perhaps starting with the deep work of von Helmholtz, educated an experienced as a military surgeon, who mastered classical music, acoustics, physiology, vision, hearing (Helmholtz, 1863b), and, most important of all, mathematics. Kirchhoff frequently expanded on Helmholtz’s contributions. It is reported that Lord Rayleigh learned German so he could read Helmholtz’s great works. The history during this time is complex. For example, Lord Kelvin wrote a letter to Stokes, suggesting that Stokes try to prove what is today known as “Stokes theorem.” As a result, Stokes posted a reward (the Smith Prize), searching for a prove of “Lord Kelvin’s theorem,” which was finally proved by Hankel (1839-73) (https://en.wikipedia.org/wiki/Hermann_Hankel). Many new concepts were being proved and appreciated over this productive period. In 1863-65, Maxwell published his famous equations, followed by a reformating in modern vector notation by Heaviside, Gibbs and Hertz. The vertical red lines connect mentor-student relationships. This figure should put to rest the idea that ones best work is done in the early years. Many of these scientists were fully productive to the end of old age. Those that were not, died early, due to poor health or accidents. Figure 1.4 (p. 19) (Newton–Gauss) gives a closer look at the 15-18 CE, and Fig. 1.17 (p.76) (Bombelli–Einstein) for the full view from 16-20 CE. [fig:TimeLine19CE]

Finding roots of polynomials  The problem of factoring polynomials has a history more than a millennium in the making. While the quadratic (degree \(N = 2\)) was solved by the time of the Babylonians (i.e., the earliest recorded history of mathematics), the cubic solution was finally published by Cardano in 1545. The same year, Cardano’s student solved the quartic (\(N = 4\)). In 1826 (281 years later) it was proved that the quintic (\(N = 5\)) could not be factored by analytic methods.

As a concrete example we begin with trivial but important case of the quadratic

\[
P_2(x) = ax^2 + bx + c. \tag{1.20}
\]

The roots are those values of \(x\) such that \(P_2(x_k) = 0\). One of the first results (recorded by the Babylonians, c2000 BCE) was the factoring of this equation by completing the square (Stillwell, 2010,
CHAPTER 1. INTRODUCTION

p. 93). One may isolate $x$ by rewriting Eq. 1.20 as

$$\frac{1}{a} P_2(x) = (x + b/2a)^2 - (b/2a)^2 + c/a. \quad (1.21)$$

This is easily verified by expanding the squared term and canceling $(b/2a)^2$

$$\frac{1}{a} P_2(x) = [x^2 + (b/a)x + (b/2a)^2] - (b/2a)^2 + c/a.$$

Setting Eq. 1.21 to zero and solving for the two roots $x_\pm$, gives the quadratic formula\(^{46}\)

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (1.22)$$

If $b^2 - ac > 0$, the two roots are real ($x_\pm \in \mathbb{R}$). If it is exactly zero, the roots are identical. Otherwise, they are a complex pair.

In the important and common case that the coefficients are real and positive, and $b$ is small (i.e., $c > b^2$). Since $a \neq 0$, the polynomial may be divided by $a$, removing it from the equation. Thus we may set $a = 1$ with no loss of generality. In this special, but common case, the roots are given by\(^{47}\)

$$x_\pm = -b/2 \pm 2j\sqrt{c - (b/2)^2} \approx -b/2 \pm j\sqrt{c}.$$

No insight is gained by memorizing the quadratic formula (Eq. 1.22). On the other hand, an important concept is gained by learning Eq. 1.21, which can be very helpful when doing analysis. It seems obvious, that instead of memorizing Eq. 1.22, learn Eq. 1.21. Arguably, the factored form (Eq. 1.21) is easier to remember, as a method, rather than as a formula that must be memorized. Importantly, the term $b/2a$ has significance $[P_2(-b/2a) = c/a - (b/2a)^2]$, the sign of which determines if the nature of the roots (real vs. complex).

In third grade I learned the trick\(^{48}\)

$$9 \cdot n = (n - 1) \cdot 10 + (10 - n). \quad (1.23)$$

With this simple rule I did not need to depend on my memory for the 9 times tables. By expanding the above, one can see why it works: $9n = n10 - 10 + 10 - n = n(10 - 1)$. Learning an algorithm is much more powerful than memorization of the 9 times tables. How one thinks about a problem can have great impact.

**Working with polynomials in Matlab/Octave:** In Matlab/Octave there are five functions that work together as an important set, and you need to become familiar with these:

1. \texttt{re}\texttt{root}(A) Vector \(A = \{a_N, a_{N-1}, \ldots, a_0\}\) are the coefficients of polynomial \(P_n(z) = \sum_{n=0}^{N} a_n z^n\), with $N$ is the degree of the polynomial. It is convenient to force $a_N = 1$, corresponding to dividing the polynomial by this value, when it is not 1, guaranteeing it cannot be zero, as mentioned above. Define \(R = \{z_1, z_2, \ldots, z_n\}\) as the vector of roots, such that \texttt{polyval(A,R)=0}.

2. \texttt{y=polyval(A,x)} This evaluates the polynomial defined by vector \(A\) at vector values of \(x\), returning vector \(y(x)\).

3. \texttt{A=poly(R)} This is the inverse of \texttt{root()}, returning a vector of polynomial coefficients \(A\), given a vector of roots \(R\). Due to IEEE-754 scaling issues, this can give strange results, that are numerically correct, but withing the limits of IEEE-754 accuracy.

\(^{46}\)By direct substitution demonstrate that Eq. 1.22 is the solution of Eq. 1.20.

\(^{47}\)This is the case for mechanical and electrical circuits having small damping. Physically $b > 0$ is the damping coefficient and $\sqrt{c} > 0$ is the resonant frequency. One may then simplify the form as $x^2 + 2bx + c^2 = (x + b + \sqrt{c})(x + b - \sqrt{c})$.

\(^{48}\)E.g.: $9 \cdot 7 = (7 - 1) \cdot 10 + (10 - 7) = 60 + 3$ and $9 \cdot 3 = (3 - 1) \cdot 10 + (9 - 3) = 20 + 7$. As a check, note that the two terms $(n - 1)$ and $(10 - n)$, add to 9.
4. \([K,R]=\text{residue}(A,B)\): Given the ratio of two polynomials \(A, B\), \(\text{residue}(A,B)\) returns vectors \(K, R\) such that
\[
\frac{A(s)}{B(s)} = \sum_k \frac{K_k}{s-s_k}.
\]
where \(s_k\) are the roots of \(B\).

5. \(C=\text{conv}(A,B)\): Vector \(C\) contain the polynomial coefficients of the convolution of the two vector of coefficients of polynomials \(A, B\). For example \([1, 2, 1]=\text{conv}([1, 1], [1, 1])\).

The use of \(\text{residue()}\) will be discussed in Sect. 1.3.3 (p. 54), and again, in more detail, in Sect. 1.4.10 (p. 89).

**Analytic Series:** A polynomial of degree \(N\) has coefficients related to its derivatives. There is an important distinction, between \(N\) finite and \(N \to \infty\). When the degree of the polynomial is in finite (i.e., \(n=\infty\)), \(P_\infty(x), x, x_o, a_n \in \mathbb{R}\), the series is said to be analytic
\[
P(x) = \sum_{0}^{\infty} a_n(x-x_o)^n.
\]
As for the finite polynomial, the series also has its coefficients related to its derivatives.

For values of \(x\) where the power series converges, \(P(x)\) is said to be an analytic function in the neighborhood of the expansion point \(x_o\), within the region of convergence (RoC), also known as the radius of convergence (RoC), for cases where the argument \(x\) is complex.

When the coefficients are determined by derivatives of \(P(x)\) evaluated at \(x = x_o\), then \(P(x)\) is called a Taylor series, where
\[
a_n = \frac{1}{n!} \frac{d^n}{dx^n} P(x) \bigg|_{x=x_o}.
\]
This formula shows, in a general way, how to uniquely define the coefficients, and without the Taylor series formula, we would have no general way of finding \(a_n\). The proof of the Taylor formula is transparent, simply by taking the term by term derivative of Eq. 1.25 and then evaluating the result at the expansion point.

**Exercise:** Verify that \(a_o\) and \(a_1\) of Eq. 1.25 follow from Eq. 1.26.

Taylor series play a special role in mathematics, as the coefficients of the series uniquely determine a function (e.g., via its derivatives). The implications, and limitations of the series representation are very specific. First, if the series fails to converge (i.e., outside the RoC), it is essentially meaningless. Second, the analytic function must be single valued. This follows from the fact that each term in Eq. 1.25 is single valued. Third, analytic functions are very “smooth,” since the may be differentiated an \(\infty\) number of times, and the sum still converges. There can be no jumps or kinks in these functions.

But these properties are both their curse and their blessing, as they represent a good starting point for solving differential equations, which is exactly how they were used, by Newton and others.

Two well known analytic functions are the geometric series
\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n
\]
and exponential series
\[
e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{3 \cdot 2} x^3 + \frac{1}{4 \cdot 3 \cdot 2} x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.
\]

**Exercise:** Verify that the coefficients of the above functions are given by Eq. 1.26.
**Region of convergence:** Determining the RoC for a given analytic function is quite important, and may not always be obvious. In general the RoC is a circle having a radius, centered on the expansion point, out to the nearest pole. Thus when the expansion point is moved, the RoC changes, since the location of the pole is fixed.

For the geometric series (Eq. 1.27), if the expansion point is taken as $x_0 = 1$, the RoC is $|x| < 1$, since $1/(1 - x)$ has a pole at $x = 1$. We may move the expansion point by a linear transformation. For example, by replacing $x$ with $z + 3$. Then the series becomes $1/((z+3) - 1) = 1/(z + 2)$, so the RoC becomes 2, because in the $z$ plane, the pole has moved to $-2$. A second important example is the function $1/(x^2 + 1)$, which has the same RoC as the geometric series, since it may be expressed as the partial fraction expansion

$$
\frac{1}{x^2+1} = \frac{1}{(x + 1j)(x - 1j)} = \frac{1}{2j} \left( \frac{1}{x - 1j} - \frac{1}{x + 1j} \right).
$$

This form of expansion is called a partial fraction expansion.

Each term has an RoC of $|x| < |1j| = 1$. In other words, it is the sum of two geometric series, with poles at $\pm 1j$ which are not as obvious because the roots are complex, and conjugate. Once factored, it becomes clear what is going on.

**Exercise:** Verify the above expression is correct, and show that the residues are $\pm 1/2j$.

The exponential series converges for every finite value of $x \in \mathbb{R}$ (the RoC is the entire real line), thus the exponential is called an entire function.

**Analytic functions:** Any function that has a Taylor series expansion is called an analytic function. Within the RoC, the series expansion defines a single valued function. Polynomials, $1/(1 - x)$ and $e^x$ are examples of analytic functions that are real functions of their real argument $x$. This is not the entire story. Because analytic functions are easily manipulated term by term, they may be used to find solutions of differential equations, since the derivatives of a series are uniquely determined within the RoC, due to Eq. 1.26.

Every analytic function has a corresponding differential equation, that is determined by the coefficients $a_k$ of the analytic power series. An example is the exponential, which has the property that it is the eigenfunction of the derivative operation

$$
\frac{d}{dx} e^{ax} = ae^{ax},
$$

which may be verified using Eq. 1.28. This relationship is a common definition of the exponential function, which is a very special, because it is the eigenfunction of the derivative.

The complex analytic power series (i.e, complex analytic functions) may also be integrated, term by term, since

$$
\int f(x)dx = \sum \frac{a_k}{k+1}x^{k+1}.
$$

Newton took full advantage of this property of the analytic function and used the analytic series (Taylor series) to solve analytic problems, especially for working out integrals, allowing him to solve differential equations. To fully understand the theory of differential equations, one must master single valued analytic functions and their analytic power series.

**Single- vs. multi-valued functions:** Polynomials, and their $\infty$-degree extensions (analytic functions) are single valued: for each $x$ there is a single value of $P_N(x)$. The set of $x$ values of a function are called the domain and the set of $y(x)$ values are called the codomain.\(^{49}\) The roles of the domain and codomain may be swapped, to obtain an inverse function, which is typically quite different in its

\(^{49}\)The codomain is also called the range, image or the ordinate.
properties compared to the function. For example \( y(x) = x^2 + 1 \) has the inverse \( x = \pm \sqrt{y - 1} \), which is double valued, and complex when \( y < 1 \). Periodic functions, such as \( y(x) = \sin(x) \) are even more "exotic," since \( x(y) = \arcsin(x) = \sin^{-1}(x) \) has an \( \infty \) number of \( x(y) \) values for each \( y \). This problem was first addressed in Riemann’s 1851 PhD thesis, while working with Gauss.

**Exercise:** Let \( y(x) = \sin(x) \). Then \( dy/dx = \cos(x) \). Show that \( dx/dy = -1/\sqrt{1-x^2} \). Hint: \( x(y) = \cos^{-1}(y) = \arccos(y) \).

**Exercise:** Let \( y(x) = \sin(x) \). Then \( dy/dx = \cos(x) \). Show that \( dx/dy = -j/\sqrt{1+x^2} \).

**Exercise:** Find the Taylor series coefficients of \( y = \sin(x) \) and \( x = \sin^{-1}(y) \).

**Complex analytic functions:** When the argument of an analytic function \( F(x) \) is complex, that is, \( x \in \mathbb{R} \) is replaced by \( s = \sigma + \omega j \in \mathbb{C} \) (recall that \( \mathbb{R} \subset \mathbb{C} \))

\[
F(s) = \sum_{n=0}^{\infty} c_n (s - s_0)^n,
\]

with \( c_n \in \mathbb{C} \), that function is said to be a *complex analytic*.

For example, when the argument of the exponential becomes complex, it is periodic on the \( \omega \) axis, since

\[
e^{st} = e^{(\sigma+\omega j)t} = e^{\sigma t} e^{\omega jt} = e^{\sigma t} [\cos(\omega t) + j \sin(\omega t)].
\]

Taking the real part gives

\[
\Re \{ e^{st} \} = e^{\sigma t} \frac{e^{\omegajt} + e^{-\omega jt}}{2} = e^{\sigma t} \cos(\omega t),
\]

and \( \Im \{ e^{st} \} = e^{\sigma t} \sin(\omega t) \). Once the argument is allowed to be complex, it becomes obvious that the exponential and circular functions are fundamentally related. This exposes the family of *entire circular functions* [i.e., \( e^s, \sin(s), \cos(s), \tan(s), \cosh(s), \sinh(s) \) and their inverses \( \ln(s), \arcsin(s), \arccos(s), \arctan(s), \cosh^{-1}(s), \sinh^{-1}(s) \)], first fully elucidated by Euler (c1750) (Stillwell, 2010, p. 315). Note that because a function, such as \( \sin(\omega t) \), is periodic, its inverse must be multi-valued. What is needed is some systematic way to account for this multi-valued property. That methodology was provided by Riemann 100 years later, in this 1851 PhD Thesis, supervised by Gauss, in the final years of Gauss’ life.

Given a complex analytic function of a complex variable, one must resort to the *extended complex plane, Riemann sheets and branch cuts*, as discussed in Section 1.3.7 (p. 64). The extended complex plane is a tool that extends the domain of complex analytic to include the point at infinity. This topic is critically important in engineering mathematics, and will be discussed in length in Sections 1.3.7-1.3.10 (pp. 64-70).

**Definition of the Taylor series of a complex analytic function:** However there is a fundamental problem, since we cannot formally define the Taylor series for the coefficients \( c_k \), since we have not defined \( dF(s)/ds \), the derivative with respect to the complex variable \( s \in \mathbb{C} \). Thus simply substituting \( s \) for \( x \) in an analytic function is leaving a major hole in our understanding of the complex analytic function.

To gain a feeling of the nature of the problem, we make take derivatives of a function with respect to various variables. For example,

\[
\frac{d}{dt} e^{st} = se^{st}.
\]

Also

\[
e^{\omega jt} \frac{d}{d\sigma} e^{\sigma t} = \sigma e^{st}.
\]
and
\[ e^{s t} \frac{d}{d \omega j} e^{\omega j} = \omega j e^{s t}. \]

are straightforward.

It was the work of Cauchy (1814) (Fig. 1.11), who uncovered much deeper relationships within complex analytic functions (Sect. 1.3.8, p. 66) by defining differentiation and integration in the complex plane, leading to several fundamental theorems of complex calculus, including the Fundamental theorem of complex integration, and Cauchy’s formula. We shall explore this in and several fundamental theorems in Sect. 1.4.1 (p. 77).

There seems to be some disagreement as to the status of multi-valued functions: Are they functions, or is a function strictly single valued? If so, then we are missing out on a host of interesting possibilities, including all the inverses of nearly every complex analytic function. For example, the inverse of a complex analytic function is a complex analytic function (e.g., \( e^z \) and \( \log(z) \)).

Impact on Physics: It seems likely, if not obvious, that the success of Newton was his ability to describe physics by the use of mathematics. He was inventing new mathematics at the same time as he was explaining new physics. The same might be said for Galileo. It seems likely that Newton was extending the successful techniques and results of Galileo. Galileo died on Jan 8, 1642, and Newton was born Jan 4, 1643, just short of one year later. Certainly Newton was well aware of Galileo’s great success, and naturally would have been influenced by them.

The application of complex analytic functions to physics was dramatic, as may be seen in the six volumes on physics by Arnold Sommerfeld (1868-1951), and from the productivity of his many (36) students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg\(^50\) and Seebach, to name a few), notable coworkers (i.e., Leon Brillouin) and others (i.e., John Bardeen), upon whom Sommerfeld had a strong influence. Sommerfeld is known for having many students who were awarded the Nobel Prize in Physics, yet he was not (the prize is not awarded in mathematics). Sommerfeld brought mathematical physics (the merging of physical and experimental principles with mathematics) to a new level with the use of complex integration of analytic functions to solve otherwise difficult problems, thus following the lead of Newton who used real integration of Taylor series to solve differential equations (Brillouin, 1960, Ch. 3 by Sommerfeld, A.).

1.3.2 Lec 12: Polynomial root classification by convolution

Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem was being formulated by d’Alembert. The idea behind this theorem is that every polynomial of degree \( N \) (Eq. 1.19) has at least one root. This may be written as the product of the root and a second polynomial of degree of \( N - 1 \). By the recursive application of this concept, it is clear that every polynomial of degree \( N \) has \( N \) roots. Today this result is known as the fundamental theorem of algebra:

Every polynomial equation \( P(z) = 0 \) has a solution in the complex numbers. As Descartes observed, a solution \( z = a \) implies that \( P(z) \) has a factor \( z - a \). The quotient

\[ Q(z) = \frac{P(z)}{z - a} = \frac{P(z)}{a} \left[ 1 + \frac{z}{a} + \left( \frac{z}{a} \right)^2 + \left( \frac{z}{a} \right)^3 + \cdots \right] \quad (1.31) \]

is then a polynomial of one lower degree. … We can go on to factorize \( P(z) \) into \( n \) linear factors.


The ultimate expression of this theorem is given by Eq. 1.19 (p. 44), which indirectly states that an \( n^{th} \) degree polynomial has \( n \) roots.

\(^{50}\)https://www.aip.org/history-programs/niels-bohr-library/oral-histories/4661-1
**1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)**

**Exercise:** Explore expressing Eq. 1.31 in terms of real 2x2 matrices, as described in Section 1.2.1, p. 24. **Sol:** Consider the representation of polynomials as a special case of products of 2x2 complex numbers, as discussed in Eq. 1.2, p. 25.

Today this theorem is so widely accepted we fail to appreciate it. Certainly about the time you learned the quadratic formula, you were prepared to understand the concept of polynomials having roots. The simple quadratic case may be extended a higher degree polynomial. The Matlab/Octave command `roots([1,a2,a1,a0])` provides the roots `[s1,s2,s3]` of the cubic equation, defined by the coefficient vector `[1,a2,a1,a0]`. The command `poly([s1,s2,s3])` returns the coefficient vector. I don’t know the largest degree that can be accurately factored by Matlab/Octave, but I’m sure its well over $N = 10^3$. Today, finding the roots numerically is a solved problem.

**Factorization versus convolution:** The best way to gain insight into the polynomial factorization problem is through the inverse operation, multiplication of monomials. Given the roots $x_k$, there is a simple algorithm for computing the coefficients $a_k$ of $P_n(x)$ for any $n$, no matter how large. This method is called convolution. Convolution is said to be a trap-door since it is easy, while the inverse, factoring (deconvolution), is hard, and analytically intractable for degree $N \geq 5$ (Stillwell, 2010, p. 102).

**Convolution of monomials**

As outlined by Eq. 1.19, a polynomial has two descriptions, first as a series with coefficients $a_n$ and second in terms of its roots $x_r$. The question is “What is the relationship between the coefficients and the roots?” The simple answer is that they are related by convolution.

Let us start with the quadratic

$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$

where in vector notation $[-a, -b]$ are the roots and $[1, a + b, ab]$ are the coefficients.

To see how the result generalizes, we may work out the coefficients for the cubic ($N = 3$). Multiplying the following three factors gives (p. 145)

$$(x - 1)(x - 2)(x - 3) = (x^2 - 3x + 2)(x - 3) = x(x^2 - 3x + 2) - 3(x^2 - 3x + 2) = x^3 - 6x^2 + 11x - 6.$$

When the roots are $[1, 2, 3]$ the coefficients of the polynomial are $[1, -6, 11, -6]$. To verify, substitute the roots into the polynomial, and show that they give zero. For example $r_1 = 1$ is a root since $P_3(1) = 1 - 6 + 11 - 6 = 0$.

As the degree increases, the algebra becomes more difficult. Imagine trying to work out the coefficients for $N = 100$. What is needed is an simple way of finding the coefficients from the roots. Fortunately, convolution keeps track of the book-keeping, by formalizing the procedure.

**Convolution of two vectors:** To get the coefficients by convolution, write the roots as two vectors $[1, a]$ and $[1, b]$. To find the coefficients we must convolve the root vectors, indicated by $[1, a] \ast [1, b]$, where $\ast$ denotes convolution. Convolution is a recursive operation. The convolution of $[1, a] \ast [1, b]$ is done as follows: reverse one of the two monomials, padding unused elements with zeros. Next slide one monomial against the other, forming the local dot product (element-wise multiply and add):

<table>
<thead>
<tr>
<th>$a$</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>$a$</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>$a$</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>$a$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$b$</td>
<td>0</td>
<td>1</td>
<td>$b$</td>
<td>1</td>
<td>$b$</td>
<td>0</td>
<td>1</td>
<td>$b$</td>
<td>0</td>
<td>0 ,</td>
</tr>
<tr>
<td>$= 0$</td>
<td>$= x^2$</td>
<td>$= (a + b)x$</td>
<td>$= abx^0$</td>
<td>$= 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

resulting in coefficients $[\cdots, 0, 0, 1, a + b, ab, 0, 0, \cdots]$.

By reversing one of the polynomials, and then taking successive dot products, all the terms in the sum of the dot product correspond to the same power of $x$. This explains why convolution of the coefficients gives the same answer as the product of the polynomials.
As seen by the above example, the position of the first monomial coefficients are reversed, and then slid across the second set of coefficients, the dot-product is computed, and the result placed in the output vector. Outside the range shown, all the elements are zero. In summary,

\[ [1, -1] \ast [1, -2] = [1, -1 - 2, 2] = [1, -3, 2]. \]

In general

\[ [a, b] \ast [c, d] = [ac, bc + ad, bd], \]

Convolving a third term \([1, -3]\) with \([1, -3, 2]\) gives (p. 145)

\[ [1, -3] \ast [1, -3, 2] = [1, -3 - 3, 9 + 2, -6] = [1, -6, 11, -6], \]

which is identical to the cubic example, found by the algebraic method.

By convolving one monomial factor at a time, the overlap is always two elements, thus it is never necessary to compute more than two multiplies and an add for each output coefficient. This greatly simplifies the operations (i.e., they are easily done in your head). Thus the final result is more likely to be correct. Comparing this to the algebraic method, convolution has the clear advantage.

**Exercise:**

1. Given the coefficient vector \([\alpha, \beta, \gamma] = [1, a + b, ab]\), find the formula for the roots \([a, b]\). **Sol:** To solve this one must complete the square of the polynomial:

\[
0 = x^2 + \beta x + \gamma = (x + \beta/2)^2 - (\beta/2)^2 + \gamma \\
(x + \beta/2)^2 = (\beta/2)^2 - \gamma \\
x + \beta/2 = \pm \sqrt{(\beta/2)^2 - \gamma} \\
x = -\beta/2 \pm \sqrt{(\beta/2)^2 - \gamma}.
\]

In terms of our assumed coefficient vector \([1, a + b, ab]\), this gives (See Eq.1.21, p. 46).

\[
x_\pm = -\frac{a + b}{2} \pm \sqrt{\left(\frac{a + b}{2}\right)^2 - ab}. \tag{1.32}
\]

While it is not obvious that \(x_\pm = [a, b]\), it must be true. One may prove this several ways. The most obvious is working backwards from what we did above. However this gives no insight (try it), other than the fact that we did the algebra correctly. Let’s try something else:

\[
x_\pm + \frac{a + b}{2} = \pm \sqrt{\left(\frac{a + b}{2}\right)^2 - ab} \\
= \pm \sqrt{\frac{a^2 + b^2}{4} + ab - ab} \\
= \pm \frac{1}{2} \sqrt{(a + b)(a - b)}
\]

No end in sight here, and there is obvious reason (that I can see) why this is.
Let’s try the product

\[ x_+ \cdot x_- = \left( -\frac{a+b}{2} + \sqrt{\left(\frac{a+b}{2}\right)^2 - ab} \right) \left( -\frac{a+b}{2} - \sqrt{\left(\frac{a+b}{2}\right)^2 - ab} \right) \]

\[ = \left( \frac{a+b}{2} \right)^2 - \left( \sqrt{\left(\frac{a+b}{2}\right)^2 - ab} \right)^2 \]

\[ = \left( \frac{a+b}{2} \right)^2 - \left( \frac{a+b}{2} \right)^2 - ab. \]

We might conclude that we have simultaneously solved two nonlinear equations in the roots \([a, b]\), given the coefficients of the polynomial \([1, \beta, \gamma]\):

\[ \beta = (a + b)/2 \]

\[ \gamma = ab \]

which are the mean and square of the geometric mean \(\sqrt{ab}\).

2. What are the three nonlinear equations that one would need to solve to find the roots of a cubic? **Sol:** From our formula for the convolution of three monomials we may find the nonlinear “deconvolution” relations between the roots\(^{51}\) \([-a, -b, -c]\) and the cubic’s coefficients \([1, \alpha, \beta, \gamma]\)

\[ (x + a) \star (x + b) \star (x + c) = (x + c) \star (x^2 + (a + b)x + ab) \]

\[ = x \cdot (x^2 + (a + b)x + ab) + c \cdot (x^2 + (a + b)x + ab) \]

\[ = x^3 + (a + b + c)x^2 + (ab + ac + cb)x + abc \]

so that the nonlinear equations must be

\[ \alpha = a + b + c \]

\[ \beta = ab + ac + bc \]

\[ \gamma = abc. \]

Clearly these are solve by the classic cubic solution. How that works would be interesting to add to this discussion.

**The product of monomial** \(P_1(x)\) **with a polynomial** \(P_N(x)\) **gives** \(P_{N+1}(x)\): This statement is another way of stating the **fundamental theorem of algebra**. Each time we convolve a monomial with a polynomial of degree \(N\), we obtain a polynomial of degree \(N + 1\). The convolution of two monomials results in a quadratic (degree 2 polynomial). The convolution of three monomials gives a cubic (degree 3). In general, the degree \(k\), of the product of two polynomials, of degree \(n, m\), is the sum of the degrees \((k = n + m)\). For example, if the degrees are each 5 \((n = m = 5)\), then the resulting degree is 10.

In summary, the product of two polynomials of degree \(m, n\) having \(m\) and \(n\) roots, gives a polynomial of degree \(m + n\) having \(m + n\) roots. This is an analysis process, of merging polynomials, by coefficient convolution. Multiplying polynomials is a merging process, into a single polynomial.

\(^{51}\)By working with the negative roots we may avoid an unnecessary and messy alternating sign problem.
CHAPTER 1. INTRODUCTION

Note that the degree of a polynomial is one less than the length of the vector of coefficients. The coefficient on the lead term should always be set to 1 since it cannot be zero, resulting in an illogical result. Always normalize the lead term of $P_n(x)$ to 1 (i.e., $a_N = 1$). This has no effect on the roots.

While you already know this theorem from high school algebra class, it is important to explicitly state the fundamental theorem of algebra.

**Composition of polynomials:** Convolution is not the only operation between two polynomials. Another is composition, which may be defined for two functions $f(z), g(z)$. Then the composition $c(z) = f(z) \circ g(z) = f(g(z))$. As a specific example, suppose $f(z) = 1 + z + z^2$ and $g(z) = e^{2z}$. With these definitions

$$f(z) \circ g(z) = 1 + e^{2z} + (e^{2z})^2 = 1 + e^{2z} + e^{4z}.$$  

Note that $f(z) \circ g(z) \neq g(z) \circ f(z)$.

**Exercise:** Find $g(z) \circ f(z)$. **Sol:** $e^{2f(z)} = e^{2(1+z+z^2)} = e^{2e^{(1+z+z^2)}} = e^{3e^z e^{2z}}$.

1.3.3 Lec 13 Residue expansions

As discussed in Section 1.3.1, p. 46, there are 5 important Matlab/Octave routines that are closely related: `root()`, `polyval()`, `poly()`, `residue()`, `conv()`. The function `residue()` is more complex that the others, and it remains to be discussed.

When lines and planes are defined, the equations are said to be linear in the independent variables. In keeping with this definition of linear, we say that the equations are non-linear when the equations have degree greater than 1 in the independent variables. The term bilinear has a special meaning, in that both the domain and codomain are linearly related by lines (or planes). As an example, an impedance is defined in frequency as the ratio of the voltage over the current $V/I$.

The impedance is typically specified as the ratio of two polynomials, $N(s) = [a_n, a_{n-1}, \ldots , a_0]$ and $D(s) = [b_K, b_{K-1}, \ldots , b_0]$, as functions of complex Laplace frequency $s = \sigma + j\omega$, having simple roots. The bilinear function may be written as $D(s)V = N(s)I$. Since $D(s)$ and $N(s)$ are both polynomials in $s$, this is called bilinear, which comes from a corresponding scalar differential equation, of the form

$$\sum_{k=0}^{K} b_k \frac{d^k}{dt^k} i(t) = \sum_{n=0}^{N} a_n \frac{d^n}{dt^n} v(t)$$

$$\leftrightarrow I(\omega) \sum_{k=0}^{K} b_k s^k = V(\omega) \sum_{n=0}^{N} a_n s^n$$

Equation 1.33 follows from the Laplace transform (on right) of the differential equation (on left), by forming the impedance $Z(s) = V/I = A(s)/B(s)$. This form of the differential equation follows from Kirchhoff’s voltage and current laws (KCL, KVL) or from Newton’s laws (mechanics).

**The physical properties of an impedance:** Based on d’Alembert’s observation that the solution to the wave equation is the sum of forward and backward traveling waves, the impedance may be rewritten in terms of forward and backward traveling waves

$$Z(s) = \frac{V}{I} = \frac{V^+ + V^-}{I^+ - I^-} = r_o 1 + \Gamma(s) \frac{1}{1 - \Gamma(s)}.$$   

(1.34)

where $r_o = P^+/I^+$ is called the surge impedance of the transmission line (e.g., wire) connected to the load impedance $Z(s)$, and $\Gamma(s) = P^-/P_+ = I^-/I^+$ is the reflection coefficient corresponding to $Z(s)$. Like $Z(s)$, $\Gamma(s)$ is causal and complex analytic. Note that the impedance and the reflectance function
must both be complex analytic, since they are connected by the bilinear (aka, Möbius) transformation, which assures the mutual complex analytic properties.

Due to the bilinear transformation, the physical properties of $Z(s)$ and $\Gamma(s)$ are very different. Specifically, the real part of the load impedance must be non-negative ($\Re\{Z(\omega)\} \geq 0$), if and only if $|\Gamma(s)| \leq 1$. In the time domain, the impedance $z(t) \leftrightarrow Z(s)$ must have a value of $r_0$ at $t = 0$. Correspondingly, the time domain relectance $\gamma(t) \leftrightarrow \Gamma(s)$ must be zero at $t = 0$.

This is the basis of conservation of energy, which may be traced back to the properties of the relectance $\Gamma(s)$.

Exercise: Show that if the $\Re\{Z(s)\} \geq 0$ then $|\Gamma(s)| \leq 1$. Sol: Their two equivalent proofs, both of which start from the relation between $Z(s)$ and $\Gamma(s)$. Taking the real part of Eq. 1.34, which must be $\geq 0$, we find

$$\Re\{Z(s)\} = \frac{r_0}{2} \left[ \frac{1 + \Gamma(s)}{1 - \Gamma(s)} + \frac{1 + \Gamma^*(s)}{1 - \Gamma^*(s)} \right] = \frac{r_0}{2} \frac{1 - |\Gamma(s)|^2}{1 + |\Gamma(s)|^2} \geq 0.$$ 

Thus $|\Gamma| \leq 1$.

1.3.4 Lec 14: Introduction to Analytic Geometry

Analytic geometry is the natural consequence of Euclid’s Geometry (which deals with conical geometry (e.g., points, lines, triangles, circles, spheres, ellipses, cones, etc., in two and three physical dimensions), merged with algebra (which deals with simultaneous equations, roots of polynomials, analytic functions, and ultimately, solutions of differential equations). The combination of Euclid’s (323 BCE) geometry and al-Khwarizmi’s (830 CE) algebra provides a new powerful tool, analytic geometry.

There are many important relationships between Euclidean geometry and 16th century algebra. Important similarities include vectors, their Pythagorean lengths $[a, b, c]$

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \quad (1.35)$$

$a = x_2 - x_1$ and $b = y_2 - y_1$, and the angles. Euclid’s geometry had no concept of coordinates, thus of vectors. One of the main differences is the ability of analytic geometry is that one may compute with numbers. A detailed comparison is given in Table 1.3.

There are several new concepts that come with the development of analytic geometry:

1. Composition of functions: If $y = f(x)$ and $z = g(y)$ then the composition of functions $f$ and $g$ is denoted $z(x) = g \circ f(x) = g(f(x))$.

2. Elimination: Given two functions $f(x, y)$ and $g(x, y)$, elimination removes either $x$ or $y$. This procedure, called Gaussian Elimination, was known to the Chinese. This is not always possible.

3. Intersection: While one may speak of the intersection of two lines to give a point, or two planes to give a line, the term intersection is also an important but very different concept in set theory. This is a special case of elimination when the functions $f(x, y), g(x, y)$ are linear in their arguments.

4. Vectors: Euclidean geometry provides the new concept of a vector, as a line with length and orientation (i.e., direction). Analytic geometry defines a vector, in any number of dimensions, as an ordered set of points $\mathbf{x} \in \mathbb{C}$.

5. Analytic geometry extends the ideas of geometry with the introduction of the product of two vectors, the scalar (dot) product $\mathbf{f} \cdot \mathbf{g}$ and the vector (cross) product $\mathbf{f} \times \mathbf{g}$ (Fig. 1.12).

What algebra added to geometry was the ability to compute with complex numbers. For example, the length of a line (Eq. 1.35) was measured in Geometry with a compass: numbers played no role.
Table 1.3: Comparison between Euclidean geometry and analytic geometry. I am uncertain about the items in the third column.

<table>
<thead>
<tr>
<th>Euclidean geometry</th>
<th>analytic geometry</th>
<th>Uncertain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof, Vector, Length, Point, Direction, ≤ 3 dimensional, Intersection, Conic Section, Dot product, Square Roots: on real line lengths.</td>
<td>&gt; 3 dimensions, Numbers, Analytic series and functions, Composition, Elimination, Fund. Thm. Algebra, sin(θ), cos(θ), $e^{i\theta}$, log(z), Derivative, Calculus, Polynomial &amp; Roots: on complex algebra.</td>
<td>Cross product, Recursion, Iteration on complex planes and circles (Newton’s method).</td>
</tr>
</tbody>
</table>

Once algebra was available, the line’s Euclidean length could be computed from the coordinates of the two ends, defined by the 3-vector

$$e = x\hat{i} + y\hat{j} + z\hat{k} = [x, y, z]^T,$$

which represents a point at $(x, y, z) \in \mathbb{R}^3 \subset \mathbb{C}^3$ in three dimensions, having direction, from the origin $(0, 0, 0)$ to $(x, y, z)$. An alternative matrix notation is $e = [x, y, z]^T$, a column vector of three numbers. These two notations are different ways of representing exactly the same thing. View them as equivalent notations.

By defining the vector, analytic geometry allows Euclidean geometry to become quantitative, beyond the physical drawing of an object (e.g., a sphere, triangle or line). With analytic geometry we have the Euclidean concept of a vector, a line having a magnitude (length) and direction, but analytic defined in terms of physical coordinates (i.e., numbers). The difference between two vectors defines a third vector, a concept already present in Euclidean geometry. For the first time, complex numbers were allowed into geometry (but rarely used until Cauchy and Riemann).

As shown in Fig. 1.12, there are two types of products, the 1) scalar $A \cdot B$ and 2) vector $A \times B$ products.

**Scalar product of two vectors:** When using algebra, many concepts in geometry are made precise. There are many examples of how algebra extends Euclidean geometry, the most basic being the scalar product (aka dot product) between two vectors

$$x \cdot \kappa = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}), \quad \in \mathbb{C}$$

$$= ax + by + cz.$$

In matrix notation the scalar product is written as

$$x \cdot \kappa = \begin{bmatrix} x^T & \alpha \\ y \\ z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = [x, y, z] \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = ax + by + cz.$$

The dot product takes the character of $\kappa$. For example, if $\kappa(s) \in \mathbb{C}$ is a function of complex frequency $s$, then the dot product is complex. If $\zeta \in \mathbb{R}$ is real, then the dot product is real.

**Norm (length) of a vector:** The norm of a vector

$$||e|| \equiv +\sqrt{e \cdot e} \geq 0.$$
is defined as the positive square root of the scalar product of the vector with itself. This is a generalization of the length, in any number of dimensions, forcing the sign of the square-root to be non-negative. The length is a concept of Euclidean geometry, and it must always be positive and real. A complex length is not physically meaningful. A zero-length vector is a point. More generally, the Euclidean length of a line is given as the norm of the difference between two real vectors $e_1, e_2 \in \mathbb{R}$

$$
\|e_1 - e_2\|^2 = (e_1 - e_2) \cdot (e_1 - e_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \geq 0.
$$

This derivation is an abbreviated version of a related discussion in Section 3.2.1 (p. 142).

**Pythagorean theorem and the Schwarz inequality:** Regarding Fig. 1.12, suppose we compute the difference between vector $A \in \mathbb{R}$ and $\alpha B \in \mathbb{R}$ as $L = \|A - \alpha B\| \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a scalar that modifies the length of $B$. We seek the value of $\alpha$, which we denote as $\alpha^*$, that minimizes the length of $L$. From simple geometrical considerations, $L(\alpha)$ will be minimum when the difference vector is perpendicular to $B$, as shown in the figure by the dashed line from the tip of $A \perp B$.

To show this algebraically we write out the expression for $L(\alpha)$ and take the derivative with respect to $\alpha$, and set it to zero, which gives the formula for $\alpha^*$. The argument does not change, but the algebra greatly simplifies, if we normalize $A, B$ to be unit vectors $a = A/\|A\|$ and $b = B/\|B\|$, which have norm $= 1$.

$$
L^2 = (a - \alpha b) \cdot (a - \alpha b) = 1 - 2\alpha a \cdot b + \alpha^2.
$$

Thus the length is shortest ($L = L^*$, as shown in Fig. 1.12) when

$$
\frac{d}{d\alpha} L^2 = -2a \cdot b + 2\alpha^* = 0.
$$

Solving for $\alpha^*$ we find $\alpha^* = a \cdot b$. Since $L^* > 0 (a \neq b)$, Eq. 1.37 becomes

$$
1 - 2|a \cdot b|^2 + |a \cdot b|^2 = 1 - |a \cdot b|^2 > 0.
$$

In conclusion $\cos \theta \equiv |a \cdot b| < 1$. In terms of $A, B$ this is $|A \cdot B| < \|A\| \|B\| \cos \theta$, as shown next to $B$ in Fig. 1.12. Thus the scalar product between two vectors is their direction cosine. Furthermore since this forms a right triangle, the Pythagorean theorem must hold. The triangle inequality says that the lengths of the two sides must be greater than the hypotenuse. Note that $\Theta \in \mathbb{R} \not\in \mathbb{C}$.

This derivation is an abbreviated version of a related discussion in Section 3.2.1 (p. 142).
Vector product of two vectors: As shown in Fig. 1.12, the vector product (aka, cross-product) \( \mathbf{a} \times \mathbf{b} \) is the second type of product between two vectors. The vector product defines a vector, perpendicular (\( \perp \)) to the plane of the two vectors being multiplied. The formula for computing the cross product is

\[
\mathbf{a} \times \mathbf{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}.
\]

For example, if the two vectors are in \( \hat{i} \) and \( \hat{j} \), then the cross-product is \( \hat{k} \). It is strictly in \( \hat{k} \) if the two vectors are perpendicular to each other (i.e., \( \hat{k} = \hat{i} \times \hat{j} = -\hat{j} \times \hat{i} \)). The vector product of a vector with itself (or the difference between two vectors) is zero. For example \( \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \). Typically \( \mathbf{a}, \mathbf{b} \in \mathbb{R} \). If they are complex, the definition must be modified to be consistent with the physics.

The scalar product of a third vector \( \mathbf{c} \) with the vector product \( \mathbf{a} \times \mathbf{b} \), with \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R} \),

\[
\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix}
c_1 & c_2 & c_3 \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix} \in \mathbb{R},
\]

is called the triple product, which represents the volume of a parallelepiped.

Impact of Analytic Geometry: The most obvious impact of analytic geometry, was a detailed analysis of the conic sections, using algebra, rather than drawings via a compass and ruler. An important example is the composition of the line and circle, a venerable construction, presumably going back to before Diophantus (250CE). Once algebra was invented, the composition could be done using formulas. With this analysis came complex numbers.

The first two mathematicians to appreciate this mixture of Euclid’s geometry and the new algebra were Fermat and Descartes (Stillwell, 2010, p. 111-115); soon Newton contributed to this effort, by the addition of physics (calculations in acoustics, orbits of the planets, and the theory of gravity and light Stillwell (2010, p. 115-117)), significant concepts for 1687.

Given these new methods, many new solutions to problems emerged. The complex roots of polynomials continued to appear, without any obvious physical meaning. Complex numbers seem to have been viewed as more of an inconvenience than a problem. Newton’s solution to this dilemma was to simply ignore the imaginary cases (Stillwell, 2010, p. 119).

Development of Analytic Geometry

Intersection and Gaussian elimination: The first “algebra” (al-jabr) is credited to al-Khwarizmi (830 CE). Its invention advanced the theory of polynomial equations in one variable, Taylor series, and composition versus intersections of curves. The solution of the quadratic equation had been worked out thousands of year earlier, but with algebra a general solution could be defined. The Chinese had found the way to solve several equations in several unknowns, for example, finding the values of the intersection of two circles. With the invention of algebra by al-Khwarizmi, a powerful tool became available to solve the difficult problems.

Composition and Elimination In algebra there are two contrasting operations on functions: composition and Elimination.
Composition: Composition is the merging of functions, by feeding one into the other. If the two functions are \( f, g \) then their composition is indicated by \( f \circ g \), meaning the function \( y = f(x) \) is substituted into the function \( z = g(y) \), giving \( z = g(f(x)) \).

Composition is not limited to linear equations, even though that is where it is most frequently applied. To compose two functions, one must substitute one equation into the other. That requires solving for that substitution variable, which is not always possible in the case of nonlinear equations. However many tricks are available that may work around this restrictions. For example if one equation is in \( x^2 \) and the other in \( x^3 \) or \( \sqrt{x} \), it may be possible to multiply the first by \( x \) or square the second.

The point is that one of the variables must be isolated so that when it is substituted into the other equations, the variable is removed from the mix.

Examples: Let \( y = f(x) = x^2 - 2 \) and \( z = g(y) = y + 1 \). Then

\[
g \circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1.
\]

In general composition does not commute (i.e., \( f \circ g \neq g \circ f \)), as is easily demonstrated. Swapping the order of composition for our example gives

\[
f \circ g = f(g(y)) = (y + 1)^2 - 2 = y^2 + 2y - 1.
\]

Intersection: Complimentary to composition is intersection (i.e., decomposition) (Stillwell, 2010, pp. 119, 149). For example, the intersection of two lines is defined as the point where they meet. This is not to be confused with finding roots. A polynomial of degree \( N \) has \( N \) roots, but the points where two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a function (equation) of lower degree, implemented with Gaussian elimination.

Intersection of two lines: Unless they are parallel, two lines meet at a point. In terms of linear algebra this may be written as 2 linear equations (left) along with the intersection point \([x_1, x_2]^T\), given by the inverse of the 2x2 set of equations (right).\(^{52}\)

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
=
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}
=\frac{1}{\Delta}
\begin{bmatrix}
  d & -b \\
  -c & a
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix},
\]

where \( \Delta = ad - bc \) is called the determinant. Note that the determinant may be written as

\[
\frac{b}{a} = \frac{d}{c},
\]

which says that the ratio of the slopes of the two lines are zero. Thus if \( \Delta = 0 \) the two lines are parallel.

By substituting the right expression into the left, and taking the inverse we obtain the intersection point. If \( \Delta = 0 \) there can be no solution, in which case the two lines are parallel (they meet at infinity.) Note the structure of the inverse: 1) The diagonal values are swapped, 2) the off-diagonal values are negative and 3) the matrix is divided by the determinant.

---

\(^{52}\)It is very important to note when writing the equation \( Ax = y \) in matrix format, the unknowns are \( x_1, x_2 \) whereas in the original equations they were \( y, x \). The starting two equations are \( ay + bx = c \) and \( dy + ex = f \) (where \( a, b, c, d, e, f \in \mathbb{C} \) are constants), two linear equations in \( (x, y) \). In matrix format the names are changed, where the two equations are \( ax_1 + bx_2 = y_1 \) and \( dx_1 + ex_2 = y_2 \) with unknowns \((x_1, x_2)\). Each pair of equations has two unknowns. In matrix equations the coordinates of the graph are \((x_1, x_2)\) rather than the scalars \((x, y)\). The first time you meet this scrambling of terminology it may be very confusing.
Exercise: Show that the equation on the right is the solution of the equation on the left. **Sol:** By a direct substitution of the left equation into the right equation

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -cb + ad \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix},
\]

which is the identity matrix.

Algebra will give the solution when geometry cannot. When the two curves fail to intersect on the real plane, the solution still exists, but is complex valued. In such cases, geometry, which only considers the real solutions, fails. For example, when the coefficients \([a, b, c, d]\) are complex, the solution exists, but the determinant can be complex. Thus algebra is much more general than geometric. Geometry fails when the solution has a complex intersection.

A system of linear \(Ax = y\) equations has many interpretations, and one should not be biased by the notation. As engineers we are trained to view \(x\) as the input and \(y\) as the output, in which case then \(y = Ax\) seems natural, much like the functional relation \(y = f(x)\). But what does the linear relation \(x = Ay\) mean, when \(x\) is the input? The obvious answer is that \(y = A^{-1}x\). But when working with systems of equations, there are many uses of equations, and we need to become more flexible in the our interpretation. For example \(y = A^2x\) has a useful meaning, and in fact we saw this type of relationship when working with Pell’s equation (p. 39) and the Fibonacci sequence (p. 41). As another example consider

\[
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_{1x} & a_{1y} \\ a_{2x} & a_{2y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

which is reminiscent of a three-dimensional surface \(z = f(x, y)\). We shall find that such generalizations are much more than a curiosity.

### 1.3.5 Lec 15 Gaussian Elimination

The method for finding the intersection of equations is based on the recursive elimination of all the variables but one. This method, known as **Gaussian elimination**, works across a broad range of cases, but may be defined as a systematic algorithm when the equations are linear in the variables. Rarely do we even attempt to solve problems in several variables of degree greater than 1. But Gaussian eliminations may still work in such cases (Stillwell, 2010, p. 90).

In Appendix B.2 (p. 190) the inverse of a 2x2 linear system of equations is derived. Even for a 2x2 case, the general solution requires a great deal of algebra. Working out a numeric example of Gaussian elimination is more instructive. For example, suppose we wish to find the intersection of the two equations

\[
x - y = 3 \\
2x + y = 2.
\]

This 2x2 system of equations is so simple that you may immediately visualize the solution: By adding the two equations, \(y\) is eliminated, leaving \(3x = 5\). But doing it this way takes advantage of the specific example, and we need a method for larger systems of equations. We need a generalized (algorithmic) approach. This general approach is called **Gaussian elimination**.

Start by writing the equations in **matrix** format (note this is not in the standard form \(Ax = y\))

\[
\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.
\]

Next, eliminate the lower left term (2\(x\)) using a scaled version of the upper left term (\(x\)). Specifically, multiply the first equation by -2, add it to the second equation, replacing the second equation with the
result. This gives
\[
\begin{bmatrix}
1 & -1 \\
0 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
2 - 3 \cdot 2
\end{bmatrix} = 
\begin{bmatrix}
3 \\
-4
\end{bmatrix}.
\]
(1.42)

Note that the top equation did not change. Once the matrix is “upper triangular” (zero below the diagonal) you have the solution. Starting from the bottom equation, \( y = -4/3 \). Then the upper equation then gives \( x - (-4/3) = 3 \), or \( x = 3 - 4/3 = 5/3 \).

In principle, Gaussian elimination is easy, but if you make a calculation mistake along the way, it is very difficult to find the error. The method requires a lot of mental labor, with a high probability of making a mistake. Thus you do not want to apply this method every time. For example suppose the elements are complex numbers, or polynomials in some other variable such as frequency. Once the coefficients become more complicated, the seeming trivial problem becomes highly error prone. There is a much better way, that is easily verified, which puts all the numerics at the end, in a single step.

The above operations may be automated by finding a carefully chosen upper-diagonalization matrix \( U \) that does the same operation. For example let
\[
U = \begin{bmatrix}
1 & 0 \\
-2 & 1
\end{bmatrix}.
\]
(1.43)

Multiplying Eq. 1.41 by \( U \) we find
\[
\begin{bmatrix}
1 & 0 \\
-2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
1 & -1 \\
0 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = 
\begin{bmatrix}
3 \\
-4
\end{bmatrix}.
\]
(1.44)

we obtain Eq. 1.42. At this point we can either back-substitute and obtain the solution, as we did above, or find a matrix \( L \) that finishes the job, by removing elements above the diagonal.

**Exercise:** Show that \( \det(U) = \det(UA) = 1 \). **Sol:** Since
\[
\det\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = ad - bc,
\]
and \( \det(UA) = detU \cdot det(A) \), and \( \det(U) = 1 - 0 = 1 \), the result follows naturally. Common notation is to let \( \det(A) = |A| \).

In Appendix B.2 the inverse of a general 2x2 matrix is summarized in terms of three steps: 1) swap the diagonal elements, 2) reverse the signs of the off-diagonal elements and 3) divide by the determinant \( \Delta = ab - cd \). Specifically
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}.
\]
(1.45)

There are very few things that you must memorize, but the inverse of a 2x2 is one of them. It needs to be in your tool-bag of tricks, like the quadratic formula.

While it is difficult to compute the inverse matrix from scratch (Appendix B), it takes only a few seconds to verify it (steps 1 and 2)
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix} = \begin{bmatrix}
ad - bc & -ab + ab \\
-cd - cd & -bc + ad
\end{bmatrix} = \begin{bmatrix}
\Delta & 0 \\
0 & \Delta
\end{bmatrix}.
\]
(1.46)

Thus dividing by the determinant gives the 2x2 identity matrix. A good strategy, when you don’t trust your memory, is to write down the inverse as best you can, and then verify.

Using the 2x2 matrix inverse on our example (Eq. 1.41), we find
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \frac{1}{1 + 2} \begin{bmatrix}
1 & 1 \\
-2 & 1
\end{bmatrix} \begin{bmatrix}
3 \\
2
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
5 \\
-6 + 2
\end{bmatrix} = \begin{bmatrix}
5/3 \\
-4/3
\end{bmatrix}.
\]
(1.47)

If you use this method, you will rarely (never) make a mistake, and the solution is easily verified. Either you can check the numbers in the inverse, as was done in Eq. 1.46, or you can substitute the solution back into the original equation.
1.3.6 Lec 16: Transmission (ABCD) matrix composition method

In this section we shall derive the method of linear composition of systems, known by several names as the *ABCD Transmission matrix method*, or in the mathematical literature as the Möbius (bilinear) transformation. Using the method of *matrix composition*, a linear system of 2x2 matrices can represent a large and important family of networks. By the application of Ohm’s law to the circuit shown in Fig. 1.13, we can model a cascade of such cells.

Figure 1.13: A a single LC cell of the LC transmission line (see Fig. 2.2 (p. 132). Every cell of any transmission line may be modeled by the ABCD method, as the product of two matrices. For the example shown here, the inductance \( L \) of the coil and the capacitance \( C \) of capacitor are in units of [Henry/m] and [Farad/m], thus they depend on length \( \Delta x \) [m] that the cell represents. Note the flows are always defined as into the + node.

**Matrix composition:** Matrix multiplication represents a composition of 2x2 matrices, because the input to the second matrix is the output of the first (this follows from the definition of composition: \( f(x) \circ g(x) = f(g(x)) \)). Thus the ABCD matrix is also known as the *transmission* matrix method, or occasionally the *chain matrix*. The general expression for a transmission matrix \( T(s) \) is

\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.
\]

(1.48)

The four coefficients \( A(s), B(s), C(s), D(s) \) are all complex functions of the Laplace frequency \( s = \sigma + j \omega \). The derivation is repeated with more detail in Section 3.3.2 (p. 151). It is a standard convention to always define the current into the node, but since the input current (on the left) is the same as the output current on the right (\( I_2 \)), the sign must be reversed to meet the convention of current into every node. When transmission matrices are cascaded, all the signs match.

We have already used 2x2 matrix composition in representing complex numbers (p. 25), and for computing the \( \text{gcd}(m, n) \) of \( m, n \in \mathbb{N} \) (p. 34), Pell’s equation (p. 39) and the Fibonacci sequence (p. 41).

**Definitions of \( A, B, C, D \):** The definitions of the four functions of Eq. 5.11 are easily read off of the equation, as

\[
A = \frac{V_1}{V_2} \bigg|_{I_2=0}, \quad B = -\frac{V_1}{I_2} \bigg|_{V_2=0}, \quad C = \frac{I_1}{V_2} \bigg|_{I_2=0}, \quad D = -\frac{I_1}{I_2} \bigg|_{V_2=0}.
\]

(1.49)

These definitions follow trivially from Eq. 5.11. These relations have general names. \( A, D \) are called voltage and current *transfer functions*, since they are the ratio of an output over an input, whereas \( B, C \) are known as the *transfer impedance* and *transfer conductance*. Each express an output (port 2) in terms of an input input at port 1.

**Exercise:** Explain why \( C \) is given as above. **Sol:** Writing out the lower equation \( I_1 = CV_2 - DI_2 \). Setting \( I_2 = 0 \), we obtain the equation for \( C \).
Thévenin parameters of a source: A very important concept in circuit theory is that of the Thévenin parameters (p. 172), the open-circuit voltage and the short-circuit current, the ratio of which define the Thévenin impedance. The open circuit voltage is defined as the voltage when the current is zero. In mechanics this is called the isometric force, the maximum force given zero velocity.

It trivially follows that the Thévenin voltage (open circuit voltage) is

\[ V_{\text{Thév}} = \frac{1}{C} \equiv \frac{V_2}{I_1|_{I_2=0}}. \]

For example, the voltage measured across a battery having no load is its Thévenin voltage.

We may find the impedance \( Z_1 = \frac{V_1}{I_1|_{I_2=0}} \) that would be seen looking into port 1, with nothing connected to port 2 \((I_2 = 0)\) by taking the ratio of the upper and lower equations

\[ Z_1 = \frac{AV_2 - BI_2}{CV_2 - DI_2} \Big|_{I_2=0} = \frac{A}{C} = AV_{\text{Thév}}. \]

Using this relation we may find a second expression for the Thévenin voltage in terms of the a source voltage \( V_1 \) rather than in terms of the source current.

\[ V_{\text{Thév}} = \frac{Z_1}{A} \equiv Z_1 \frac{V_2}{V_1|_{I_2=0}}. \]

The choice of the appropriate expression depends on the physics of the actual source.

Example of the use of the ABCD matrix composition: In Fig. 1.13 we see the network is composed of a series inductor (mass) having an impedance \( Z_l = sL \), and a shunt capacitor (compliance) having an impedance \( Z_c = 1/sC \), where \( s \in \mathbb{C} \). As determined by Ohm’s Law, each impedance is describe by a linear relation between the current and the voltage. For the inductive impedance, applying Ohm’s law, we find

\[ V_1 - V_2 = Z_l I_1 \]

where \( Z_l \) is the complex impedance of the inductor. For the capacitive impedance, applying Ohm’s law we find

\[ V_2 = (I_1 + I_2)Z_c, \]

where \( Z_c \) is the complex impedance of the capacitor.

Each of these linear impedance relations may be written in matrix form. The series inductor equation gives (note \( I_1 = -I_2 \))

\[ \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \tag{1.50} \]

while the shunt capacitor equation yields (note \( V_1 = V_2 \))

\[ \begin{bmatrix} V_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \tag{1.51} \]

where \( Y_c = 1/Z_c \) is called the admittance.

When the second matrix equation for the shunt admittance (Eq. 1.51) is substituted into the series impedance equation (Eq. 1.50), we find the ABCD matrix \((T_1 \circ T_2)\), for the cell is simply the product of two matrices

\[ \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 + Z_l Y_c & Z_l \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{1.52} \]

Note that the determinant of the matrix \( \Delta = AD - BC = 1 \).

Thus \( A(s) = 1 + Z_L u_c = 1 + s^2 L C, \ B(s) = Z_l, \ C(s) = Y_c \) and \( D = 1 \). This equation characterizes every possible relation between the input and output voltage and current of the cell.
CHAPTER 1. INTRODUCTION

For example, the ratio of the output to input voltage with the output unloaded \( (I_2 = 0) \), known as the voltage divider relation may be found from the upper equation with \( I_2 = 0 \). Writing this out gives

\[
\left. \frac{V_2}{V_1} \right|_{I_2=0} = \frac{1}{A(s)} = \frac{1}{1 + \frac{Z_l Y_c}{Z_c + Z_i}}.
\]

To derive the formula for the current divider equation, use the lower equation and set \( V_2 = 0 \).

\[
-\frac{I_2}{I_1} \bigg|_{V_2=0} = \frac{1}{D} = 1.
\]

1.3.7 Lec 17: Riemann Sphere: 3rd extension of chord and tangent method

Once algebra was formulated c830 CE, mathematicians were able to expand beyond the limits placed on it by geometry on the real plane, and the verbose descriptions of each problem in prose (Stillwell, 2010, p. 93). The geometry of Euclid’s Elements had paved the way, but after 2000 years, the addition of the language of algebra changed everything. The analytic function was a key development, heavily used by both Newton and Euler. Also the investigations of Cauchy made important headway with his work on complex variables. Of special note was integration and differentiation in the complex plane of complex analytic functions, which is the topic of stream 3.

It was Riemann, working with Gauss in the final years of Gauss’ life, who made the breakthrough, with the concept of the extended complex plane.\(^{54}\) This concept was based on the composition of a line with the sphere, similar to the derivation of Euclid’s formula for Pythagorean triplets (Fig. 2.3, p. 134). While the importance of the extended complex plane was unforeseen, it changed analytic mathematics forever, along with the physics it supported. It unified and thus simplified many important integrals, to the extreme. This idea is captured by the fundamental theorem of complex integral calculus (Table 1.7 p. 75) and 1.4, p. 75.

\[\text{Figure 1.14: The left panel shows how the real line may be composed with the circle. Each real } x \text{ value maps to a corresponding point } x' \text{ on the unit circle. The point } x \to \infty \text{ maps to the north pole } N. \text{ This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane } z \to \infty \text{ maps onto the north pole } N. \text{ This construction is important because while the plane is open (does not include } z \to \infty), \text{ the sphere is analytic at the north pole. Thus the sphere defines the closed extended plane. Figure from Stillwell (2010, pp. 299-300).}\]

The idea is outlined in Fig. 1.14. On the left is a circle and a line. The difference between this case and the derivation of the Pythagorean Triplets is, that the line starts at the north pole, and ends on the real \( x \in \mathbb{R} \) axis at point \( x \). At point \( x' \), the line cuts through the circle. Thus the mapping from \( x \) to \( x' \) takes every point on \( \mathbb{R} \) to a point on the circle. For example, the point \( x = 0 \) maps to the south pole (not indicated). To express \( x' \) in terms of \( x \) one must compose the line and the circle, similar to the composition used in Fig. 2.3 (p. 134). The points on the circle, indicated here by \( x' \), require a

\(^{54}\)“Gauss did lecture to Riemann but he was only giving elementary courses and there is no evidence that at this time he recognized Riemann’s genius.” [Link to source]

“In 1849 he [Riemann] returned to Göttingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851.” [Link to source]
traditional polar coordinate system, having a unit radius and an angle defined between the radius and a vertical line passing through the north pole. When $x \to \infty$ the point $x' \to N$, known as the point at infinity. But this idea goes much further, as shown on the right half of Fig. 1.14.

Here the real tangent line is replaced by the a tangent complex plane $z \in \mathbb{C}$, and the complex puncture point $z' \in \mathbb{C}$, in this case on the complex sphere, called the extended complex plane. This is a natural extension of the chord/tangent method on the left, but with significant consequences. The main difference between the complex plane $z$ and the extended complex plane, other than the coordinate system, is what happens at the north pole. The point at $|z| = \infty$ is not defined on the plane, whereas on the sphere, the point at the north pole is simply another point, like every other point on the sphere.

**Open vs. closed sets:** Mathematically the plane is said to be an open set, since the limit $z \to \infty$ is not defined, whereas on the sphere, the point $z'$ is a member of a closed set, since the north pole is defined. The distinction between an open and closed set is important, because the closed set allows the function to be complex-analytic at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

The $z$ plane may be replaced with another plane, say the $w = F(z) \in \mathbb{C}$ plane, where $w$ is some function $F$ of $z \in \mathbb{C}$. For the moment we shall limit ourselves to complex analytic functions of $z$, namely $w = F(z) = u(x, y) + v(x, y)j = \sum_{n=0}^{\infty} c_n z^n$.

In summary, given a point $z = x + yj$ on the open complex plane, we map it to $w = F(z) \in \mathbb{C}$, the complex $w = u + vj$ plane, and from there to the closed extended complex plane $w'(z)$. The point of doing this is that it allows us to allow the function $w'(z)$ to be analytic at the north pole, meaning it can have a convergent Taylor series at the point at infinity $z \to \infty$. Since we have not yet defined $dw(z)/dz$, the concept of a complex Taylor series remains undefined.

### Möbius bilinear transformation

In mathematics the Möbius transformation has special importance because it is linear in its action. In the engineering literature this transformation is known as the bilinear transformation. Since we are engineers we shall stick with the engineering terminology. But if you wish to read about this on the internet, be sure to also search for the mathematical term, which may be better supported.

When a point on the complex plane $z = x + yj$ is composed with the bilinear transform $(a, b, c, d \in \mathbb{C})$, the result is $w(z) = u(x, y) + v(x, y)j$

$$w = \frac{az + b}{cz + d} \quad (1.53)$$

the transformation from $z \to w$ is a cascade of four independent compositions

1. translation $(w = z + b; \; a = 1, b \in \mathbb{C}, c = 0, d = 1)$,
2. scaling $(w = |a|z; \; a \in \mathbb{R}, b = 0, c = 0, d = 1)$
3. rotation $(w = \frac{a}{\bar{a}}z; \; a \in \mathbb{C}, b = 0, c = 0, d = |a|)$ and
4. inversion $(w = \frac{1}{z}; \; a = 0, b = 1, c = 1, d = 0)$.

Each of these transformations are a special case of Eq. 1.53, with the inversion the most complicated. A wonderful video showing the effect of the bilinear (Möbius) transformation on the plane is available that I highly recommend: [https://www.youtube.com/watch?v=0z1fIsUNh04](https://www.youtube.com/watch?v=0z1fIsUNh04)

When the extended plane (Riemann sphere) is analytic at $z = \infty$, one may take the derivatives there, defining a Taylor series with the expansion point at $\infty$ and then integrate through $\infty$. When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing the analytic nature at infinity. A second way to access the point not defined.
CHAPTER 1. INTRODUCTION

at infinity is by inversion, swapping poles with zeros. Thus a zero at infinity is the same as a pole at the origin.

This construction of the Riemann sphere and the M"{o}bius (bilinear) transformation allow us to understand the point at infinity, and treat it like any other point. If you felt that you never understood the meaning of the point at $\infty$ (likely), this should help.

### 1.3.8 Lec 18: Complex analytic mappings (Domain-coloring)

One of the most difficult aspects of complex functions of a complex variable is understanding what’s going on. For example, $w(z) = \sin(x)$ is trivial when $z = x + yj$ is real (i.e., $y = 0$), because $\sin(x)$ is real. Likewise for the case where $x = 0$, where

$$\sin(yj) = \frac{e^{-y} - e^{y}}{2j} = jsinh(y)$$

is purely imaginary. But the general case, $w(z) = \sin(z) \in \mathbb{C}$, is not easily visualized. Note that

$$w(z) = \sin(z) = \sin(xj - y) = jsinh(z).$$

Thus when $u(z)$ and $v(z)$ are not well known, $w(z)$ are can be much more difficult to visualize.

Fortunately with computer software today, this problem can be solved by adding color to the chart. A Matlab/Octave script `zviz.m` has been used to make the charts shown here.\(^{55}\) This tool is known as Domain-coloring. Rather than plotting $u(\sigma, \omega)$ and $v(\sigma, \omega)$ separately, domain-coloring allows us to display the entire function on one chart. Note that for this visualization we see the complex polar form of $w(s) = |w|e^{\omega j}$, rather than a four dimensional Cartesian $w(x + yj) = u(x, y) + v(x, y)j$ graph.

To further complicate the issue, mathematicians typically use the more abstract (i.e., non–physical) notation $w(z)$, where $w = u + vj$ and $z = x + yj$, whereas engineers think in terms of physical complex impedance $Z(s) = R(s) + X(s)j$, having resistance $R(s)$ and reactance $X(s)$ (Ohms), as function of the complex Laplace radian frequency $s = \sigma + \omega j$ [rad], as used, for example, with the Laplace transform (Sect. 1.3.10, p. 70).

In Fig. 1.15 we use both notations, with $Z(s) = s$ on the left and $w(z) = z - \sqrt{j}$ on the right, where we show this color code as a 2x2 dimensional domain-coloring graph. Intensity (dark to light) represents the magnitude of the function, while hue (color) represents the phase, where red is $0^\circ$, seaweed is $90^\circ$, blue-green is $135^\circ$, blue is $180^\circ$, and violet is $-90^\circ$ (or $270^\circ$). The function $w = s$ has a dark spot (a zero) at $s = 0$, and becomes brighter away from the origin. On the right is $w(z) = z - \sqrt{j}$, which shifts the zero to $z = \sqrt{j}$. Thus domain–coloring gives the full picture of the complex analytic function mappings $w(x, y) = u(x, y) + v(x, y)j$ in colorized polar coordinates.

Two additional examples are given in Fig. 1.16 to help you interpret the two complex mappings $w = e^{s}$ (left) and its inverse $s = \ln(w)$. The exponential is relatively easy to understand because $w(s) = |e^\sigma e^{\omega j}| = e^\sigma$. The red region is where $\omega \approx 0$ in which case $w \approx e^\sigma$. As $\sigma$ becomes large and negative, $w \to 0$, thus the entire field becomes dark on the left. The field is becoming light on the right.

\(^{55}\) URL for `zviz.m`: `http://jontalle.web.engr.illinois.edu/uploads/298/zviz.16.m`
where \( w = e^\sigma \to \infty \). If we let \( \sigma = 0 \) and look along the \( \omega \) axis, we see that the function is changing phase, sea-green (90\(^\circ\)) at the top and violet (-90\(^\circ\)) at the bottom.

In the right panel note the zero for \( \ln(w) = \ln|w| + \omega j \) at \( w = 1 \). The root of the log function is \( \log(w_\sigma) = 0 \) is \( w_\sigma = 1, \phi = 0 \), since \( \log(1) = 0 \). More generally, the log of \( w = |w|e^{\phi j} \) is \( s = \ln|w| + \phi j \). Thus \( s(w) \) can be zero only when the angle of \( w \) is zero.

The \( \ln(w) \) function has a branch cut along the \( \phi = 180^\circ \) axis. As one crosses over the cut, the phase goes above 180\(^\circ\), and the plane changes to the next sheet of the log function. The only sheet with a zero is the principle value, as shown. All others, the log function is either increasing or decreasing monotonically, and there is no zero, as seen on sheet 0 (the one showing in Fig. 1.16).

### 1.3.9 Lec 19: Signals: Fourier and Laplace transforms

Two basic transformations in engineering mathematics are the Fourier and the Laplace transforms, which deal with time–frequency analysis.

The Fourier transform takes a time domain signal \( f(t) \in \mathbb{R} \) and transforms it to the frequency domain \( \omega \in \mathbb{R} \), where it is complex \( (F(\omega) \in \mathbb{C}) \). For the Fourier transform, both the time \(-\infty < t < \infty\) and frequency \( \infty < \omega < \infty \) are strictly real. The relationship, between \( f(t) \) and its transform \( F(\omega) \), is indicated by the double arrow symbol

\[
 f(t) \leftrightarrow F(\omega).
\]

Since the FT obeys superposition, it is possible to define a FT of a complex time function \( f(t) \in \mathbb{C}, t \in \mathbb{R} \). This is useful in certain engineering applications (i.e., Hilbert envelope, Hilbert transforms).

The Laplace transform takes a real signals \( f(t) \in \mathbb{R} \), as a function of real time \( t \in \mathbb{R} \), that are causal, i.e., strictly zero for negative time \( f(t) = 0 \) for \( t < 0 \), and transforms them into complex functions \( F(s) \in \mathbb{C} \) of complex frequency \( s = \sigma + \omega j \). As for the Fourier transform, there is the notation \( f(t) \leftrightarrow F(s) \).

When a signal is zero for negative time \( f(t < 0) = 0 \), it is said to be causal, and the resulting transform \( F(s) \) is then complex analytic over significant regions of the \( s \) plane. For a function of time to be causal, time must be real \( (t \in \mathbb{R}) \), since if it were complex, it would lose the order property (thus it could not be causal). It is helpful to emphasize the causal nature of \( f(t)u(t) \) by indicating the causal nature, using the Heaviside step function \( u(t) \).

Restrictions on a function (e.g., real, causal, periodic, positive real part, etc.) are called a symmetry property. There are many forms of symmetry (Section 1.3.11, p. 73). The concept of symmetry is very
CHAPTER 1. INTRODUCTION

general and widely used in both mathematics and physics, where it is more generally known as Group theory. We shall restrict ourselves to only a few very basic symmetries (Section 3.5.1, p. 160).

**Periodic signals:** Besides these two basic types of time–frequency transforms, there are several variants that depend on the symmetry in time and frequency. For example, when the time signal is sampled (discrete in time), the frequency response becomes periodic, leading to the Discrete-time Fourier transform (DTFT). When a time response is periodic, the frequency response is sampled (discrete in frequency), leading to the Fourier Series. These two symmetries may be simply characterized as periodic in time ⇒ discrete in frequency, and periodic in frequency ⇒ discrete in time. In Section ?? (p. ??) we shall explain this concept with examples. When a function is both discrete in time and frequency, it is necessarily periodic in time and frequency, leading to the Discrete Fourier Transform (DFT). The DFT is typically computed with an algorithm called the Fast Fourier Transform (FFT), which can dramatically speed up the calculation when the data is a power of 2 in length.

A very important symmetry is when functions that are causal (in time) and periodic (in frequency). The best known example is the class of signals that have z transforms, which are causal (one-sided in time) discrete-time signals. The harmonic series (Eq. 1.27, p. 47) is the z-transform of the discrete-time step function, thus, by symmetry, analytic within the RoC in the complex frequency (z) domain.

**Summary:** The definitions of the FT ($\mathcal{F}$) and LT ($\mathcal{L}$) transforms are superficially similar. The key difference is that the time response of the Laplace transform is causal, leading to a complex analytic frequency responses. The frequency response of the Fourier transform is real, thus typically not analytic. These are not superficial differences. The concept of symmetry is helpful in understanding the many different types of time-frequency transforms. Two fundamental types of symmetry are causality and periodicity.

The $\mathcal{F}$ characterizes the steady-state response while the $\mathcal{L}$ characterizes both the transient and steady-state response. Given a system response $H(s) \leftrightarrow h(t)$ having input $x(t)$, then the output $y(t) = h(t) \ast x(t) \leftrightarrow Y(\omega) = H(s)|_{s=j\omega} X(\omega)$.

**Definition of the Fourier transform:** The forward transform takes $f(t)$ to $F(\omega)$ while the inverse transform takes $F(\omega)$ to $\tilde{f}(t)$. The tilde symbol indicates that in general recovered inverse transform signal can be slightly different from $f(t)$. We will give examples of this below.

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \tag{1.54}
\]

\[
F'(\omega) \leftrightarrow \tilde{f}(t) \quad \tilde{f}(t) \leftrightarrow F'(\omega). \tag{1.55}
\]

**Properties of Fourier Transforms:**

1. Both time $t$ and frequency $\omega$ are real.
2. For the forward transform (time to frequency), the sign of the exponential is negative.
3. The limits on the integrals in both the forward and reverse FTs are $[-\infty, \infty]$.
4. When taking the inverse FT (IFT), the normalization factor of $1/2\pi$ is required to cancel the $2\pi$ in the differential of the integral $d\omega/2\pi = df$, where $f$ is frequency in [Hz], and $\omega$ is the radian frequency [rads].
5. The Fourier step function may be defined by the use of superposition of 1 and $\text{sgn}(t) = t/|t|$ as

\[
\tilde{u}(t) = \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & \text{if } t > 0 \\ 1/2 & t = 0 \\ 0 & \text{if } t < 0 \end{cases}
\]
Table 1.4: The following table provides a short table of simple Fourier Transforms. Note $a > 0 \in \mathbb{R}$ has units [rad/s]. To flag this necessary condition, we use $|a|$ to assure this condition will be met. The other constant $T_o \in \mathbb{R}$ [s] has no restrictions, other than being real. Complex constants may not appear as the argument to a delta function, since complex numbers do not have the order property.

<table>
<thead>
<tr>
<th>$f(t)$ ↔ $F(\omega)$</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\delta}(t)$ ↔ $1(\omega) \equiv 1 \forall \omega$</td>
<td>Dirac</td>
</tr>
<tr>
<td>$1(t) \equiv 1 \forall t$ ↔ $2\pi \tilde{\delta}(\omega)$</td>
<td>Dirac</td>
</tr>
<tr>
<td>$\text{sgn}(t) = \frac{t}{</td>
<td>t</td>
</tr>
<tr>
<td>$\tilde{u}(t) = \frac{1(t) + \text{sgn}(t)}{2}$ ↔ $\pi \tilde{\delta}(\omega) + \frac{1}{j\omega} \equiv \tilde{U}(\omega)$</td>
<td>step</td>
</tr>
<tr>
<td>$\tilde{\delta}(t - T_o)$ ↔ $e^{-j\omega T_o}$</td>
<td>delay</td>
</tr>
<tr>
<td>$\tilde{\delta}(t - T_o) \star f(t)$ ↔ $F(\omega)e^{-j\omega T_o}$</td>
<td>delay</td>
</tr>
<tr>
<td>$\tilde{u}(t)e^{-</td>
<td>a</td>
</tr>
<tr>
<td>$\text{rec}(t) = \frac{1}{T_o} [\tilde{u}(t) - \tilde{u}(t - T_o)]$ ↔ $\frac{1}{T_o} \left(1 - e^{-j\omega T_o}\right)$</td>
<td>pulse</td>
</tr>
<tr>
<td>$\tilde{u}(t) \star \tilde{u}(t)$ ↔ $\delta^2(\omega)$</td>
<td>NaN</td>
</tr>
</tbody>
</table>

**FT Properties**

| $\frac{d}{dt}v(t)$ ↔ $j\omega V(\omega)$ | deriv |
| $f(t) \star g(t)$ ↔ $F(\omega)G(\omega)$ | conv |
Taking the FT of a delayed step function

\[ \tilde{u}(t - T_o) \leftrightarrow \frac{1}{2} \int_{-\infty}^{\infty} [1 - \text{sgn}(t - T_o)] e^{-j\omega t} dt = \pi \tilde{\delta}(\omega) + \frac{e^{-j\omega T_o}}{j\omega} \]

Thus the FT of the step function has the term \( \pi \delta(\omega) \) due to the 1 in the definition of the Fourier step. This term introduces a serious flaw with the FT of the step function: While it appears to be causal, it is not.

6. The convolution \( \tilde{u}(t) \ast \tilde{u}(t) \) is not defined because both \( 1 \ast 1 \) and \( \delta^2(\omega) \) do not exist (and cannot be defined).

7. The inverse FT has convergence problems whenever there is a discontinuity in the time response. This we indicate with a hat over the reconstructed time response. The error between the target time function and the reconstructed is zero in the root-mean sense, but not point-wise. Specifically, at the discontinuity point for the Fourier step function \( (t = 0) \), \( \tilde{u}(t) \neq u(t) \), yet \( \int |\tilde{u}(t) - u(t)|^2 dt = 0 \). At the point of the discontinuity the reconstructed function displays Gibbs ringing (it oscillates around the step, hence does not converge at the jump).\(^{56}\) The L does not exhibit Gibbs ringing. More about this in Section ??.

8. The FT is not always analytic in \( \omega \), as in this example of the step function. The step function cannot be expanded in a Taylor series about \( \omega = 0 \), because \( \tilde{\delta}(\omega) \) is not analytic in \( \omega \).

9. The Fourier \( \delta \) function is denoted \( \tilde{\delta}(t) \), to differentiate it from the Laplace delta function \( \delta(t) \). They differ because the step functions differ, due to the convergence problem described above.

10. One may define

\[ \tilde{u}(t) = \int_{-\infty}^{t} \tilde{\delta}(t) dt, \]

and define the somewhat questionable notation

\[ \tilde{\delta}(t) =: \frac{d}{dt} \tilde{u}(t), \]

since the Fourier step function is not analytic.

11. The \( \text{rec}(t) \) function is defined as

\[ \text{rec}(t) = \frac{\tilde{u}(t) - \tilde{u}(t - T_o)}{T_o} = \begin{cases} 0 & \text{if } t > 0 \\ 1/T_o & 0 < t < T_o \\ 0 & \text{if } t < 0 \end{cases} \]

It follows that \( \tilde{\delta}(t) = \lim_{T_o \to 0} \). Like \( \tilde{\delta}(t) \), the \( \text{rec}(t) \) has unit area.

1.3.10 Lec 20: Laplace transforms

When dealing with engineering problems it is convenient to separate the signals we use from the systems that process them. We do this by treating signals, such as a music signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has no mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that they remain physical). These physical restrictions are described in terms of the Network Postulates, which are briefly discussed in Sect. 1.3.11, and in greater detail in Sect. 3.5.1.

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56\[https://en.wikipedia.org/wiki/Gibbs_phenomenon\]
1.3. STREAM 2: ALGEBRAIC EQUATIONS (12 LECTURES)

Table 1.5: The following table provides a brief table of useful Laplace Transforms. Assume that: \( f(t), \delta(t), u(t), \text{rect}(t), T_o, p, e, \in \mathbb{R} \) and \( F(s), G(s), s, a \in \mathbb{C} \). Note that: \( K_1 \) is the Bessel function of the second kind (i.e., \( J_o \) is of the first kind).

<table>
<thead>
<tr>
<th>( f(t) \leftrightarrow F(s) )</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(t) \leftrightarrow 1 )</td>
<td>Dirac</td>
</tr>
<tr>
<td>( \delta\left</td>
<td>a\right</td>
</tr>
<tr>
<td>( \delta(t - T_o) \leftrightarrow e^{-stT_o} )</td>
<td>delay</td>
</tr>
<tr>
<td>( \delta(t - T_o) \ast f(t) \leftrightarrow F(s)e^{-stT_o} )</td>
<td>–</td>
</tr>
<tr>
<td>( u(t) \leftrightarrow \frac{1}{s} )</td>
<td>pure step</td>
</tr>
<tr>
<td>( e^{-at}u(t) \leftrightarrow \frac{1}{s + a} )</td>
<td>Damped step</td>
</tr>
<tr>
<td>( u(at) \leftrightarrow \frac{a}{s} )</td>
<td>( a &gt; 0 ); dilate</td>
</tr>
<tr>
<td>( u(t - T) \leftrightarrow \frac{1}{s}e^{-st} )</td>
<td>time-shift</td>
</tr>
<tr>
<td>( u(t) \ast u(t) \leftrightarrow 1/s^2 )</td>
<td>ramp</td>
</tr>
<tr>
<td>( u(t) \ast u(t) \ast u(t) = \frac{1}{2}t^2u(t) \leftrightarrow 1/s^3 )</td>
<td>Triple convolution</td>
</tr>
<tr>
<td>( t^pu(t) \leftrightarrow \frac{\Gamma(p+1)}{sp^{p+1}} )</td>
<td>( p \in \mathbb{R} \geq 0 )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{\pi t}}u(t) \leftrightarrow \frac{1}{\sqrt{s}} )</td>
<td>fractional integral</td>
</tr>
<tr>
<td>( e^{\mp at}u(t) \leftrightarrow \frac{1}{s \pm a} ), ( a \in \mathbb{C} )</td>
<td>modulate/damped</td>
</tr>
<tr>
<td>rect(t) = ( \frac{1}{T_o} \left[u(t) - u(t - T_o]\right] \leftrightarrow \frac{1}{T_o} \left(1 - e^{-sT_o}\right) )</td>
<td>pulse</td>
</tr>
<tr>
<td>( \sum_{n=0}^{\infty} \delta(t - nT_o) = \frac{1}{1 - \delta(t - T_o)} \leftrightarrow \frac{1}{1 - e^{-stT_o}} = \sum_{n=0}^{\infty} e^{-snT_o} )</td>
<td>–</td>
</tr>
<tr>
<td>( J_o(at)u(t) \leftrightarrow \frac{1}{s^2 + a^2} )</td>
<td>Bessel</td>
</tr>
<tr>
<td>( J_1(t)u(t)/t \leftrightarrow \sqrt{s^2 + 1 - s} )</td>
<td></td>
</tr>
<tr>
<td>( J_1(t)u(t)/t + 2u(t) \leftrightarrow \sqrt{s^2 + 1 + s} = e^{\sinh^{-1}(s)} )</td>
<td></td>
</tr>
<tr>
<td>( \delta(t) + J_1(t)u(t)/t \leftrightarrow \sqrt{s^2 + 1} )</td>
<td></td>
</tr>
<tr>
<td>( I_o(t)u(t) \leftrightarrow 1/\sqrt{s^2 - 1} )</td>
<td></td>
</tr>
<tr>
<td>( u(t)/\sqrt{t + 1} \leftrightarrow e^s \sqrt{\frac{\pi}{s}} \text{erfc}(\sqrt{s}) )</td>
<td></td>
</tr>
<tr>
<td>( \sqrt{t}u(t) \ast \sqrt{t}u(t) \leftrightarrow e^{s/2}K_1(s/2)/2s )</td>
<td></td>
</tr>
</tbody>
</table>
Definition of the Laplace transform: The forward and inverse Laplace transforms are

\[ F(s) = \int_{t=0}^{\infty} f(t) e^{-st} dt \quad f(t) = \frac{1}{2\pi j} \int_{t=0}^{\infty} F(s) e^{st} ds \]  
(1.56)

\[ F(s) \leftrightarrow f(t) \]  
(1.57)

1. Time \( t \in \mathbb{R} \) and Laplace frequency is defined as \( s = \sigma + \omega j \in \mathbb{C} \).

2. When taking the forward transform \( t \to s \), the sign of the exponential is negative. This is necessary to assure that the integral converges when the integrand converges as \( t \to \infty \). For example, if \( f(t) = e^{\sigma t} u(t) \) (i.e., without the negative \( \sigma \) exponent), the integral does not converge.

3. The target time function \( f(t < 0) = 0 \) (i.e., it must be causal). The time limits are \( 0^- < t < \infty \). Thus the integral must start from slightly below \( t = 0 \) to integrate over a delta function at \( t = 0 \). For example if \( f(t) = \delta(t) \), the integral must include both sides of the impulse. If you wish to include non-causal functions such as \( \delta(t + 1) \) it is necessary to extend the lower time limit. In such cases simply set the lower limit of the integral to \( -\infty \), and let the integrand \( (f(t)) \) determine the limits.

4. The limits on the integrals of the forward transform are \( t : (0^-, \infty) \in \mathbb{R} \), and the reverse LTs are \( [\sigma_o - \infty j, \sigma_o + \infty j] \in \mathbb{C} \). These limits will be further discussed in Section 1.4.9 (p. 88).

5. When taking the inverse FT (IFT), the normalization factor of \( 1/2\pi j \) is required to cancel the \( 2\pi j \) in the differential \( ds \) of the integral.

6. The frequency for the LT must be is complex, and in general \( F(s) \) is complex analytic for \( \sigma > \sigma_o \). It follows that the real and imaginary parts of \( F(s) \) are related. Given \( \Re \{ F(s) \} \) it is possible to find \( \Im \{ F(s) \} \) (Boas, 1987). More on this in Section ?? (p. ??).

7. To take the inverse Laplace transform, we must learn how to integrate in the complex \( s \) plane. This will be explained in Sections 1.4.5-1.4.9 (p. 84-88).
8. The Laplace step function is defined as

\[ u(t) = \int_{-\infty}^{t} \delta(t) \, dt = \begin{cases} 
1 & \text{if } t > 0 \\
\text{NaN} & \text{if } t = 0 \\
0 & \text{if } t < 0 
\end{cases} \]

Alternatively, one could define \( \delta(t) = du(t)/dt \).

9. It is easily shown that \( u(t) \leftrightarrow 1/s \) by direct integration,

\[ F(s) = \int_{0}^{\infty} u(t) e^{-st} \, dt = \left. \frac{-e^{-st}}{s} \right|_{0}^{\infty} = \frac{1}{s} . \]

With the LT step \( (u(t)) \) there is no Gibbs ringing effect.

10. In many physical applications, the Laplace transform takes the form of a ratio of two polynomials. In such case, the roots of the numerator polynomial are called the zeros while the roots of the denominator polynomial are called the poles. For example, the LT of \( u(t) \leftrightarrow 1/s \) has a pole at \( s = 0 \), which represents integration, since

\[ u(t) \ast f(t) = \int_{-\infty}^{t} f(\tau) \, d\tau \leftrightarrow \frac{F(s)}{s} . \]

11. The LT is quite different from the FT in terms of its analytic properties. For example, the step function \( u(t) \leftrightarrow 1/s \) is complex analytic everywhere, except at \( s = 0 \). The FT of \( 1 \leftrightarrow 2\pi \delta(\omega) \) is not analytic anywhere.

Once complex integration in the complex plane has been defined (Section 1.4.2, p. 78), we can justify the definition of the inverse LT (Eq. 1.56).

1.3.11 Lec 21: Ten network postulates

Solutions of differential equations, such as the wave equation, are conveniently described in terms of mathematical properties, which we present here in terms of 10 network postulates:

(P1) causality (non-causal/acausal): Causal systems respond when acted upon. Virtually all physical systems obey causality. An example of a causal system is an integrator, which has a response of a step function. Filters are also examples of causal systems. Signals represent acausal responses. They do not have a clear beginning or end, such as the sound of the wind or traffic noise.

(P2) linearity (nonlinear): Linear systems obey superposition. If two signals \( x(t) \) and \( y(t) \) are the inputs to a linear system, producing outputs \( x'(t) \) and \( y'(t) \), then if the inputs are presented together as \( ax(t) + by(t) \), with weights \( a, b \in \mathbb{R} \), then the output will be \( ax'(t) + by'(t) \). If either \( a, b \) are zero, that signal is removed from the output.

Nonlinear systems mix two inputs, thereby producing other signals not present in the input. For example, if the inputs to a nonlinear system are two sine waves, the output will contain distortion components, having frequencies not present at the input. An example of a nonlinear system is one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current flow only in one direction. Most physical systems have some degree of nonlinear response, but this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

(P3) passive (active): An active system has a power source, such as a battery while a passive system has no power source. While you may consider a transistor amplifier to be active, it is only so when connected to a power source.
(P4) real (complex) time response: Typically systems are “real in, real out.” They do not naturally have complex responses (real and imaginary parts). While a Fourier transform takes real inputs and produces complex outputs, this is not an example of a complex time response. P4 is a characterization of the input signal, not its Fourier transform.

(P5) time-invariant (time varying): For a system to be time varying system its properties must depend on when the signal starts and stops. If the output, relative to the input, is independent of the starting time, then the system is time-invariant.

(P6) reciprocal (non- or anti-reciprocal): In many ways this is the most difficult property to understand. It is best characterized by the ABCD matrix. If $B = C$ it is said to be reciprocal. If $B = -C$ it is said to be anti-reciprocal. The loudspeaker is anti-reciprocal, which is why it is modeled by the gyrator rather than a transformer. All non-reciprocal systems are modeled by such gyrator, which swap the force and flow variables. In some ways this property is beyond the scope of this book.

(P7) reversibility (non-reversible): If the system can be flipped, between input and output, the it is said to be reversible. In terms of the ABCD parameters, if $A = D$ is is reversible.

(P8) space-invariant (space-variant): If a system operates independently as a function of where it physically is in space, then it is space-invariant. When the parameters that characterize the system depend on position, it is space-variant.

(P9) Deterministic (random): Predictability depends on the light cone. Given the wave equation, along with the boundary conditions, the solution may be deterministic, or not, depending on the extent of the “light cone.” Consider the wave with a wave propagating out into uncharted territory. When the wave hits an object, the reflection will return a wave that might considered not be predicted. This is an example of the boundary condition being the radiation impedance, which is not known in advance. One must do the experiment to do solve the problem, and the radiation impedance is not known in advance. To define the impedance requires that it be time invariant (P5). One cannot look beyond the light-cone, which depends on the system to be time invariant.

(P10) quasi-static (multi-modal): Quasi-statics follows from systems that are small compared to the wavelength. This is a very general assumption, that must be false when the frequency is raised and the wavelength becomes short. Thus this is also known as the long-wavelength approximation. It is a very powerful tool in modeling systems such as transmission lines.

Each postulate has two (or more) categories. For example for (P2) a system is either linear or non-linear and for (P1) is either causal, non-causal or acausal. P6 and P9 only apply to 2-port networks (those having an input and an output). The others can apply to both a 2- or 1-port networks (e.g., an impedance is a 1-port).

Examples. Related forms of these postulates had been circulating in the literature for many years, widely accepted in the network theory literature (Van Valkenburg, 1964a,b; Ramo et al., 1965). The first six of these were formally introduced Carlin and Giordano (1964), and (P7-P9) were added by Kim et al. (2016).

1.3.12 Lec 22: Exam II (Evening Exam)
1.4 Stream 3a: Scalar (i.e., Ordinary) Differential Equations

Stream 3 is $\infty$, a concept which typically means unbounded (immeasurably large), but in the case of calculus, $\infty$ means *infinitesimal* (immeasurably small), since taking a limit requires small numbers. Taking a limit means you may never reaching the target, a concept that the Greeks called *Zeno’s paradox* (Stillwell, 2010, p. 76).

When speaking of the class of *ordinary* (versus *vector*) differential equations, the term *scalar* is preferable, since the term “ordinary” is vague, if not a meaningless label. There are a special subset of fundamental theorems for scalar calculus, all of which are about integration, as summarized in Table 1.7, starting with Leibniz’s theorem. These will be discussed below, and more extensively in Lec. 1.4.1 (p. 77) and Chapters 4 (p. 167) and 5 (p. 167).

Table 1.7: *Summary of the fundamental theorems of integral calculus, each of which deals with integration. There are at least two main theorems related to scalar calculus, and three more for vector calculus.*

<table>
<thead>
<tr>
<th>Name</th>
<th>Mapping</th>
<th>p.</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leibniz</td>
<td>$\mathbb{R}^1 \rightarrow \mathbb{R}^1$</td>
<td>77</td>
<td>Area under a real curve.</td>
</tr>
<tr>
<td>Cauchy</td>
<td>$\mathbb{C}^1 \rightarrow \mathbb{C}^0$</td>
<td>77</td>
<td>Residue integration and complex analytic functions.</td>
</tr>
<tr>
<td>Gauss’s Law</td>
<td>$\mathbb{R}^3 \rightarrow \mathbb{R}^2$</td>
<td>115</td>
<td>Conservation of mass and charge crossing a closed surface.</td>
</tr>
<tr>
<td>Stokes</td>
<td>$\mathbb{R}^3 \rightarrow \mathbb{R}^1$</td>
<td>115</td>
<td>Line integral $\equiv \nabla \times$ flux crossing an open surface $\in \mathbb{R}^3$.</td>
</tr>
<tr>
<td>Green</td>
<td>$\mathbb{R}^2 \rightarrow \mathbb{R}^0$</td>
<td>78</td>
<td>Special case of Stokes for a plane</td>
</tr>
<tr>
<td>Helmholtz</td>
<td></td>
<td>117</td>
<td>Vector decomposed as dilatation and rotation.</td>
</tr>
</tbody>
</table>

Following the integral theorems on scalar calculus, are those on vector calculus, without which there could be no understanding of Maxwell’s equations. Of these, the *fundamental theorem of complex calculus* (aka, Helmholtz decomposition), Gauss’s Law and Stokes theorem, form the cornerstone of modern vector field analysis. These theorems allow one to connect the differential (point) and macroscopic (integral) relationships. For example, Maxwell’s equations may be written as either vector differential equations, as shown by Heaviside (along with Gibbs and Hertz), or in integral form. It is helpful to place these two forms side-by-side, to fully appreciate their significance. To understand the differential (microscopic) view, one must understand the integral (macroscopic) view. These are presented in Sections 1.144 (p. 114) and Fig. 1.25 (p. 116).

The beginning of modern mathematics

The birth of mathematics as we know it today, occurred during the 16th to 18th centuries, perhaps starting with Galileo, Descartes, Fermat, Newton, the Bernoulli family, and Euler, as outlined in Fig. 1.17 (p. 76). Galileo was formidable, due to his fame, fortune, and his “successful” stance against the powerful Catholic establishment. His creativity in scientific circles was certainly well known due to his many skills and accomplishments. Fermat and Descartes were at the forefront of merging algebra and geometry, but while Fermat kept meticulous notebooks, he did not publish what he knew, and tended to be secretive.

Regarding the development of calculus, much was yet to be done by Newton and Leibniz, using term by term integration of functions, based on a Taylor series representation. This was a powerful technique, but as stated, incomplete, because the series is only valid for single-valued functions, within the RoC. But more importantly, Newton (and others) failed to recognize the powerful generalization to complex analytic functions. The first breakthrough was Newton’s publication of Principia (1687), and a second was Riemann (1851), with his thesis, advised by Gauss.

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CHAPTER 1. INTRODUCTION

Chronological history post 16th century

16th

Bombelli 1526-1572

17th

Galileo 1564-1642, Kepler 1571-1630, Newton 1642-1727, Principia 1687; Mersenne; Huygens; Pascal; Fermat; Descartes (analytic geometry); Bernoullis Jakob, Johann & son Daniel

18th

Euler 1707-1783; Student of Johann Bernoulli; d’Alembert 1717-1783; Kirchhoff; Lagrange; Laplace; Gauss 1777-1855

19th


20th

Hilbert; Einstein; . . .

Time Line

<table>
<thead>
<tr>
<th>1525</th>
<th>1600</th>
<th>1700</th>
<th>1800</th>
<th>1875</th>
<th>1925</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bombelli</td>
<td>Descartes</td>
<td>Newton</td>
<td>Johann Bernoulli</td>
<td>Mersenne</td>
<td>Riemann</td>
</tr>
<tr>
<td>Mersenne</td>
<td>Huygens</td>
<td>Euler</td>
<td>Gauss</td>
<td>Daniel Bernoulli</td>
<td>Cauchy</td>
</tr>
<tr>
<td>Galileo</td>
<td>Jacob Bernoulli</td>
<td>d’Alembert</td>
<td>Lagrange</td>
<td>Mozart</td>
<td>US Civil War</td>
</tr>
</tbody>
</table>

Figure 1.17: Time-line of the four centuries from the 16th and 20th CE. Given the time line, it is likely that Bombelli’s discovery of Diophantus’ book on “Arithmetic” in the Vatican library, triggered many of the ideas presented by Descartes and Fermat, followed by others (i.e., Newton). Bombelli’s discovery might be considered as a magic moment in mathematics. The vertical red lines connect mentor-student relationships. Figure 1.4 (p. 19) for a closer look at the 15-18c, and Fig. 1.11 (p.45) for a closer look at the 18-20c. ![fig:TimeLineCE]

Following Newton’s lead, the secretive and introverted behavior of the typical mathematician dramatically changed with the Bernoulli family (Fig. 1.17, p. 76). The oldest brother, Jacob, taught his much younger brother Johann, who then taught his son Daniel, and Johann’s star pupil, Euler. Euler first mastered all the tools and then published, with a prolific voracity previously unknown.

Euler and the circular functions: The first major task was to understand the family of analytic circular functions, $e^x$, $\sin(x)$, $\cos(x)$, and $\log(x)$, a task begun by the Bernoulli family, but mastered by Euler. Euler sought relations between these many functions, some of which may not be thought of as being related, such as the log and sin functions. The connection that may “easily” be made is through their Taylor series (Eq. 1.26, p. 47). By the manipulation of the analytic series representations, the relationship between $e^x$, and the $\sin(x)$ and $\cos(x)$ was precisely captured by

$$e^{\pi j} = \cos(x) + j \sin(x),$$

and its analytic inverse (Greenberg, 1988, p. 1135)

$$\ln(x) = \tan^{-1} \left( \frac{x - 1j}{x + 1j} \right).$$

These not only relate the exponential function to the circular functions, but also contains $j = \sqrt{-1}$. While every high school student has learned Euler’s relation, it is doubtful (i.e., rare) that they fully appreciate the significance of complex analytic functions (Eq. 1.30, p. 49). While Euler found these relationships using real power series (i.e., analytic functions), neither he, nor those who followed, seemed to fully appreciated the importance of complex analytic, and its potential role. For example, Newton famously ignored imaginary numbers, and called them imaginary in a disparaging (pejorative) way. Given Newton’s prominence, that certainly must have attenuated any interest in any complex algebra,
even though it had been previously quantified by Bombelli in 1525, likely based on his serendipitous finding of Diophantus’ book *Arithmetic* in the Vatican library.

While Euler was fluent with \( j = \sqrt{-1} \), he did not consider functions to be complex analytic. That concept was first explored by Cauchy, almost a century later. The missing link to the concept of complex analytic is the definition of the derivative with respect to the complex argument (i.e., \( dF(s)/ds \) with \( s = \sigma + \omega j \)), without which the complex analytic Taylor coefficients may not be defined.

### 1.4.1 Lec 23: Fundamental theorems of calculus

**History of the fundamental theorem of calculus:** In some sense, the story of calculus begins with the fundamental theorem of calculus (FTC), also known generically as Leibniz’s formula. The simplest integral is the length of a line \( L = \int_0^L dx \). If we label a point on a line as \( x = 0 \) and wish to measure the distance to any other point \( x \), we form the line integral between the two points. If the line is straight, this integral is simply the Euclidean length given by the difference between the two ends (Eq. 1.36, p. 57).

If \( F(x) \) describes a height above the line, then the area under \( F(x) \) is

\[
f(x) - f(0) = \int_{x=0}^x F(\chi) d\chi,
\]

(1.58)

where \( \chi \) is a dummy variable of integration. Thus the area under \( F(x) \) (\( F(x), x \in \mathbb{R} \)) only depends on the difference of the area evaluated at the end points. This makes intuitive sense, by viewing \( f(x) \) as the area of a graph of \( F(x) \).

This property of the area as an integral only depending on the end points, has important consequences in physics in terms of conservation of energy. Thus generalizations are important.

**Fundamental theorems of real calculus:**

If \( F(x) \) is analytic (Eq. 1.25, p. 47), then

\[
F(x) = \frac{d}{dx} f(x),
\]

(1.59)

which is known as the fundamental theorem of (real) calculus (FTC). Thus Eq. 1.59 may be viewed as an anti-derivative, or exact differential. This is easily shown by evaluating

\[
\frac{d}{dx} f(x) = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta} = F(x),
\]

starting from Eq. 1.58. If \( F(x) \) is not analytic then the limit may not exist, so this seems like a necessary condition. There are many important variations on this very basic theorem (see Sect. 1.4).

**Exercises:** Learn to use Matlab’s related functions \texttt{root()}, \texttt{poly()}, \texttt{polyval()}, \texttt{residue()}, \texttt{conv()}, manipulating roots, polynomials and residues.

**Case of a complex path** The fundamental Theorem of complex calculus (FTCC) states that for any complex analytic function \( F(s) \in \mathbb{C} \) with \( s = \sigma + \omega j \in \mathbb{C} \)

\[
f(s) - f(s_0) = \int_{s_0}^s F(\zeta) d\zeta.
\]

(1.60)

(Greenberg, 1988, p. 1197). If we compare this to Eq. 1.59, they differ in that the path of the integral is complex, meaning that the integral is over \( s \in \mathbb{C} \), rather than a real integral over \( x \in \mathbb{R} \). The fundamental theorems of complex calculus (FTCC) states that the integral only depends on the end points, since

\[
F(s) = \frac{d}{ds} f(s).
\]

(1.61)

Comparing Eq. 1.59 (FTC) with Eq. 1.61 (FTCC), it would appear that this can only be true if \( F(s) \in \mathbb{C} \) is complex analytic, meaning it must have a Taylor series in powers of \( s \in \mathbb{C} \).
Complex analytic functions: The definition of a complex analytic function $F(s)$ of $s \in \mathbb{C}$ is that the function may be expanded in a Taylor series (Eq. 1.30, p. 49) about an expansion point $s_o \in \mathbb{C}$. This definition follows the same logic as the FTC. Thus we need a definition for the coefficients $c_n \in \mathbb{C}$, which most naturally follow from Taylor’s formula

$$c_n = \frac{1}{n!} \frac{d^n F(s)}{ds^n} \bigg|_{s=s_o}.$$  \hfill (1.62)

The requirement that $F(s)$ have a Taylor series naturally follows by taking derivatives with respect to $s$ at $s_o$. The problem is that both integration and differentiation of functions of complex Laplace frequency $s = \sigma + j\omega$ have not yet been defined.

Thus the question is: “What does it mean to take the derivative of a function $F(s) \in \mathbb{C}$, $s = \sigma + j\omega \in \mathbb{C}$, with respect to $s$, where $s$ defines a plane rather than a real line?” We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?

The answer is affirmative. The question may be resolved by applying the rules of the real derivative when defining the derivative in the complex plane. However for the complex case, there is an issue, regarding direction. Given any analytic function $F(s)$, is the partial derivative with respect to $\sigma$ different from the partial derivative with respect to $j\omega$? For complex analytic functions, the FTCC states that the integral is independent of the path in the $s$ plane. Based on the chain rule, the derivative must also be independent of direction at $s_o$. This directly follows from the FTCC. If the integral of a function of a complex variable is to be independent of the path, the derivative of a function with respect to a complex variable must be independent of the direction. This follows from Taylor’s formula, Eq. 1.62 for the coefficients of the complex analytic formula. The FTC is an integral over an area. The FTCC is an integral over a complex function along a complex path. Both relate an area under an integral. But what is the meaning of “area” for the complex line integral? The Cauchy-Riemann conditions provide the answer.

1.4.2 Lec 24: Cauchy-Riemann conditions

For the integral of $Z(s) = R(\sigma, \omega) + X(\sigma, \omega)j$ to be independent of the path, the derivative of $Z(s)$ must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the Cauchy-Riemann conditions. This is an important generalization of Eq. 1.1, p. 15 which goes from real integration ($x \in \mathbb{R}$) to complex integration ($s \in \mathbb{C}$), based on lengths, thus on area.

Taking partial derivatives of $Z(s)$, with respect to $\sigma$ and $j\omega$, and then equating the real and imaginary parts, gives

$$CR-1: \frac{\partial R(\sigma, \omega)}{\partial \sigma} = j \frac{\partial X(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad CR-2: \frac{\partial R(\sigma, \omega)}{\partial \omega} = j \frac{\partial X(\sigma, \omega)}{\partial \sigma}.$$  \hfill (1.63)

The $j$ cancels in CR-1, but introduces a $j^2 = -1$ in CR-2. These equations are known as the Cauchy-Riemann (CR) conditions. They may also be written in polar coordinates.

The CR conditions are necessary conditions that the integral of $Z(s)$, and thus its derivative, be independent of the path, expressed in terms of conditions on the real and imaginary parts of $Z$. This is a very strong condition on $Z(s)$, which follows assuming that $Z(s)$ may be approximated by a Taylor series in $s$

$$Z(s) = Z_o + Z_1 s + \frac{1}{2} Z_2 s^2 + \cdots,$$  \hfill (1.64)

where $Z_n \in \mathbb{C}$ are complex constants given by the Taylor series formula (Eq. 1.62, p. 78).

Every function that may be expressed as a Taylor series about a point $s$ is said to be complex analytic at that point. This series, which is single valued, is said to converge within a radius of convergence (RoC). This highly restrictive condition has significant physical consequences. As an important example, every impedance function $Z(s)$ obeys the CR conditions over large regions of the $s$ plane, including the entire right half plane (RHP) ($\sigma > 0$). This condition is summarized by the Brune condition $\Re \{Z(\sigma > 0)\} \geq 0$ (Eq. 1.71, Section 1.4.3).
When this condition is generalized to volume integrals, it is called Green’s theorem, which is a special case of both Gauss’s and Stokes’s Laws (Greenberg, 1988, p. 819), used heavily in the solution of boundary value problems in engineering and physics (e.g., solving problems that begin with Maxwell’s equations). The last chapter of this course shall depend heavily on these concepts.

We may merge these equations into a pair of second order equations by taking a second round of partials. Specifically, eliminating the real part \( R(\sigma, \omega) \) of Eq. 1.63 gives

\[
\frac{\partial^2 R(\sigma, \omega)}{\partial \sigma \partial \omega} = \frac{\partial^2 X(\sigma, \omega)}{\partial^2 \omega} = -\frac{\partial^2 X(\sigma, \omega)}{\partial^2 \sigma},
\]

which may be compactly written as \( \nabla^2 X(\sigma, \omega) = 0 \). Eliminating the imaginary part gives

\[
\frac{\partial^2 X(\sigma, \omega)}{\partial \omega \partial \sigma} = \frac{\partial^2 R(\sigma, \omega)}{\partial^2 \sigma} = -\frac{\partial^2 R(\sigma, \omega)}{\partial^2 \omega},
\]

may be written as \( \nabla^2 R(\sigma, \omega) = 0 \).

In summary, for a function \( Z(s) \) to be complex analytic, the derivative \( dZ/ds \) must be independent of direction (path), which requires that the real and imaginary parts of the function obey Laplace’s equation, i.e.,

\[
\nabla^2 R(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 X(\sigma, \omega) = 0.
\]

The CR equations are easier to work with because they are first order, but the physical intuition is best understood by noting that the real and imaginary parts of the function each obey Laplace’s equation. As you learn about solutions to Laplace’s equation, this will become more clear.

As we shall see in the next few lectures, analytic functions must be very smooth in magnitude, like and area or volume, due to the above condition. They must be maximum on their boundary. For example, when you stretch a rubber sheet over a jagged form the height sheet obeys Laplace’s equation. Nowhere does the height of the sheet rise above the value at the boundary. Such functions are very smooth, having no second derivatives.

Such functions define Conservative fields, which means that energy (like an area) is conserved. The work done in moving a mass from \( a \) to \( b \) in such a field is conserved. If you return the mass from \( b \) back to \( a \), the energy is retrieved, and zero net work has been done.

### 1.4.3 Lec 25: Complex Analytic functions and Brune Impedance

It is rarely stated that the variable that we are integrating over, either \( x \) (space) or \( t \) (time), is real \( x, t \in \mathbb{R} \), since that fact is implicit, due to the physical nature of the formulation of the integral. But this intuition must be refined once complex numbers are included with \( s \in \mathbb{C} \), where \( s = \sigma + \omega \), as first developed by Cauchy (1789-1857), with the aid of Gauss (1777-1855).

The fact that time and space are real variables is more than an assumption, rather it is a requirement, due to the order property. Real numbers have order. For example, if \( t = 0 \) is now (the present), then \( t < 0 \) is the past and \( t > 0 \) is the future. The order property of time and space allows one to order these along a real axis. If the axis were complex, as in frequency \( s \), the order property is lost. It follows that if we desire order, time and space must be real (\( t, x \in \mathbb{R} \)). Interestingly, it has was shown by d’Alembert (1747) that time and space are related by the wave speed \( c \). To obtain a solution to the governing wave equation, that Newton first proposed for sound waves, \( x, t \in \mathbb{R}^3 \) may be combined as

\[
\zeta = t \pm x/c_o,
\]

where \( c \) [m/s] is the phase velocity of the waves. The d’Alembert solution to the wave equation, describing waves on a string under tension, is

\[
u(x, t) = F(t - x/c_o) + G(t + x/c_o),
\]

which describes the transverse velocity (or displacement) of two independent waves \( F(\xi), G(\xi) \) on the string, which represent forward and backward traveling waves. For example, starting with a string at
rest, if one displaces the left end, at \( x = 0 \), by a step function \( u(t) \), then that step displacement will propagate to the right as \( u(t - x/c_o) \), arriving at location \( x_o \) at time \( c_o t_o \). Before this time, the string will not move to the right of the wave-front, at \( x_o \), and after \( t_o \) it will have displacement 1. Since the wave equation obeys superposition (postulate P1, p. 73), it follows that the “plane-wave” eigen-functions of the wave equation for \( x, k \in \mathbb{R}^3 \) are given by

\[
\Psi_{\pm}(x, t) = e^{st \pm jk\cdot x}. \tag{1.69}
\]

where \( s = \sigma + \omega j \) and \( k = \omega / \lambda \) is the wave number, \( \lambda = |\lambda| \) is the wavelength, and \( |s/k| = \lambda f = c \). When propagation losses are considered, we must replace \( jk \) with a complex analytic wave number \( \kappa(s) = k_o + jk \), known as the dispersion relation. An important example is the case of electron waves propagating in crystals (i.e., silicon). For this special case, \( \kappa(s) \) describes the crystal’s dispersion relations, known as Brillouin zones (Brillouin, 1953).

### Complex impedance functions

Conservation of energy (or power) is a corner stone of modern physics. It may have first been under consideration by Galileo Galilei (1564-1642). Today the question is not if it is true, the questions should be why. Specifically, what is the physics behind conservation of energy. Surprisingly, the answer is straight forward, by looking at the definition of power and impedance. The power is given by the product of two variables, sometimes called conjugate variables, the force and the flow. In electrical terms, these are voltage (force) \((v(t) \leftrightarrow V(\omega))\) and current (flow) \((i(t) \leftrightarrow I(\omega))\), thus the electrical power is

\[
\mathcal{P}(t) = v(t)i(t) \leftrightarrow V(\omega)I(\omega). \tag{1.70}
\]

One of the most important and obvious applications of complex functions of a complex variable is the impedance function.

### Every impedance must obey conservation of energy (P3)

According to Postulate P3 Section 1.3.11 (p. 73), a system is passive if it does not contain a power source. Drawing power from an impedance violates conservation of energy. This propriety is also called positive real, which is defined as (Brune, 1931a,b)

\[
\Re\{Z(s \geq 0)\} \geq 0, \tag{1.71}
\]

namely the real part of every impedance must be non-negative for \( \sigma \geq 0 \). When this condition holds, one cannot draw more power than stored in the impedance. A second condition requires that the impedance has simple poles. If there were a pole in the region \( \sigma > 0 \), then the first condition would not be met. Therefore there can only be simple poles (degree of 1) in the region \( \sigma \leq 0 \). The region \( \sigma \leq 0 \) is called the left half s plane (LHP), and the complementary region, \( \sigma > 0 \), is called the right half s plane (RHP). The condition on the simple poles is sufficient, but not necessary, as \( Z(s) = 1/\sqrt{s} \) is a physical impedance, but is not a first order pole. The impedance function \( Z(s) = R(\sigma, \omega) + jX(\sigma, \omega) \) has resistance \( R \) and reactance \( X \) as a function of complex frequency \( s = \sigma + j\omega \). The function \( z(t) \leftrightarrow Z(s) \) are defined by a Laplace transform pair. From the causality postulate (P1) of Sections 1.3.11 and 3.5.1 (p. 160), \( z(t < 0) = 0 \).

As an example, a resistor \( R_o \) in series with an capacitor \( C_o \) has an impedance given by (Table 1.5, p. 71)

\[
Z(s) = R_o + 1/sC_o \leftrightarrow R_o \delta(t) + 1/C_o u(t) = z(t), \tag{1.72}
\]

with constants \( R_o, C_o \in \mathbb{R} > 0 \). In mechanics an impedance composed of a dash-pot (damper) and a spring have the same form. A resonant system has an inductor, resistor and a capacitor, with an impedance given by (Table 1.6, p. 72)

\[
Z(s) = \frac{sC_o}{1 + sC_oR_o + s^2C_oM_o} \leftrightarrow C_o \frac{d}{dt} (c_+ e^{s+t} + c_- e^{s-t}) = z(t). \tag{1.73}
\]
is a second degree polynomial with two complex resonant frequencies $s_{\pm}$. When $R_o > 0$ these roots are in the left half $s$ plane, with $z(t) \leftrightarrow Z(s)$.

Systems (networks) containing many elements, and transmission lines can be much more complicated, yet still have a simple frequency domain representation. This is the key to understanding how these physical systems work, as will be described below.

**Complex analytic functions:** To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However he only used real functions of a real variable, due to the fundamental lack of appreciation of the complex analytic series. This same method is how one finds solutions to scalar differential equations today, but using an approach that makes the solution method less obvious. Rather than working directly with the Taylor series, today we use the complex exponential, since the complex exponential is the eigenfunction of the derivative $d/dt e^{st} = se^{st}$.

Since $e^{st}$ may be expressed as a Taylor series, having coefficients $c_n = 1/n!$, in some real sense the modern approach is a compact way of doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency $s$, by looking for solutions of the form $A(s)e^{st}$, transforming the differential equation into a polynomial $A(s)$ in complex frequency. For example

$$\frac{d}{dt} f(t) + af(t) \leftrightarrow (s + a)F(s).$$

The root of $A(s_r) = s_r + 0 = 0$ is the eigenvalue of the differential equation. The key to understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain. The Taylor series has been replaced by $e^{st}$, transforming Newton’s real Taylor series into the complex exponential eigenfunction. In some sense, these are the same method, since

$$e^{st} = \sum_{n=0}^{\infty} \frac{(st)^n}{n!}. \quad (1.74)$$

Taking the derivative with respect to time gives

$$\frac{d}{dt} e^{st} = se^{st} = \sum_{n=0}^{\infty} \frac{(st)^n}{n!}, \quad (1.75)$$

which is also complex analytic. Thus if the series for $F(s)$ is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key here is “Where does the series fail to converge?” in which case, the entire representation fails. This is the main message behind the FTCC (Eq. 1.61).

The FTCC (Eq. 1.61) is formally the same as the FTC (Eq. 1.59) (Leibniz formula), the key (and significant) difference being that the argument of the integrand $s \in \mathbb{C}$. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depends on the path of integration.

But, according to FTCC, it does not. In fact it is indistinguishable from its much simple cousin, the fundamental theorem of calculus. And the reasoning is the same. If $F(s) = df(s)/ds$ is complex analytic (i.e., has a power series $f(s) = \sum c_k s^k$, with $f(s), c_k, s \in \mathbb{C}$), then it may be integrated, and the integral does not depend on the path. This is sort of amazing. The key is that $F(s)$ and $f(s)$ must be complex analytic, which means they are differentiable. This all follows from the Taylor series formula Eq. 1.62 (p. 78) for the coefficients of the complex analytic series. For Eq. 1.61 to hold, the derivatives
must be independent of the direction, as discussed in Section 1.4.2. The concept of a complex analytic function therefore has eminent consequences.

The use of the complex Taylor series formula in the complex plane, generalizes the functions they describe, with unpredictable consequences, as nicely shown by the domain coloring diagrams presented in Section 1.3.8 (p. 66). The tool of complex integration was first exploited in physics by Sommerfeld (1952), to explain the onset transients in waves, as explained in detail in Brillouin (1960, Chap. 3). Up to 1910, when Sommerfeld first published his results using complex analytic signals and saddle point integration in the complex plane, there was a poor understanding of the implications of the causal wave-front. It would be reasonable to say that his insights changed our understanding of wave propagation, for both light and sound. Sadly this insight has not been fully appreciated, even to this day. If you question this summary, please read Brillouin (1960, Chap. 1).

The full power of the complex analytic function was first appreciated by Bernard Riemann (1826-1866), in his University of Göttingen PhD Thesis of 1851, under the tutelage of Carl Friedrich Gauss (1777-1855), and drawing heavily on the work of Cauchy.

The key definition of a complex analytic function is that it has a Taylor series representation over a region of the complex frequency plane \( s = \sigma + j\omega \), that converges in a region of convergence (RoC) about the expansion point, with a radius determined by the nearest pole of the function. A further surprising feature of all analytic functions is that within the RoC, the inverse of that function also has a complex analytic expansion. Thus given \( w(s) \), one may also determine \( s(w) \) to any desired accuracy, critically depending on the RoC.

1.4.4 Lec 26: Multi-valued functions

This is a time-out for an entire lecture to answer your questions that have been building, on the many new concepts presented over the last nine weeks.

The best question of the week: “What is a multi-valued function?” Second best question: “What is a line integral?,” “What is the difference between the impedance of a mass and inductor?” and “How are physics and math related?”

Branch Cuts: The concept of a branch cut allows one to manipulate (and visualize) multi-valued functions, by breaking each region into a single valued sheets. The concepts of the branch cut and the sheets, along with the extended plane, were first devised by Riemann, working with Gauss (1777-1855), first described in his thesis of 1851. Of course it was these three mathematical constructions that provide the deep insight to complex analytic functions, supplementing the important earlier work of Cauchy (1789-1857), on the calculus of complex analytic functions.

The branch cut is a line that separates the various single valued parts of a multi-valued function. For example, in Fig. 1.18 we see the double-valued mapping of \( w(z) = \pm \sqrt{z} \). Since the square root function has two overlapping regions, corresponding to the \( \pm \) of the radical, there must be two connected regions, sort of like mathematical Siamese-twins, distinct, yet the same.

This concept of analytic inverses becomes rich when the inverse function is multi-valued. For example, if \( F(s) = s^2 \) then \( s(F) = \pm \sqrt{F} \). Riemann dealt with such extensions with the concept of a branch-cut with multiple sheets, labeled by a sheet number. Each sheet describes an analytic function (Taylor series), that converges within some RoC, with a radius out to the nearest pole of that function. This Riemann’s branch cut and sheets explicitly deal with the need to define unique single valued inverses of multi-valued functions.

Branch cuts emanate from poles that have non-integral degrees, and terminate at either another pole of the same degree, or at \( \infty \). For example, suppose that in the neighborhood of the pole, at \( s_o \), the function is

\[
f(s) = \frac{w(s)}{(s - s_o)^k},
\]

where \( w, s, K, k \in \mathbb{C} \). Here \( w(s_o) \) is the residue, when \( k = 1 \), \( s = \sigma + \omega j \), \( s_o = \sigma_o + \omega_o j \) is the pole.
1.4. STREAM 3A: SCALAR CALCULUS (11 LECTURES)

Figure 1.18: Here we see the mapping for the square root function \( w(z) = \pm \sqrt{z} \) which has two single-valued sheets, corresponding to the two signs of the square root. The lower sheet is \(+\sqrt{z}\), and the upper sheet is \(-\sqrt{z}\). The location of the branch cut may be moved by rotating the \( z \) coordinate system. For example, \( w(z) = \pm j\sqrt{z} \) and \( w(z) = \pm \sqrt{-z} \) have a different branch cuts, as may be easily verified using the Matlab/Octave commands \( j*zviz(z) \) and \( zviz(-z) \). A function is analytic on the branch cut, since the cut may be moved. If a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the complex analytic series (i.e., the Taylor formula, Eq. 1.62) does not depend on the location of a branch cut, as it only describes the function uniquely (as a single valued function), valid in its local region of convergence. This figure has been taken from Stillwell (2010, p. 303).

location, and \( k \) is some complex or real constant that defines the order of the pole (not the degree, as in the degree of a polynomial).

Up to this point we have assumed that the pole order is an integer \( k \in \mathbb{Z} \) or fractional \( (k \in \mathbb{F}) \). When \( k \in \mathbb{Z} \) there is no branch cut. When \( k \in \mathbb{F} \) there must be a branch cut, of order \( k \). For example, if \( k = 1/2 \), the pole is of order 1/2, and there are two Riemann sheets, as shown in Fig. 1.18. An important example is the Bessel function

\[
\delta(t) + \frac{1}{\pi} J_1(t) u(t) \leftrightarrow \sqrt{s^2 + 1},
\]

which is related to the solution to the wave equation in two-dimensional cylindrical coordinates (Table 1.5, p. 71). Bessel functions are the solutions to guided acoustic waves in round pipes, or surface waves on the earth (seismic waves) or waves on the surface of a pond.

It is important to understand that the function is analytic on the branch cut, but not at the branch point (the pole). One is free to move the branch cut, almost at will. It does not need to be on a line, it could be cut as a spiral. The only rule is that it must start and stop at two poles of the same order, or at \( \infty \), which must have a pole of order \( k \).

When the pole has an irrational order \( (k \in \mathbb{I}) \), the branch cut has the same irrational order. Accordingly there must be an infinite number of Riemann sheets, as in the case of the log function. An example is \( k = \pi \), for which

\[
F(s) = \frac{1}{s^\pi} = e^{-\log(s^\pi)} = e^{-\pi \log(s)} = e^{-\pi \log(\rho)} e^{-\pi \theta j},
\]

where the domain is expressed in polar coordinates \( s = \rho e^{\theta j} \). When the irrational number is very close to 1, the branch cut could be very subtle (it could even be unnoticed), but it would have an impact on the nature of the function, and of course, on the inverse Laplace transform. In this course we shall not attempt to deal with irrational pole order.

**Multivalued functions:** The two basic functions we review, to answer the questions about multivalued functions and branch cuts, are \( w(s) = s^2 \) and \( w(s) = e^s \), along with their inverse functions \( w(s) = \sqrt{s} \) and \( w(s) = \log(s) \). For uniformity we shall refer to the abscissa \( (s = \sigma + \omega j) \) and the ordinate \( w(s) = u + v j \). The proper, but less well known terms are the *domain* and *co-domain*, today’s nomenclature.
CHAPTER 1. INTRODUCTION

Square root function: The multi-valued nature of the square root is best understood by working with the function in polar coordinates. Let

\[ s_k = re^{\theta_j e^{2\pi kj}}, \]

where \( k \) is the sheet-index, and

\[ w = \rho e^{\psi_j} = \sqrt{r} e^{\theta/2} e^{\pi k}, \]

where \( r, \rho \in \mathbb{R} \) are the magnitudes and \( \theta, \psi \in \mathbb{R} \) are the angles. The domain-coloring program `zviz.m` assumes that the angles go from \(-\pi < \theta < \pi\), with \( \theta = 0 \) being a light red and \( \pm\pi \) a blue color. This angle to color map is shown in Fig. 1.15 (p. 66). The first Riemann sheet of \( \sqrt{s} \) is defined by \(-\pi < \theta < \pi\). The second sheet picks up at \( \theta = \pi \) and continues on to \( \pi + 2\pi = 3\pi \). The first sheet maps the angle of \( w \) (i.e., \( \phi = \angle w = \theta/2 \)) from \(-\pi/2 < \phi < \pi/2 \) \((w = \sqrt{r} e^{\theta/2})\). This corresponds to \( u = \Re\{w(s)\} > 0 \). The second sheet maps \( \pi/2 < \psi < 3\pi/2 \) (i.e., \( 90^\circ \) to \( 270^\circ \)), which is \( \Re\{w\} = u < 0 \). In summary, twice around the \( s \) plane is once around the \( w(s) \) plane, because the angle is half due to the \( \sqrt{s} \). This then describes the multi-valued nature of the square root function.

Log function: Next we discuss the multi-valued nature of the log function. In this case there are an infinite number of Riemann sheets, not well captured by Fig. 1.16 (p. 67), which only displays the principal sheet. However if we look at the formula for the log function, the nature is easily discerned. The abscissa \( s \) may be defined as multi-valued since

\[ s_k = re^{\theta_j e^{2\pi kj}}. \]

Here we have extended the angle of \( s \) by \( 2\pi k \), where \( k \) is the sheet index \( \in \mathbb{Z} \). Taking the log

\[ \log(s) = \log(r) + (\theta + 2\pi k)j. \]

When \( k = 0 \) we have the principal value sheet, which is zero when \( s = 1 \). For any other value of \( k \) \( w(s) \neq 0 \), even when \( r = 1 \), since the angle is not zero, except for the \( k = 0 \) sheet.

1.4.5 Lec 27: Three Cauchy Integral Theorems

Cauchy’s theorems for integration in the complex plane

There are three basic definitions related to Cauchy’s integral formula. They are closely related, and can greatly simplify integration in the complex plane.

1. Cauchy’s (Integral) Theorem:

\[ \oint_{\mathcal{C}} F(s)ds = 0, \quad (1.76) \]

if and only if \( F(s) \) is complex-analytic inside of the closed curve \( \mathcal{C} \) (Boas, 1987, p. 45),(Stillwell, 2010, 319). The FTCC (Eq. 1.61) says that the integral only depends on the end points if \( F(s) \) is complex-analytic. By closing the path (contour \( \mathcal{C} \)) the end points are the same, thus the integral must be zero, as long as \( F(s) \) is complex analytic.
2. Cauchy’s Integral Formula:

\[ \frac{1}{2\pi j} \oint_{C} \frac{F(s)}{s - s_{o}} ds = \begin{cases} F(s_{o}), & s_{o} \in C \text{ (inside)} \\ 0, & s_{o} \notin C \text{ (outside)} \end{cases} \]  

(1.77)

Here \( F(s) \) is required to be analytic everywhere within (and on) the contour \( C \) (Greenberg, 1988, p. 1200), (Boas, 1987, p. 51), (Stillwell, 2010, p. 220). The value \( F(s_{o}) \in C \) is called the residue of the pole \( s_{o} \) of the function \( f(s) = F(s)/(s - s_{o}) \). The Taylor series (Eq. 1.62, p. 78) is defined with \( c_{-1} = 0 \). A modified series, having the \( c_{-1} \neq 0 \), is known as a Laurent series. Note the residue is the coefficient of a Taylor series expansion \( c_{-1} \).

3. The (Cauchy) Residue Theorem (Greenberg, 1988, p. 1241), (Boas, 1987, p. 73)

\[ \oint_{C} f(s) ds = 2\pi j \sum_{k=1}^{K} c_{k} = \sum_{k=1}^{K} \oint_{C} \frac{F(s)}{s - s_{k}} ds, \]  

(1.78)

where the residues \( c_{k} \in C \), corresponding to the \( k \)th poles of \( f(s) \), enclosed by the contour \( C \). By the use of Cauchy’s integral formula, the last form of the residue theorem is equivalent to the residue form.\(^{58}\)

**How to calculate the residue:** The case of first order poles, while special, is important, since the Brune impedance only allows simple poles and zeros, increasing the utility of this special case. The residues for simple poles are \( F(s_{k}) \), which is complex analytic in the neighborhood of the pole, but not at the pole.

Consider the function \( f(s) = F(s)/(s - s_{k}) \), where we have factored \( f(s) \) to isolate the first-order pole at \( s = s_{k} \), with \( F(s) \) analytic at \( s_{k} \). Then the residue of the poles at \( c_{k} = F(s_{k}) \). This coefficient is computed by removing the singularity, by placing a zero at the pole frequency, and taking the limit as \( s \rightarrow s_{k} \), namely

\[
 c_{k} = \lim_{s \rightarrow s_{k}} [(s - s_{k})F(s)]
\]


When the pole is an \( N \)th order, the procedure is much more complicated, and requires taking \( N - 1 \) derivatives of \( f(s) \), followed by the limit process (Greenberg, 1988, p. 1242). Higher order poles are rarely encountered, thus it is good to know that this formula exists, but perhaps it is not worth the effort to memorize it.

**Summary and examples:** These three theorems, all attributed to Cauchy, collectively are related to the fundamental theorems of calculus. Because the names of the three theorems are so similar, they are easily confused.

1. In general it makes no sense to integrate through a pole, thus the poles (or other singularities) must not lie on \( C \).

2. The Cauchy integral theorem (Eq. 1.76), follows trivially from the fundamental theorem of complex calculus (Eq. 1.61, p. 77), since if the integral is independent of the path, and the path returns to the starting point, the closed integral must be zero. Thus Eq. 1.76 holds when \( F(s) \) is complex analytic within \( C \).

3. Since the real and imaginary parts of every complex analytic function obey Laplace’s equation (Eq. 1.67, p. 79), it follows that every closed integral over a Laplace field, i.e., one defined by Laplace’s equation, must be zero. In fact this is the property of a conservative system, corresponding to many physical systems. If a closed box has fixed potentials on the walls, with any

\(^{58}\)This theorem is the same as a 2D version of Stokes thm (citations).
distribution what so ever, and a point charge (i.e., an electron) is placed in the box, then a force equal to \( F = qE \) is required to move that charge, thus work is done. However if the point is returned to its starting location, the net work done is zero.

4. Work is done in charging a capacitor, and energy is stored. However when the capacitor is discharged, all of the energy is returned to the load.

5. Soap bubbles and rubber sheets on a wire frame, obey Laplace’s equation.

6. These are all cases where the fields are Laplacian, thus closed line integrals must be zero. Laplacian fields are commonly observed, because they are so basic.

7. We have presented the impedance as the primary example of a complex analytic function. Physically, every impedance has an associated stored energy, and every system having stored energy has an associated impedance. This impedance is usually defined in the frequency \( s \) domain, as a force over a flow (i.e., voltage over current). The power \( P(t) \) is defined as the force times the flow and the energy \( \mathcal{E}(t) \) as the time integral of the power

\[
\mathcal{E}(t) = \int_{-\infty}^{t} P(t) dt, \tag{1.79}
\]

which is similar to Eq. 1.58 (p. 77) [see Section 3.2.1, Eq. 3.5 (p. 144)]. In summary, impedance and power and energy are all fundamentally related. Recall that the force and flow are also known as conjugate variables.

1.4.6 Lec 28: Cauchy Integral Formula & Residue Theorem

The Cauchy integral formula (Eq. 1.77) is an important extension of the Cauchy integral theorem (Eq. 1.76) in that a pole has been explicitly injected into the integrand at \( s = s_o \). If the pole location is outside of the curve \( \mathcal{C} \), the result of the integral is zero, in keeping with Eq. 1.76. However, when the pole is inside of \( \mathcal{C} \), the integrand is no longer complex analytic, and a new result follows. By a manipulation of the contour in Eq. 1.77, the pole can be isolated with a circle around the pole, and then taking the limit, the radius goes to zero.

For the related Cauchy residue theorem (Eq. 1.78) the same result holds, except it is assumed that there are \( K \) simple poles in the function \( F(s) \). This requires the repeated application of Eq. 1.77, \( K \) times, so it represents a minor extension of Eq. 1.77. The function \( F(s) \) may be written as \( f(s)/P_K(s) \), where \( f(s) \) is analytic in \( \mathcal{C} \) and \( P_K(s) \) is a polynomial of degree \( K \), with all of its roots \( s_k \in \mathcal{C} \).

Non-integral degree poles  The key point is that this theorem applies when \( n \in \mathbb{I} \), including fractionals \( n \in \mathbb{F} \). The function \( 1/\sqrt{s} \) has a zero residue, which is strictly the amplitude of \( 1/s \) (Boas, 1987, p. 73). When \( n \in \mathbb{F} \), the residue is, by definition, zero. When \( n \in \mathbb{I} \), the residue is, by definition, zero. When \( n = 1 \), the residue is \( \epsilon_{-1} \). For an intuitive discussion of Riemann sheets and branch cuts, see Boas (1987, Section 29, pp. 221-225).

This point is equally important when defining the inverse Laplace transform. When integrating over \( \omega \in \mathbb{R} \), the value passes through all possible exponents, including all rational and irrational numbers. The only value of \( \omega \) that has a residue, are those at the poles of the integrand.

1.4.7 Lec 29: Inverse Laplace transform (Cauchy residue theorem)

The inverse Laplace transform Eq. 1.57 transforms a function of complex frequency \( F(s) \) and returns a causal function of time \( f(t) \)

\[
f(t) \leftrightarrow F(s),
\]
where \( f(t) = 0 \) for \( t < 0 \). Examples are provided in Table 1.5 (p. 71). We next discuss the details of finding the inverse transform by use of the Cauchy residue theorem, and how the causal requirement \( f(t < 0) = 0 \) comes about.

The integrand of the inverse transform is \( F(s)e^{st} \) and the limits of integration are \(-\sigma, \pm \omega \). To find the inverse we must close the curve, at infinity, and show that the integral at \( \omega \rightarrow \infty \). There are two ways to close these limits, to the right \( \sigma > 0 \) (RHP), and to the left \( \sigma < 0 \) (LHP), but there needs to be some logical reason for this choice. That logic is the sign of \( t \). For the integral to converge the term \( e^{st} \) must go to zero as \( \omega \rightarrow \infty \). In terms of the real and imaginary parts of \( s = \sigma + \omega \), the exponential may be rewritten as \( e^{st}e^{\pm \omega \tau} \). Note that both \( t \) and \( \omega \) go to \( \infty \). So it is the interaction between these two limits that determines how we pick the closure, RHP vs. LHP.

**Case for causality \((t < 0)\):** Let us first consider negative time, including \( t \rightarrow -\infty \). If we were to close \( \mathcal{C} \) in the left half plane \((\sigma < 0)\), then the product \( \sigma t \) is positive \((\sigma < 0, t < 0, \text{ thus } \sigma t > 0)\). In this case as \( \omega \rightarrow \infty \), the closure integral \(|s| \rightarrow \infty \) will diverge. Thus we may not close in the LHP for negative time. If we close in the RHP \( \sigma > 0 \) then the product \( \sigma t < 0 \) and \( e^{st} \) will go to zero as \( \omega \rightarrow \infty \).

This then justifies closing the contour, allowing for the use of the Cauchy theorems.

If \( F(s) \) is analytic in the RHP, the FTCC applies, and the resulting \( f(t) \) must be zero, and the inverse Laplace transform must be causal. This argument holds for any \( F(s) \) that is analytic in the RHP \((\sigma > 0)\).

**Case of unstable poles:** An important but subtle point arises: If \( F(s) \) has a pole in the RHP, then the above argument still applies if we pick \( \sigma_o \) to be to the right of the RHP pole. This means that the inverse transform may still be applied to unstable poles (those in the RHP). This explains the need for the \( \sigma_o \) in the limits. If \( F(s) \) has no RHP poles, then \( \sigma_o = 0 \) is adequate, and this factor may be ignored.

**Case for zero time \((t = 0)\):** When time is zero, the integral does not, in general, converge, leaving \( f(t) \) undefined. This is most clear in the case of the step function \( u(t) \leftrightarrow 1/s \), where the integral may not be closed, because the convergence factor \( e^{st} = 1 \) is lost for \( t = 0 \).

### 1.4.8 Lec 30: Inverse Laplace transform and the Cauchy Residue Theorem

**Case of \( t > 0 \):** Next we investigate the convergence of the integral for positive time \( t > 0 \). In this case we must close the integral in the LHP \((\sigma < 0)\) for convergence, so that \( st < 0 \) \((\sigma \leq 0 \text{ and } t > 0)\).

When there are poles on the \( \omega \) axis, \( \sigma_o > 0 \) assures convergence by keeping the on-axis poles inside the contour. At this point the Cauchy residue theorem (Eq. 1.78) is relevant. If we restrict ourselves to simple poles (as required for a Brune impedance), the residue theorem may be directly applied.

The most simple example is the step function, for which \( F(s) = 1/s \), and thus

\[
u(t) = \int_{\text{LHP}} e^{st} \frac{ds}{s} \leftrightarrow \frac{1}{s},
\]

which is a direct application of Eq. 1.78. The forward transform of \( u(t) \) is straightforward, as discussed in Section 1.3.10 (p. 70). This is true of most if not all of the elementary forward Laplace transforms. In these cases, causality is built into the integral by the limits, so is not a result, as it must be in the inverse transform. An interesting problem is proving that \( u(t) \) is not defined at \( t = 0 \).

The inverse Laplace transform of \( F(s) = 1/(s + 1) \) has a residue of 1 at \( s = -1 \), thus that is the only contribution to the integral. A case that is more demanding is the Laplace transform pair

\[
\frac{1}{\sqrt{t}} u(t) \leftrightarrow \sqrt{\pi s} \quad \text{and} \quad J_o(t)u(t) \leftrightarrow \frac{1}{s^2 + 1} = \frac{1}{(s + j)(s - j)}.
\]
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Many of these are easily proved in the forward direction, but are much more difficult in the inverse direction due to the properties at \( t = 0 \), unless of course, the residue theorem (Eq. 1.78, p. 85) is invoked. The last \( \mathcal{L} \)-pair helps us understand the basic nature of the Bessel functions \( J_0(z) \), and \( H_0^{(1)}(z^2) \), with a branch cut along the negative axis (see Fig. 3.2, p. 154).

Some open questions: Without the use of the CRT (Eq. 1.78) it is difficult to see how evaluate do the inverse Laplace transform of \( 1/s \) directly. For example, how does one show that the above integral is zero for negative time (or that it is 1 for positive time)? The CRT neatly solves this difficult problem, by the convergence of the integral for negative and positive time. Clearly the continuity of the integral at \( \omega \to \infty \) plays an important role. Perhaps the Riemann sphere plays a role in this, that has not yet been explored.

1.4.9 Lec 31: Properties of the LT (e.g., Linearity, convolution, time-shift, modulation, etc.)

As shown in the table of Laplace transforms, there are integral (i.e., integration, not integer) relationships, or properties, that are helpful to identify. The first of these is a definition not a property: 59

\[
f(t) \leftrightarrow F(s).
\]

When taking the LT, the time response is given in lower case (e.g., \( f(t) \)) and the frequency domain transform is denoted in upper case (e.g., \( F(s) \)). It is required, but not always explicitly specified, that \( f(t < 0) = 0 \), that is, the time function must be causal (P1: Section 1.3.11).

Linearity: A key property so basic that it almost is forgotten, is the linearity property of the LT. These properties are summarized as P2 of Section 1.3.11, 73).

Convolution property: One of the most basic and useful properties is that the product of two LTs in frequency, results in convolution in time

\[
f(t) \star g(t) = \int_0^t f(\tau)g(t - \tau)d\tau \leftrightarrow F(s)G(s),
\]

where we use the \( \star \) operator as a shorthand for the convolution of two time functions. A key application of convolution is filtering, which takes on many forms. The most basic filter is the moving average, the moving sum of data samples, normalized by the number of samples. Such a filter has very poor performance. It also introduces a delay of half the length of the average, which may, or may not constitute a problem, depending on the application. Another important example is a low-pass filter, that removes high frequency noise, or a notch filter that removes line-noise (i.e., 60 [Hz] in the US, and its 2d and 3d harmonics, 120 and 180 [Hz]). Such noise is typically a result of poor grounding and ground loops. It is better to solve the problem at its root, than to remove it with a notch filter. Still, filters are very important in engineering.

By taking the LT of the convolution we can derive this relationship

\[
\int_0^\infty [f(t) \star g(t)]e^{-st}dt = \int_0^\infty \left[ \int_0^t f(\tau)g(t - \tau)d\tau \right]e^{-st}dt
\]

\[
= \int_0^t f(\tau) \left( \int_{t=0}^\infty g(t - \tau)e^{-st}d\tau \right)d\tau
\]

\[
= \int_0^t f(\tau) \left( e^{-s\tau} \int_{\tau'=0}^\infty g(t')e^{-s\tau'}dt' \right)d\tau
\]

\[
= G(s) \int_0^t f(\tau)e^{-s\tau}d\tau
\]

\[
= G(s)F(s)
\]

59 Put this notional property in Appendix A.
We encountered this relationship in Section 1.3.3 (p. 54) in the context of multiplying polynomials, which was the same as convolving their coefficients. Hopefully the parallel is obvious. In the case of polynomials, the convolution was discrete in the coefficients, and here it is continuous in time. But the relationships are the same.

**Time-shift propriety:** When a function is time-shifted by time $T_o$, the LT is modified by $e^{sT_o}$, leading to the propriety

$$f(t - T_o) \leftrightarrow e^{-sT_o} F(s).$$

This is easily shown by applying the definition of the LT to a delayed time function.

**Time derivative:** The key to the eigen-function analysis provided by the LT, is the transformation of a time derivative on a time function, that is

$$\frac{df}{dt}(t) \leftrightarrow sF(s).$$

Here $s$ is the eigen value corresponding to the time derivative of $e^{st}$. Given the definition of the derivative of $e^{st}$ with respect to time, this definition seems trivial. Yet that definition was not obvious to Euler. It needed to be extended to the space of complex analytic function $e^{st}$, which did not happen until at least Riemann (1851).

*Given a differential equation of order $K$, the LT results in a polynomial in $s$, of degree $K$. It follows that this LT property is the corner-stone of why the LT is so important to scalar differential equations, as it was to the early analysis of Pell’s equation and the Fibonacci sequence, as presented in earlier chapters. This property was first uncovered by Euler. It is not clear if he fully appreciated its significance, but by the time of his death, it certainly would have been clear to him. Who first coined the term *eigen value* and *eigen function*? The word *eigen* is a German word meaning *of one*. It seem likely that this term became popular somewhere between the 19th and 20th century.*

**Initial and final value theorems:** There are much more subtle relations between $f(t)$ and $F(s)$ that characterize $f(0^+)$ and $f(t \to \infty)$. While these properties can be very important in certain application, they are are beyond the scope of the present treatment. These relate to so-called *initial value theorems*. If the system under investigation has potential energy at $t = 0$, then the voltage (velocity) need not be zero for negative time. An example is a charged capacitor or a moving mass. These are important situations, but better explored in a more in depth treatment.

1.4.10  **Lec 32: Properties of the Brune impedance**

**Impedance**

Impedance plays a crucial role in the solution of differential equations, and in the theory and practice of complex variables. As a theoretical concept, it is the best example of a complex analytic function, thus plays a key role in potential theory.

Theoretically, impedance plays the role of a boundary condition in the solution of differential equations, accounting for the reflectance of a wave at the boundary. It plays a critical role in conservation of energy, since if energy at the boundary (surface) must always be lost, given a passive boundary.

The first person to fully appreciate the importance of impedance at the boundary may have been Arnold Sommerfeld, who expressed this in what today is known as the *radiation impedance*, which is the impedance boundary condition as a point source wave radiates to $|R| \to \infty$. 


**Brune impedance:** As the concept of impedance evolved (it is a linear relation between a force and a flow), it was incorporated into more theories, such as electrical circuit theory and mechanics. The first persons to quantify the concept of impedance was Ohm, followed by Kirchhoff and Heaviside. Kennelly, not Heaviside, was the first to express the idea as a complex variable of a complex function of frequency (Kennelly, 1893). Perhaps Heaviside fully appreciated the concept, and has been given proper credit. A very interesting development was the largely forgotten but related mathematical work of Bott and Duflin (Van Valkenburg, 1964b).

There are several important theorems here that follow from *Brune’s Theorem on positive-real functions*, defined by the relation

\[
\Re\{Z(\sigma > 0)\} \geq 0, \tag{1.80}
\]

where \(Z(s) = R(s) + jX(s) \in \mathbb{C}\) is a Brune impedance having real part \(R(s) = \Re\{Z(s)\}\) and imaginary part \(X(s) = \Im\{Z(s)\}\), each as functions of the Laplace frequency \(s = \sigma + \omega j\) (Brune, 1931a; Van Valkenburg, 1964b). Thus a Brune impedance has a positive analytic real part in the right-half plane \(\sigma > 0\).

This condition has many non-obvious ramifications, including (order is significant)

1. \(Z(s) \leftrightarrow z(t < 0) = 0\) is real and causal
2. \(Y(s) = 1/Z(s) \leftrightarrow y(t < 0) = 0\) is real and causal
3. The phase \(\angle Z(s)\) lies between \(\pm \pi/2\) (i.e., \(Z(s)\) is minimum-phase)
4. All the poles and zeros are first order (there are no higher order poles)
5. All poles and zeros are first order (there are no higher order poles)
6. \(Z(\sigma > 0)\) is complex-analytic \([Z(s)\) obeys the Cauchy-Riemann conditions in the RHP \((\sigma > 0)\)]
7. All poles and zeros of \(Z(s)\) on the \(\omega j\) or on the \(\sigma\) axis must alternate (Foster’s Theorem Van Valkenburg (1964b))
8. All Brune impedances are quasi-static (lumped-parameter, where the size of the impedance element is much smaller than the wavelength).
9. The Brune impedance is a subset of impedance, where delay has been approximated away in the long-wavelength limit. For example, transmission lines are not in the class of Brune impedances.

A Brune impedance is defined as the ratio of the force \(F(\omega)\) over the flow \(U(\omega)\), and may be expressed in residue form as

\[
Z(s) = sL_o + R_o + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}, \tag{1.81}
\]

known as the partial fraction expansion. The term \(L_o\) represents an inductance, and \(R_o\) a resistor. Coefficients \(c_k\) are the residues, and \(s_k\) the roots of \(D(s)\). It follows that

\[
D(s) = \prod_{k=1}^{K} (s - s_k) \quad \text{and} \quad c_k = \lim_{s \to s_k} (s - s_k)Z(s) = \prod_{n'=1}^{K-1} (s - s_n),
\]

where the prime on index \(n'\) means that \(n = k\) is not included in the product. Every Brune impedance may be expanded in a partial fraction expansion of the form of Eq. 1.81, and every partial fraction expansion may be converted back into the ratio of two polynomials, using the Matlab/Octave commands \([C,P,K]\texttt{=}\texttt{residue}(N,D)\) and \([N,D]\texttt{=}\texttt{residue}(C,P,K)\). These two representations are interchangeable.

Given a 2-port network, the input impedance and the transfer function share the same poles.\(^{60}\) In fact the transfer function must be all-pole.

\(^{60}\) Is \(C(s)\) of the ABCD matrix the same as the impedance denominator \(D(s)\)? Verify!
Exercise: Find the Laplace transform ($\mathcal{L}$) of the three impedance relations in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: Each classical law is a linear relation

1. Hooke’s Law $f(t) = Kx(t)$. Sol: First $\mathcal{L}$-Hooke’s Law and then write it in terms of force and velocity

   Taking the $\mathcal{L}$ gives
   
   $$ F(s) = KX(s) $$

   Since
   
   $$ v(t) = \frac{d}{dt}x(t) \leftrightarrow V(s) = sX(s) $$

   Thus the impedance of the spring is
   
   $$ Z_s(s) = \frac{F(s)}{V(s)} = \frac{K}{s} $$

   which is analogous to the impedance of an electrical capacitor. If we specifying the compliance $C$ of a spring as $C = 1/K$, the relation looks just like the electrical case.

   While Hooke’s law says the force and displacement are proportional, in terms of impedance variables force and flow,
   
   $$ f(t) = K \int^t v(t)dt $$

2. Dash-pot resistance $f(t) = Rv(t)$. Sol: From the $\mathcal{L}$ this becomes

   $$ F(s) = RV(s) $$

   and the impedance of the dash-pot is then
   
   $$ Z_r = R $$

   analogous to that of an electrical resistor.

   Exercise: If $Z_r$ is the impedance of a resistor, find the formula time domain impedance. Sol:

   $$ Z_r = R \leftrightarrow z_r(t) = R\delta(t) $$

3. Newton’s 2d Law for a constant mass $M$: $f(t) = Mdv(t)/dt$. Sol: Taking the $\mathcal{L}$ gives

   $$ F(s) = sMV(s) $$

   thus
   
   $$ Z_m(s) = \frac{F(s)}{V(s)} = sM $$

   analogous to an electrical inductor.

4. Find the total impedance seen by the net force $f(t)$ applied to the mass $M$. Sol: Summing the forces must equal the applied force, Eq. E.1, p. 197.

5. Take the Laplace transform ($\mathcal{L}$) of Eq. E.1 (p. 197), and evaluate the total impedance $Z(s)$ of the mechanical circuit. Sol: From the properties of the $\mathcal{L}$ that $dx/dt \leftrightarrow sX(s)$, we find

   $$ Ms^2X(s) + RsX(s) + KX(s) = F(s) $$

   In terms of velocity this is $(Ms + R + K/s)V(s) = F(s)$. Thus the circuit impedance is

   $$ Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s} $$
6. What are \( N(s) \) and \( D(s) \) (e.g. Eq. 1.81, p. 90)? Sol: \( D(s) = s \) and \( N(s) = K + Rs + Ms^2 \).

7. Assume that \( M = R = K = 1 \). Find the residues (e.g. Eq. 1.81, p. 90) of the admittance \( Y(s) = 1/Z(s) \), in terms of the roots \( s_\pm \). Check your answer with the Matlab/Octave command residue. Sol: First find the roots of the numerator of \( Z(s) \) (the denominator of \( Y(s) \)):

\[
s_\pm^2 + s_\pm + 1 = (s_\pm + 1/2)^2 + 3/4 = 0,
\]

which is

\[
s_\pm = \frac{-1 \pm j\sqrt{3}}{2}.
\]

Next form the partial fraction expansion

\[
\frac{s}{1 + s + s^2} = c_0 + \frac{c_+}{s - s_+} + \frac{c_-}{s - s_-} = \frac{s(c_+ + c_-) - (c_+ s_- + c_- s_+)}{1 + s + s^2}
\]

Comparing the two sides requires that \( c_0 = 0 \). We also have two equations for the residues \( c_+ + c_- = 1 \) and \( c_+ s_- + c_- s_+ = 0 \). The best way to solve this is to set up a matrix relation and take the inverse

\[
\begin{bmatrix}
1 & 1 \\
-s_- & s_+
\end{bmatrix}
\begin{bmatrix}
c_+ \\
c_-
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

thus:

\[
\begin{bmatrix}
c_+ \\
c_-
\end{bmatrix}
= \frac{1}{s_+ - s_-}
\begin{bmatrix}
s_+ & -1 \\
-s_- & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix},
\]

which gives \( c_\pm = \pm \frac{s_\pm}{s_+ - s_-} \). The denominator is \( s_+ - s_- = j\sqrt{3} \) and the numerator is \( \pm 1 + j\sqrt{3} \). Thus

\[
c_\pm = \pm \frac{s_\pm}{s_+ - s_-} = \frac{1}{2} \left( 1 \pm \frac{2}{\sqrt{3}} \right).
\]

As always, finding the coefficients is always the most difficult part. Using 2x2 matrix algebra can really simplify up the process, and will more likely give the correct answer.

8. By applying the CRT, find the inverse Laplace transform \((L^{-1})\). Use the residue form of the expression that you derived in the previous exercise. Sol:

\[
z(t) = \frac{1}{2\pi j} \oint_{\mathcal{C}} Z(s)e^{st}ds.
\]

were \( \mathcal{C} \) is the Laplace contour which encloses the entire left-half \( s \) plane. Applying the CRT

\[
z(t) = c_+ e^{s_+ t} + c_- e^{s_- t}.
\]

where \( s_\pm = -1/2 \pm j\sqrt{3}/2 \) and \( c_\pm = 1/2 \pm j/(2\sqrt{3}) \).

1.4.11 Lec 33: Exam III (Evening Exam)
1.5 Vector Calculus (Stream 3b)

1.5.1 Lec 34: Gradient ∇, divergence ∇·, Curl ∇×, and Laplacian ∇²

Before we can define the vector operations ∇(), ∇·(), ∇×(), ∇²() we must define scalar and vector fields.

Scalar and vector fields

A field is a function of \((x, y, z) \in \mathbb{R}^3\) than is analytic, namely has a Taylor series, meaning it can take derivatives with respect to \(x = (x, y, z)\). An example of a scalar field is \(T(x) = xy\), or \(\Phi(x) = e^x \log(y)z\).

In every case a scalar field is an analytic function of \(x = (x, y, z)\). Think of the scalar field for the case of a voltage \(\Phi(x, y, z)\), or temperature \(T(x, y, z)\), distributed in \(\mathbb{R}^3\), say between two finite sized capacitor plates, or the temperature \(T(x, y, z)\) in a room, or within a computer chip. If the furnace is on, blowing hot air into the room, it is not a smooth (analytic) function of \(\mathcal{A}\). Likewise, if the computer chip has heat-generating transistors inside, the temperature would not be smooth, thus not analytic. Every scalar field can play the role of a potential.

A vector field is a ordered set of scalar fields, for example

\[
A(x) = [\phi(x), \psi(x), \theta(x)]^T
\]

is a vector field in \(\mathbb{R}^3\), as long as each of the three function is analytic. For example \(A(x) = [x, xy, xyz]^T\) is a legal vector field. An excellent discussion of vector potentials may be found in (Feynman, 1970, p. NN).

The physical meaning of a potential: The important question is: “What is the physical meaning of the scalar and vector field?” This is best answered with a few examples. If a temperature field is constant (i.e., \(T(x) = T_0\)), there can be no heat flux (flow of heat), because it is the change in the temperature (potential) that causes flux. The same holds for every potential.

Note how the difference in the temperature is proportional to the heat flux. Likewise it is the voltage difference that drives an electrical current. This is simply stated by Ohm’s Law

\[
I = \frac{V_2 - V_1}{R}
\]

namely the drop in voltage divided by the resistance, defines the current (flow or electrical flux). More generally the impedance is the ratio of the voltage (a potential) drop over the current (a flow). In electrical terms it is common to define a ground point all voltages are referenced to. This allows one to abstract away (hide) the difference in voltage. In fact, it is the difference across any impedance that drives the current. Another physical example is the pressure, the gradient of which is a force density, which drives velocity. More on this in section 1.5.6 (p. 116). When the impedance is complex, the angle of the complex number represents the phase between the voltage and current.

It is very helpful to look at the units of the scalar or vector potential, to help identify that it is a potential. But a proper mathematical definition seems to be that it is must be an analytic function of \(x\) (position) or \(t\) (time).

As we have learned in earlier chapters, the impedance can be complex functions of complex frequency. A capacitor has an impedance \(Z(s) = 1/sC_o\) and an inductor \(Z(s) = sL_o\). For each of these examples, the voltage is a potential, which most generally is a complex analytic function of \(s\), as well as an analytic function of \(x\). If the potential is not analytic, then the impedance would not be well defined.
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<table>
<thead>
<tr>
<th>Name</th>
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<th>Output</th>
<th>Operator</th>
<th>Mnemonic</th>
</tr>
</thead>
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<tr>
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<td>Scalar</td>
<td>Vector</td>
<td>$-\nabla()$</td>
<td>Grad</td>
</tr>
<tr>
<td>Divergence</td>
<td>Vector</td>
<td>Scalar</td>
<td>$\nabla \cdot ()$</td>
<td>Div</td>
</tr>
<tr>
<td>Curl</td>
<td>Vector</td>
<td>Vector</td>
<td>$\nabla \times ()$</td>
<td>Curl</td>
</tr>
<tr>
<td>Laplacian</td>
<td>Scalar</td>
<td>Vector</td>
<td>$\nabla \nabla = \nabla^2()$</td>
<td>DoG</td>
</tr>
<tr>
<td>Vector Laplacian</td>
<td>Vector</td>
<td>Vector</td>
<td>$\nabla \nabla = \nabla^2()$</td>
<td>GoD</td>
</tr>
</tbody>
</table>

Table 1.8: The three vector operators manipulate scalar and vector fields, as indicated here. The gradient converts scalar fields into vector fields. The divergence eats vector fields and outputs scalar fields. Finally the curl maps vector fields into vector fields. It is helpful to have a name for second order operators (e.g., DoG, GoD: mnemonics defined in Sect. 1.5.5, p. 116).

Vector differential operators act on scalar and vector fields

There are three key vector differential operators that are necessary for understanding partial differential equations, such as the wave equation and the diffusion equation. All of these use the differential operator $\nabla$ operator, a multi-variate generalization of the derivative

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}. \quad (1.82)$$

When carefully defined, the shorthand form $\partial_x$ is convenient. The official name of this operator is nabla. It has three basic uses: 1) the gradient of a scalar field, the 2) divergence of a vector field, and the 3) curl of a vector field.

5 Basic differential vector operator definitions:

1. The gradient transforms a scalar potential field into vector field. In $\mathbb{R}^3$ the gradient of a scalar field $\nabla \phi(x)$ is defined as

$$\nabla \phi(x) = \left( \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \phi(x) = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}. \quad (1.82)$$

Important examples include: the electric field vector $E(x) = -\nabla \phi(x) \ [\text{Volts/m}]$, which is the gradient of a voltage [V], and the force density $f(x) = -\nabla \rho(x) \ [\text{N/m}]$, which is the gradient of a pressure [Pa].

2. The divergence of a vector field results in a scalar field. For example, the divergence of the electric field flux vector $D(x) \ [\text{Col/m}^2]$ equals the scalar field charge density $\rho(x) \ [\text{Col/m}^3]$

$$\nabla \cdot D(x) \equiv \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot D(x) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho(x). \quad (1.83)$$

This it is analogous to the scalar product between two vectors.

When working with guided waves, narrow tubes of flux, such that the diameter is small compared with the wavelength, the divergence may be accurately approximated as (Appendix 5.7, p. 176)

$$\nabla \cdot D(x) = \nabla_r D_r = \frac{1}{A(r)} \frac{\partial}{\partial r} A(r) D_r(r). \quad (1.84)$$

For example in spherical coordinates $A(r) = A_{sr} r^2$, and this reduces to the radial component of the divergence of $D(x)$ in spherical coordinates. In cylindrical coordinates $A(r) = A_{sr}$.

3. The curl

$$\nabla \times H(x) \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = C(x) \quad (1.85)$$
transforms the vector field \( \mathbf{H}(x) \) into the vector field \( \mathbf{C}(x) \). The notation \(| \cdot |\) indicates taking the determinant (Appendix A, p. 181).

Here the curl of the magnetic intensity \( \mathbf{H}(x) \) [Amp/m\(^2\)] is equal to the current density \( \mathbf{C}(x) \) [Amps/m\(^2\)]. As before, units can be helpful.

4. The Laplacian (\( \nabla^2 \equiv \nabla \cdot \nabla \))

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\] (1.86)

**Exercise:** Construct an example which shows that the definition of the operator agrees with the type of input vector. For Example, show that the gradient must operate on a scalar, to produce a vector. Sol: Given a scalar field \( \Phi(x) = xyz \) [Volts], the electric field is \( \mathbf{E}(x) = [\hat{i} \partial_x, \hat{j} \partial_y, \hat{k} \partial_z]xyz = [\hat{i}yz, \hat{j}xz, \hat{k}xy]\) [V/m], and that the curl must operate on a vector, and results in a vector.

**Examples of the Gradient operator \( \nabla \)**

When the gradient \( \nabla \) operates on a scalar field \( V(x) \), the result is a vector field\(^{61} \)

\[ \mathbf{E}(x) = [E_x(x), E_y(x), E_z(x)]^T = -\nabla V(x). \]

For example, if \( V(x) = 2 + x - y^2 \) then \( \mathbf{E} = -\nabla (x + 2 - y^2) = -\hat{i} + 2\hat{j}y \). To understand these three operations we therefore need to define the domain and range of their operation, as specified in Table 1.8. Recall that the voltage is the line integral of the electric field

\[
V(a) - V(b) = \int_a^b \mathbf{E}(x) \cdot dx = -\int_a^b \nabla V(x) \cdot dx = -\int_a^b \frac{dV}{dx} dx
\] (1.87)

which is simply the fundamental theorem of calculus (Leibniz Theorem). In a charge free region, this integral is independent of the path from \( a \) to \( b \), which is the propriety of a conservative system.

The simplest example of a scalar potential is the voltage between two very large (think \( \infty \)) conducting parallel planes, or plates (large so that we may ignore the edge effects). In this case the voltage varies linearly (the voltage is complex analytic) between the two plates. For example

\[ V(x, y, z) = V_o(1 - x) \] (1.88)

is a scalar potential, thus it is scalar field (i.e., potential). At \( x = 0 \) the voltage is \( V_o \) and at \( x = 1 \) the voltage is zero. Between 0 and 1 the voltage varies linearly. Thus \( V(x, y, z) \) defines a scalar field.

What is the physical configuration corresponding to \( V(x) = 2 + x - y^2 \) (the example above)?

If the same setup were used but the two plates were 1x1 [cm\(^2\)], with a 1 [mm] air gap, there will be a small “fringe” effect at the edges that would slightly modify the ideal fields. This effect can be made small by changing the air gap to area ratio, so that the ends do not impact the capacitor’s value. If we are given a set of three scalar fields, we define a vector field. If the three elements of the vector are potentials, then we have a magnetic vector potential.

Perhaps a more intuitive example is an acoustic pressure potential \( \rho(x, t) \), which defines a force density \( f(x, t) = -\nabla \rho(x, t) \) (Eq. 1.91, p. 97).

\(^{61}\)As before vectors are columns, which take up space on the page, thus we write them as rows and take the transpose to properly format them.
1.5.2 Lec 35: Scalar Wave Equation (Acoustics)

There are three distinct mathematical classes of equations used to describe physical systems: 1) The “vector” wave equation describes the evolution of a vector field, such as Maxwell’s electric field vector $\mathbf{E}(x, t)$. When these fields are restricted to a one dimensional domain they are known as guided waves constrained by wave guides. 2) The “simpler” scalar wave equation describes the evolution of a scalar field, such as a pressure or voltage. The best known examples are electrical and acoustic transmission lines. Such systems are loosely referred to as the telegraph or telephone equations, referring back to early days of their discovery. In acoustics, guided waves are called horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names can reflect their historical development, with the mathematics and the applications frequently running in close parallel. The third class 3) is systems of lumped elements, such as electrical inductors, capacitors and resistors, or their mechanical counterparts, masses, springs and dash-pots. Such systems are represented by systems of matrix equations, the size of which is equal to the number of elements in the network. Such networks are not viewed as waves, and are thus not typically viewed as guided systems. However there is an overlap with scalar waves and transmission lines, which may be modeled as lumped networks.

We shall limit our analysis here to the scalar wave equation. Acoustic wave propagation was first analyzed mathematically by Isaac Newton (electricity had yet to be discovered) in his famous book *Principia* (1687), where he was the first to calculate the speed of sound, based on conservation of mass and momentum. His early development understandably ignored viscous and thermal losses, which can be significant at acoustic frequencies when the radius of the container (i.e., the horn) becomes less than the viscous or thermal boundary layers (e.g., less than 1 mm) as first described in Helmholtz (1863b) and soon extended by Kirchhoff (1868, 1974, English) (Rayleigh, 1896; Mason, 1927, 1928).

Such loss-less horns include an important generalization of the solution of the 1-dimensional (1D) wave equation, in regions where the area $A(r)$ of the horn varies along the range axial (e.g., along the direction of wave propagation), as depicted in Fig. 1.20. Classic applications of horns include vocal tract acoustics, loudspeaker design, cochlear mechanics, and cases having wave propagation in periodic media (Brillouin, 1953).

For the 1D scalar wave equation (guided waves, aka, acoustic horns) the Laplacian may be accurately approximated as

$$\nabla_r^2 \psi(r, t) \equiv \nabla \cdot \nabla \psi(r, t) = \frac{1}{A(r)} \frac{\partial}{\partial r} A(r) \frac{\partial}{\partial r} \psi(r, t), \quad (1.89)$$

assuming that the horn’s diameter $d$ is less than first allowed “radial mode’s” (a half wavelength) (i.e., $d < \lambda/2$). The term on the right is an approximation that applies for thin tubes (e.g., spherical, cylindrical coordinates, etc.), with angular symmetry, where the potential $\psi(x, t) \leftrightarrow \Psi(x, s)$ is averaged over the iso-potential cross-sectional area (Appendix 5.3, p. 173). Such formulations, which explicitly depend on the area of the horn, are called Webster horn equations.

It follows that the 1D acoustic wave equation is

$$\frac{1}{A(r)} \frac{\partial}{\partial r} A(r) \frac{\partial}{\partial r} \varrho(r, t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \varrho(r, t) \leftrightarrow s^{2} P(r, s), \quad (1.90)$$

where $\varrho(r, t) \leftrightarrow P(r, s)$ is the average pressure (Hanna and Slepian, 1924; Mawardi, 1949; Morse, 1948), Olson (1947, p. 101), Pierce (1981, p. 360). Extensive experimental analysis for various types of horns (conical, exponential, parabolic) along with a review of horn theory is provided by Goldsmith and Minton (1924).

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62This condition may be written in forms, the most common being that for a ridged walled tube $ka < 1$, where $k = 2\pi/\lambda$ and $a$ is the radius. This may be expressed in terms of the diameter as $\frac{2\pi}{d} < 1$, which is $d < \lambda/\pi < \lambda/2$. Thus $d < \lambda/2$ is a more precise metric, by a factor of $\pi/2 \approx 1.6$. 
The limits of the Webster horn equation: It is frequently (i.e., always) stated that the Webster Horn equation (WHEN) is a fundamentally limited, and is an approximation that only applies at low frequencies. However in all these discussions it is assumed that the area function \( A(r) \) is the horn’s cross-sectional area, not the area of the iso-pressure wave-front (Morse, 1948; Shaw, 1970; Pierce, 1981).

In Section 5.1.1 (p. 169) it is shown that these “limitations” may be totally avoided, making the Webster theory of the horn equation exact (but still subject to the guided wave assumption, \( ka < 1 \)). In this case acoustic variables are defined as the average pressure and the corresponding volume velocity, each defined on the iso-pressure wave-front boundary (Webster, 1919; Hanna and Slepian, 1924).

Separable coordinate systems: The most common examples of separable coordinate systems are rectangular, cylindrical and spherical coordinates (Morse, 1948, p. 296-7). Even a slight deviation from these separable cases represents a major barrier to any understanding, blocking further insight into the general case. Of course the wave equation “knows” nothing of such coordinate systems, since these are special cases having high degrees of symmetry. What we seek is a more general understanding of such systems. When the pressure (i.e., a force potential) is not constrained to a “separable” coordinate system, the horn equation is a Sturm-Louville equation.

Early history: The study of wave propagation begins at least as early as Huygens (ca. 1678), followed soon after (ca. 1687) by Sir Isaac Newton’s calculation of the speed of sound (Pierce, 1981). To obtain a wave, one must include two basic components, the stiffness of air, and its mass. These two equations shall be denoted (1) Newton’s 2\textsuperscript{nd} Law \( (F = ma) \) and (2) Hooke’s Law \( (F = kx) \), respectively. In vector form these equations are (1) Euler’s equation (i.e., conservation of momentum density)

\[
-\nabla p(x, t) = \rho_o \frac{\partial}{\partial t} u(x, t), \quad (1.91)
\]

assuming \( \rho_o \) is independent of time and position \( x \), and (2) the continuity equation (i.e., conservation of mass density)

\[
-\nabla \cdot u(x, s) = \frac{1}{\eta_o \rho_o} \frac{\partial}{\partial t} p(x, t) \quad (1.92)
\]

(Pierce, 1981, page 15). Combining Eqs. 1.91 and 1.92 results in the 3-dimensional (3D) scalar pressure wave equation

\[
\nabla^2 \varrho(x, t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \varrho(x, t) \leftrightarrow s^2 \varrho^s(x, t), \quad (1.93)
\]

where \( c_o \) is the sound velocity.

Primitive solutions \( \varrho^\pm(r, t) \)

Because the wave equation (Eq. 1.93) is 2d order in time, there are two causal independent primitive solutions. Likewise there are two causal independent primitive solutions of the homogeneous (i.e.,
undriven) Webster horn equation, for each choice of area function $A(x)$, identified as $g^\pm(x,t)$ an *outbound* (right-traveling) and *inbound* (left-traveling) wave, denoted as the Laplace transform pair

$$g^\pm(r,s) \leftrightarrow \mathcal{P}^\pm(r,s) = \int_0^\infty g^\pm(r,t)e^{-st}dt,$$

typically normalized so that $\mathcal{P}^\pm(r_0,s) = 1$, where $r_0$ defines the input excitation point. This construction is equivalent to a separation of time and space variables.

These two primitive solutions depend on $A(r)$, which causes local reflections with any area variations. Thus there is a tight relationship between the area change $dA(r)/dr$, internal resonances (eigen-modes), and the primitive wave solutions $\mathcal{P}^\pm(r,s)$. These relationships will be explored further in Chapter 5.

**Complex vs. real frequency:** We shall continue to maintain the distinction that functions of $\omega$ are Fourier transforms and functions of Laplace frequency $s$ correspond to Laplace transforms, which, because they are causal, are necessarily complex analytic in $s$ in the *right half plane* (RHP) region of convergence (ROC). This distinction is critical, since we typically describe impedance $Z(s)$ and admittance $Y(s)$, which must be causal, in terms of their poles and zeros, as complex analytic functions in $s$.\(^6\)

**Primitive solutions of the 1D wave equation:** As best we know, Huygen (1690) was the first to gain insight into wave propagation. While his concept, today known as “Huygens’ principle” showed a deep insight, it was seriously flawed, as it ignored the backward traveling wave (Miller, 1991). In 1747 a famous French intellectual and mathematician, Jean le Rond d’Alembert, published the general form of the solution to the 1D scalar wave equation

$$\varrho(x, t) = f(t - x/c_o) + g(t + x/c_o), \quad (1.94)$$

where $f(\cdot)$ and $g(\cdot)$ are quite general functions of their argument. That this is the solution may be easily shown by use of the chain rule, by taking partials with respect to $x$ and $t$. Surprisingly, this is the solution even when the functions are not differentiable. For example $u(t \mp x/c_o)$, and $\delta(t \mp x/c_o)$, are valid solutions, even though the proof is more difficult.

**Exercise:** By the use of the chain rule, prove that d’Alembert’s formula satisfies the 1D wave equation. *Sol:* Taking a derivatives with respect to $t$ and $r$ give

- $\partial_t g(r,t) = -c_o f'(r - c_o t) + c_o g'(r + c_o t)$
- $\partial_r g(r,t) = f'(r - c_o t) + g'(r + c_o t)$,

and a second derivatives gives

- $\partial_{tt} g(r,t) = c_o^2 f''(r - c_o t) + c_o^2 g''(r + c_o t)$
- $\partial_{rr} g(r,t) = f''(r - c_o t) + g''(r + c_o t)$.

From these last two equations we have the 1D wave equation

$$\partial_{rr} g(r,t) = \frac{1}{c_o^2} \partial_{tt} g(r,t),$$

at odds with Huygens’ 1690 “principle” (Miller, 1991).

---

\(^6\)When an analytic function of complex variable $s$ includes the pole it is called a *Laurent series* in $s$. For example, the impedance of a capacitor $C$ is $Z_c(s) = 1/sC$, which is analytic in $s$ everywhere other than $s = 0$. The capacitor has a voltage time response given by the integral of the current, i.e., $v(t) = \frac{1}{C} \int_0^t i(t)dt = \frac{1}{C} u(t) \ast i(t)$, where $u(t)$ is the Heaviside step function and $\ast$ represents convolution.
In terms of the physics, d’Alembert’s general solution describes two arbitrary wave-forms \( f(\cdot), g(\cdot) \), traveling at a speed \( c_o \), one forward, and one reversed. Thus his solution is quite easily visualized. As previously discussed, Newton (1687) was the first to calculate the speed of sound \( c_o \), be it with an error of \( \sqrt{\frac{1}{4}} \). This error was not corrected for over 200 years, following the creation of thermodynamics, and the concept of an adiabatic process.

**Example:** Assuming \( f(\cdot), g(\cdot) \) are \( \delta(\cdot) \), find the Laplace transform of the solution. Using Table 1.5 (p. 71) of Laplace Transforms on Eq. 1.94 gives

\[
\varphi(x, t) = \delta(t - x/c_o) + \delta(t + x/c_o) \leftrightarrow e^{-sx/c_o} + e^{sx/c_o}.
\]  

(1.95)

Note that the delay is \( T_o = \pm x/c_o \).

**3D d’Alembert waves:** The d’Alembert solution generalizes to 3D waves due to the fact that (1D \( x \) is replace by 3D \( r \) in Eq. 1.90 with \( A(r) = A_o r \), as explained in Appendix 5.7, p. 176),

\[
\nabla^2 \psi(r, t) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \psi(r, t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \psi(r, t)
\]

resulting in the general spherical (3D) wave equation solution

\[
\psi(r, t) = \frac{f(t - r/c_o)}{r} + \frac{g(t + r/c_o)}{r},
\]

for arbitrary wave-forms \( f(\cdot) \) and \( g(\cdot) \).

**Related partial differential equations:**

Historically the wave equation was seen to be related to several other important partial differential equations, as the theory evolved, resulting in the

1. **Fourier diffusion equation**

\[
D_o \frac{\partial T(x, t)}{\partial t} = \nabla^2 T(x, t)
\]

(1.96)

which describes, for example, the temperature \( T(x, t) \) as proposed by Fourier in 1822, or for the diffusion of two miscible liquids (Fick, 1855) and Brownian motion (Einstein, 1905),

2. **Poisson’s equation**

\[
\nabla^2 \Phi(x, t) = \rho(x, t)
\]

(1.97)

which holds for gravitational fields, or the voltage around a charge, and

3. **Laplace’s equation**

\[
\nabla^2 \Phi(x, t) = 0,
\]

(1.98)

which describes, for example, the voltage inside a closed chamber with various voltages on the walls, or the steady state temperature within a closed container, given a specified temperature distribution on the walls.

Each of these four equations has properties that may be simply explained, and visualized, in simple geometries, and all contain the Laplacian \( \nabla^2() = \nabla \cdot \nabla() \).
CHAPTER 1. INTRODUCTION

**The Laplacian \( \nabla^2 \):** We first discussed the Laplacian as a 2D operator in Section 1.4.2 (p. 78), when we studied complex analytic functions. Then an approximation for horns (variable tubes) was presented as Eq. 1.89. In 3D rectangular coordinates it is defined as

\[
\nabla^2 T(x) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(x). \tag{1.99}
\]

In summary, the Laplacian operator is ubiquitous in mathematical physics, starting with simple complex analytic functions (Laplace’s equation), and progressing to the Poisson’s equation, the diffusion equation, and finally the wave equation. Only the wave equation expresses delay. The diffusion equation “wave” has an instantaneous spread (the effective “wave” velocity is infinite, i.e., its not a wave).

**1.5.3 Lec 36 The 2D Webster horn equation**

Newton’s conservation of momentum Law (Eq. 1.91), along with conservation of mass (Eq. 1.92), are modern versions of Newton’s starting point for calculating the speed of sound. Following the simplification of averaging the normal component of the particle velocity over the iso-pressure wave front (defining the volume velocity), the two equations may be written as a 2x2 matrix (Ramo et al., 1965) in terms of acoustic variables average pressure \( P(r, \omega) \) and volume velocity \( V(r, \omega) \), sometimes referred to as conjugate variables.\(^{64}\)

\[
\frac{d}{dr} \begin{bmatrix} P(r, \omega) \\ V(r, \omega) \end{bmatrix} = - \begin{bmatrix} 0 & \mathcal{Z}(s, r) \\ \mathcal{Y}(s, r) & 0 \end{bmatrix} \begin{bmatrix} P(r, \omega) \\ V(r, \omega) \end{bmatrix}. \tag{1.100}
\]

This key transformation, from particle to volume velocity, is discussed in Section 5.3 (Eqs. 5.16, 5.18, p. 173). The Fourier-transform pair of the average pressure and volume velocity are denoted \( g(r, t) \leftrightarrow P(r, \omega) \) and \( \nu(r, t) \leftrightarrow V(r, s) \).\(^{65}\) The complex Laplace frequency \( s = \sigma + j\omega \) is required when defining the per-unit-length impedance

\[
\mathcal{Z}(r, s) \equiv \frac{s \rho_o}{A(r)} = sM(r) \tag{1.101}
\]

and per-unit-length admittance

\[
\mathcal{Y}(s, r) \equiv \frac{s}{\eta_o P_o} = sC(r), \tag{1.102}
\]

to indicate that these functions are causal, and except at their poles, analytic in \( s \).\(^{66}\) Here \( M(r) = \rho_o/A(r) \) is the horn’s per-unit-length mass, \( C(r) = A(r)/\eta_o P_o \) per-unit-length compliance, \( \rho_o \) is the density of air (i.e., 1.2 \([kg/m^3]\)), \( P_o \) is the atmospheric pressure (i.e., 10\(^5\) \([N/m^2]\)), \( \eta_o = c_p/c_v \approx 1.4 \) (air). It is helpful to express the bulk properties of the waves in terms of \( \mathcal{Z} \) and \( \mathcal{Y} \), which is provided next.

**Bulk acoustic parameters:** The bulk acoustic parameters are the free-field speed of sound

\[
c_o = \sqrt{\frac{\eta_o P_o}{\rho_o}} = \sqrt{\frac{\text{stiffness}}{\text{mass}}}, \tag{1.103}
\]

the specific acoustic resistance is

\[
\rho_o c_o = \sqrt{\rho_o \eta_o P_o} = \sqrt{\text{stiffness-mass}}. \tag{1.104}
\]


\(^{65}\)Notation: Lower case variables (i.e., \( g(r, t) \), \( \nu(r, t) \)) denote time-domain functions while upper case letters (i.e., \( P(r, \omega) \), \( V(r, \omega) \), \( \mathcal{Z}(r, s) \), \( \mathcal{Y}(r, s) \)) indicate frequency domain Fourier (\( \omega \)) or Laplace (\( s \)) transforms, the latter being analytic in \( s = \sigma + j\omega \) for \( \sigma > 0 \). An exception to this notational rule is \( \kappa(s) \).

\(^{66}\)A function \( F(s) \) is said to be complex analytic in \( s = \sigma + \omega j \) at point \( a \) if it may be represented by a convergent Taylor series in \( s \) the neighborhood of \( s = a \). That is if \( F(s) = \sum_{n=0}^{\infty} f_n(a)(s - a)^n \), where \( f_n(a) = \left. \frac{d^n F(s)}{ds^n} \right|_{s=a} \).
**Characteristic admittance $\mathcal{Y}(r)$ and impedance $\mathcal{Z}(r)$:** An important definition is each primitive wave’s **characteristic admittance** $\mathcal{Y}(r)$, defined as the square root of the ratio of $\mathcal{Y}$ over $\mathcal{Z}$ (Eq. 2.8, p. 132 and Fig. 2.2, p. 132),

$$\mathcal{Y}^\pm(r) \equiv \frac{\mathcal{Y}(r,s)}{\mathcal{Z}(r,s)} = \sqrt{\mathcal{Y}(r,s)} = \sqrt{\frac{sA(r)}{\eta_o P_o}} = \sqrt{\frac{A(r)}{\rho_o c}} > 0,$$

(1.105)

which depends specifically on $A(r)$, but not on frequency $s$ (Campbell, 1903, 1910, 1922). Based on physical requirements that the admittance must be positive, only the positive square root is allowed. The **characteristic impedance** $\mathcal{Z}(r) = 1/\mathcal{Y}(r)$.

Since the horn (Eq. 1.90, p. 96) is loss less, $\mathcal{Y}^\pm(r)$ must be real (and positive). If losses are introduced, $\kappa(s)$ and $\mathcal{Y}(r,s)$ can become complex analytic functions of the Laplace frequency $s$ (Kirchhoff, 1974; Mason, 1928; Ramo et al., 1965; Pierce, 1981, p. 532-4). This more difficult case is addressed in Section 5 (p. 169).

**Propagation function $\kappa(s)$:** The primitive solutions of the horn equation always depend on the complex wave propagation function $^6\kappa(s)$, defined as the square root of the product of $\mathcal{Z}$ and $\mathcal{Y}$ (Eq. 2.7, p. 132 and Eq. 5.26, p. 179):

$$\kappa(s) \equiv \sqrt{\mathcal{Z}(r,s)\mathcal{Y}(r,s)} = \sqrt{s\rho_o A(r)} \cdot \sqrt{s\rho_o A(r)} = \frac{s}{c_o}.$$  

(1.106)

Note that since the area $A(r)$ cancels in this expression, $c_o$ is independent of $r$. While horns are generally dispersive, that is the solution can have a group velocity (and thus a group delay) that depends on frequency, the wave-front speed is always constant, defined by the free-field sound velocity $c_o$.

The relations between the phase, group and wave-front velocities (delay) were first demonstrated between 1907–1914 by Brillouin (1960, English version), while working with Arnold Sommerfeld (Pleshko and Plačz, 1969).

**Horn input admittance:** A horn’s acoustic input admittance $Y_{in}(r,s)$ is the admittance looking into a horn (Fig. 1.20, p. 97), at any point $r$. It is defined as the ratio of the volume velocity over the average pressure

$$Y_{in}(r,s) = \frac{\mathcal{V}(r,s)}{\mathcal{P}(r,s)},$$  

(1.107)

where the direction of flow $\mathcal{V}(r,s)$ is into the admittance. $Y_{in}(s)$ depends on the entire horn, and any terminating admittance, in the case of a finite horn.

The radiation admittance $Y_{rad}(s)$ is defined for an infinite horn, when there is no terminating load, as $Y_{in}(s) = \lim_{r \to \infty} Y_{in}(r,s)$. This may be viewed as the horn’s termination at $r = \infty$.

It is a physical requirement that the real part of every physical admittance (impedance) be positive (non-negative). To guarantee this physical requirement, care must be taken when fixing the flow direction. Thus the velocity (or current) for any admittance must be defined as directed into the admittance, or else the real part will automatically be negative.

From Eq. 1.100 (p. 100) (top equation), the volume velocity is

$$\mathcal{V}(r, \omega) = -\frac{1}{2\mathcal{Z}(r,s)} \frac{d}{dr} \mathcal{P}(r, \omega).$$  

(1.108)

$^6$ The wave propagation function is commonly called the wave-number (Sommerfeld, 1952, p. 152) or simply the propagation constant. However since $\kappa(s)$ is neither a number, nor constant, we have appropriately rename it. The meaning of a complex $\kappa(s)$ is addressed in (Sommerfeld, 1952, p. 154). The units of $\kappa(s)$ are reciprocal length, namely $\kappa(s) = 2\pi/\lambda(s)$.
Examples: Given the equation for the two d’Alembert pressure waves $P^\pm(x,s)$ for the case of the uniform horn (Eq. 1.95, p. 99), with $\kappa(s) = s/c_0$, the volume velocity is

$$V^\pm(x,\omega) = -\frac{1}{2^\prime(x,s)} \frac{d}{dx} e^{\mp \kappa(s)x}$$

$$= \mp \kappa(s) P^\pm(x,s)$$

$$= Y_o P^\pm(x,s)$$

$$= \frac{A_o}{\rho_o c_o} P^\pm(x,s).$$

Thus

$$Y^\pm = \frac{A_o}{\rho_o c_o} > 0$$

is the characteristic admittance of the uniform horn, and independent of the wave direction. This is the simplest case.

For the spherical geometry Eq. 1.5.2 (p. 99):

$$V^\pm(r,\omega) = -\frac{1}{2^\prime(r,s)} \frac{d}{dr} e^{\mp \kappa(s)r}$$

$$= -\frac{1}{2^\prime(r,s)} \left[ \mp \kappa(s) \frac{e^{\mp \kappa(s)r}}{r} - \frac{e^{\mp \kappa(s)r}}{r^2} \right]$$

$$= \frac{1}{2^\prime(r,s)} \left[ \pm \kappa(s) e^{\mp \kappa(s)r} + \frac{e^{\mp \kappa(s)r}}{r^2} \right]$$

$$= \pm \frac{1}{2^\prime(r,s)} \left[ \kappa(s) P^\pm \mp \frac{1}{r} P^\pm \right]$$

$$= \pm \frac{1}{2^\prime(r,s)} \left[ \kappa(s) \mp \frac{1}{r} \right] P^\pm(r,s)$$

$$= \frac{\kappa(s)}{2^\prime(r,s)} \left[ 1 \pm \frac{1}{\kappa(s)r} \right] P^\pm(r,s)$$

$$= Y(r) \left[ 1 \pm \frac{1}{\kappa(s)r} \right] P^\pm(r,s)$$

Thus

$$Y_{in}^\pm(r,s) = Y(r) \left[ 1 \pm \frac{1}{\kappa(s)r} \right]$$

where $\kappa(s) = s/c_0$. Note that $\Re\{Y_{in}\} = Y > 0$ as is physically required.

Writing $Y_{in}(r,s)$ in terms of $\ln P^\pm$ greatly simplifies the algebra, and thus provides improved insight into the properties of the radiation admittance. For example, for the spherical wave,

$$\frac{\partial \ln P^\pm}{\partial r} = \frac{\partial}{\partial r} (\mp \kappa(s)r - \ln r) = \mp \kappa(s) - \frac{1}{r}.$$ (1.111)

Frequently the velocity and pressure are specified as functions of frequency $\omega$, not complex frequency $s = \sigma + \omega j$. However since $Y_{in}(x,s)$ must be causal and be non-negative, we indicate it as a function of $s$. Since the primitive solutions must be causal and stable functions, they also must be analytic functions of $s$ for $\sigma > 0$.

Summary of four classic horns: Table 1.9 is a summary of the properties for four different cases of $A(r)$ for the uniform, parabolic, conical and exponential horns.
Table 1.9: Table of horns and their properties for 1, 2, and 3 dimensions, and the exponential horn. In this table the horn’s range variable is \( x \), having area \( A(x) \), radius \( r_o(x) = \sqrt{A(x)/\pi} \). \( F(x) \) is the coefficient on \( P_x \) (Eq. 5.22, p. 176), \( \kappa(s) \equiv s/c_o \), where \( c_o \) is the speed of sound. A dimensionless range variable is defined as \( x \equiv (\xi - \xi_o)/(L - \xi_o) \), with \( \xi \) the linear distance along the horn axis, from \( x = \xi_o \) to \( L \) corresponding to \( x = 0 \) to 1. The horn’s primitive solutions are \( P^\pm(x, \omega) \leftrightarrow \alpha^\pm(x, t) \). When \( \pm \) is indicated, the outbound solution corresponds to the negative sign. The last column is the radiation admittance normalized by \( A(x)/\rho c_o \) where \( s = \sigma + \omega j \) is the Laplace frequency.

<table>
<thead>
<tr>
<th>#D</th>
<th>Name</th>
<th>radius</th>
<th>Area/( A_o )</th>
<th>( F(x) )</th>
<th>( P^\pm(x, s) )</th>
<th>( \alpha^\pm(x, t) )</th>
<th>( Y^\pm/\Gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D</td>
<td>plane</td>
<td>( 1/x )</td>
<td>1</td>
<td>0</td>
<td>( e^{\pm \kappa(s) x} )</td>
<td>( \delta(t \mp x/c) )</td>
<td>1</td>
</tr>
<tr>
<td>2D</td>
<td>parabolic</td>
<td>( x/x_o )</td>
<td>1</td>
<td>( 1/x )</td>
<td>( H_o^\pm(-j\kappa(s)x) )</td>
<td>( -j\kappa(s)x )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>3D</td>
<td>conical</td>
<td>( x^2 )</td>
<td>2</td>
<td>( 2/x )</td>
<td>( e^{\mp \kappa(s) x / 2} )</td>
<td>( \delta(t \mp x/c) / x )</td>
<td>1 + ( c/sx )</td>
</tr>
<tr>
<td>EXP</td>
<td>exponential</td>
<td>( e^{mx} )</td>
<td>( e^{2mx} )</td>
<td>( 2m )</td>
<td>( e^{-m(\pm \sqrt{m^2 + \kappa^2})x} )</td>
<td>( e^{-mx} E(t) )</td>
<td>--</td>
</tr>
</tbody>
</table>

The quasi-static approximation: Since the velocity perpendicular to the walls of the horn must be zero, any radial wave propagation is exponentially attenuated (\( \kappa(s) \) is real and negative, i.e., Eq. 1.106 will not describe radial wave propagation), with a space constant of about 1 diameter. The assumption that these radial waves can be ignored (i.e., more than 1 diameter from their source) is called the quasi-static approximation. As the frequency is increased, once \( f \geq f_c = 2c_o/\lambda \), the radial wave can satisfy the zero normal velocity wall boundary condition, and therefore will not be attenuated. Thus above this critical frequency, radial waves (aka, higher order modes) are supported (\( \kappa \) becomes imaginary). Thus for Eq. 1.90 to describe guided wave propagation, \( f < f_c \). But even under this condition, the solution will not be precise within a diameter (or so) of any discontinuities (i.e., rapid variations) in the area.

Each horn, as determined by the area function \( A(r) \), has a distinct wave equation, and thus a distinct solution. Note that the area function determines the upper cutoff frequency via the quasi-static approximation since \( f_c = c_o/\lambda_c \), \( \lambda_c/2 > d \), \( A(r) = \pi(d/2)^2 \). Thus to satisfy the quasi-static approximation, the frequency \( f \) must be less than the cutoff frequency

\[
f < f_c(r) = \frac{c_o}{4} \sqrt{\frac{\pi}{A(r)}}. \tag{1.112}
\]

We shall discuss two alternative matrix formulations of these equations, the ABCD transmission matrix, use for computation, and the impedance matrix, used when working with experimental measurements (Pierce, 1981, Chapter 7). For each formulation reciprocity and reversibility show up as different matrix symmetries, as addressed in Section 1.3.11 (p. 73) (Pierce, 1981, p. 195-203).

Finite horns: When a finite section of horn is terminated in a load impedance, the forward wave \( P^+(r, s) \) is reflected at the load, producing a backward traveling wave \( P^-(r, s) \). The way this is treated is by introducing boundary conditions at the termination, a key concept that needs an explanation.

Boundary Conditions: The pressure and velocity at any point \( x \) be written in terms of a superposition of the two homogeneous solutions \( P^+(x, s) \) and \( P^-(x, s) \) (aka, primitive solutions) of Eq. 1.100 (p. 100), since by d’Alembert principle \( V = V^+ - V^- \) and \( P = P^+ + P^- \). The formula for the input admittance at any point \( x \) on the line is given in terms of the forward and reverse primitive solutions

\[
Y_{in}(r) \equiv \frac{V(r, \omega)}{P(r, \omega)} = \frac{V^+ - V^-}{P^+ + P^-} = \frac{V_+}{P_+} \left( \frac{1 - V^-/V^+}{1 + P^-/P^+} \right) = \frac{1 - \Gamma(r, s)}{1 + \Gamma(r, s)}.
\]

Here the reflectance \( \Gamma(r, s) \) is defined as the ratio of the forward and retrograde traveling waves

\[
\Gamma(r, s) \equiv \frac{V^-(r, \omega)}{V^+(r, \omega)} = \frac{P^-(r, \omega)}{P^+(r, \omega)}.
\]
By a rearrangement of the above terms we may recover the definition of the characteristic admittance

$$Y(r) = \frac{V^+(r, \omega)}{P^+(r, \omega)} = \frac{V^-(r, \omega)}{P^-(r, \omega)},$$

where $Y(r) = \rho_0 c_0 / A(r)$ for lossless horns.

It follows that in matrix notation d’Alembert’s principle may be written as

$$\begin{bmatrix} P(x) \\ V(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ Y(x) & -Y(x) \end{bmatrix} \begin{bmatrix} P^+(x) \\ P^-(x) \end{bmatrix}. \quad (1.113)$$

**Complex-analytic nature of $\Gamma(s)$ and $Z_{in}(s)$:** A very important assumption has been made here when expressing the complex reflectance $\Gamma(s)$ as a function of the complex frequency $s = \sigma + j\omega$, even though it is defined by the ratio of two functions of real (radian) frequency $\omega$. This assumption is based on the fact that, like the impedance, the reflectance must be causal. Namely $\gamma(t) \leftrightarrow \Gamma(s)$ is zero for $t < 0$. The same may be assumed of the time-domain impedance $\zeta(t) \leftrightarrow Z_{in}(s)$). That $\gamma(t)$ and $\zeta(t)$ are causal is required by the physics. It follows that both $\Gamma(s)$ and $Z_{in}(s)$ are complex analytic functions of $s$, which means they must have a Taylor series expansion in $s$ everywhere in the right-half $s$ plane ($\sigma > 0$). This follow from the *Cauchy’s integral formula* (aka, The Residue theorem).

The forward and retrograde waves may be any function of frequency $\omega$, since they depend on the source pressure (or velocity) at the input to the horn. The reflectance is a transfer function (thus the source term cancels), that only depends on the impedance (or reflectance) looking into the system (at any position $x$).

As an alternative way to derive $\Gamma(r, s)$, invert Eq. 1.113

$$\begin{bmatrix} P^+(x) \\ P^-(x) \end{bmatrix} = \frac{1}{2Y(x)} \begin{bmatrix} Y(x) & 1 \\ Y(x) & -1 \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & Z(x) \\ 1 & -Z(x) \end{bmatrix} \begin{bmatrix} P \\ V \end{bmatrix} \quad (1.114)$$

and form the ratio of reflected to incident waves

$$\Gamma(s) = \frac{P^-}{P^+} = \frac{P - 2V}{P + 2V} = \frac{Z_{in} - Z}{Z_{in} + Z}.$$

It is convenient to define the normalized input impedance $Z_{in}/Z$ when working with this form of $\Gamma(s)$.

Given some experience with these two forms, $Z_{in} \equiv 1/Y_{in}$ and $\Gamma(s)$, one may quickly begin to appreciate the advantage of working with the reflectance over the radiation impedance/admittance (aka immittance). The impedance has so many different forms, each of which are complicated, whereas the reflectance is easily understood, as it is closer to the physics.

**Finite length Horns:** For a horn of fixed length $L$ these expressions may be rewritten in terms of re-normalized primitive waves. If we define the forward wave $P^+(x)$ as launched from $x = 0$ and the retrograde wave $P^-(x)$ as launched from $x = L$, we may also write the pressure and velocity in terms of the primitives.

$$\begin{bmatrix} P(x) \\ V(x) \end{bmatrix} = \begin{bmatrix} P^+(x) \\ Y^+(s)P^+(x) \end{bmatrix} \begin{bmatrix} P^-(x - L) \\ -Y^-(s)P^-(x - L) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \quad (1.115)$$

The characteristic admittance $Y^\pm(x, s)$ depends on the range $x$ and Laplace frequency $s$. Coefficients $\alpha(\omega)$ and $\beta(\omega)$, which depend on frequency $\omega$ (but not $x$), are determined by the boundary condition at $x = L$. To find the four parameters $[A(s), B(s), C(s), D(s)]$ we evaluate the inverse of Eq. 1.115 at $x = L$, substitute this result into 1.115, and then evaluated the matrix product at $x = 0$.

For compactness we adopt the following simplified subscript notation: $P_o^\pm \equiv P^\pm(x = 0, s)$, $P_L^\pm \equiv P^\pm(x = L, s)$, i.e., $P_L^+ \equiv P(x = L)$ and $V_L \equiv V(x = L)$. The normalization of the primitive solutions are taken as $P_o^+ = 1$ and $P_L^- = 1.
To compute \( \Delta_L \) we start with

\[
-\Delta_T(x, s) = \mathcal{P}(x) Y^-(x-L) + \mathcal{P}(x-L) Y^+(x) = \mathcal{P}_o \end{array}
\]

evaluated at \( x = L \) (recall \( Y^\pm_{rad}(x) = L_p \), \( P^+_o = P^-_o = 1 \)),

\[
-\Delta_L = Y_{rad} \left[ \mathcal{P}_L^+ \mathcal{P}_o^- + \mathcal{P}_o^+ \mathcal{P}_L^- \right],
\]

thus

\[
-\Delta_L = Y_{rad} \left[ \mathcal{P}_L^+ \mathcal{P}_o^- + \mathcal{P}_o^+ \mathcal{P}_L^- \right].
\]

This detailed set of computations results in the following:

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = -\frac{1}{\Delta_L} \begin{bmatrix}
Y^+_{rad}(x-L) & \mathcal{P}^-(x-L) \\
Y^+_{rad}(x) & \mathcal{P}^+(x)
\end{bmatrix} \begin{bmatrix}
\mathcal{P}_L \\
\mathcal{V}_L
\end{bmatrix}, \tag{1.116}
\]

which when simplified is

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = -\frac{1}{\Delta_L} \begin{bmatrix}
Y^-_{rad}(L) & -1 \\
Y^+_{rad}(L) & \mathcal{P}_L^+
\end{bmatrix} \begin{bmatrix}
\mathcal{P}_L \\
\mathcal{V}_L
\end{bmatrix}. \tag{1.117}
\]

The subscript to the right of each matrix indicates it is evaluated at \( x = 0 \) or \( x = L \). Here \( \mathcal{P}_L^- \) is \( \mathcal{P}^-(x-L) \) at \( x = 0 \). The sign of \( \mathcal{V}_L \) must be negative to satisfy the definition of every ABCD matrix, that the output velocity (i.e., \( -\mathcal{V}_L \)) is out of the port.

The relationship of \( \beta/\alpha \) has special significance because it specifies the ratio of the reflected wave amplitude \( \beta(\omega) \) in terms of the incident wave amplitude \( \alpha(\omega) \). This ratio is known as the reflectance

\[
\gamma_L(t) \leftrightarrow \Gamma_L(s) = \frac{\beta}{\alpha}. \tag{1.119}
\]

It has a critical role in the theory of horns, as we shall see as it is determined by the relative rate of change of the impedance (i.e., area) with range (i.e., \( d\ln(Z)/dx \)).

### 1.5.4 Lec 37 Three examples of horns

The uniform horn

The 1D wave equation \([A(r) = A_o]\)

\[
\frac{d^2}{dr^2} \mathcal{P} = \frac{s^2}{c_o^2} \mathcal{P}.
\]

**Solutions:** The two primitive solutions of this equation are the two d’Alembert waves (Eq. 1.68, p. 79)

\[
\varrho(x, t) = \varrho^+(t-x/c) + \varrho^- (t+x/c) \leftrightarrow \mathcal{P}^+(r_o, s) e^{-\kappa_x r} + \mathcal{P}^-(r_o, s) e^{\kappa_x r},
\]

where \( \mathcal{P}^\pm(r_o, s) \) are Laplace transform pairs representing the causal forward and retrograde traveling wave pressure amplitudes. Thus the normalized primary solutions are

\[
\varrho^+(x, t) = \delta(t-x/c_o) \leftrightarrow \mathcal{P}^+(r_o, s) e^{-\kappa_x r}
\]

and

\[
\varrho^- (x, t) = \delta(t+x/c_o) \leftrightarrow \mathcal{P}^-(r_o, s) e^{\kappa_x r}.
\]

Thus it is convenient to normalize \( \mathcal{P}^\pm(r_o, s) = 1 \). As before, separation of time and place variables explains this relationship.
Characteristic admittance $\mathcal{Y}$: Since $\mathcal{V}^\pm = -\partial_x \ln \mathcal{P}^\pm$ we find

$$\mathcal{Y}^\pm = \frac{\mathcal{V}^\pm}{\mathcal{P}^\pm} = A_o/\rho_o c$$

(Eq. 1.109, p. 102), independent of direction. The signs must be “physically chosen,” with the velocity $\mathcal{V}^\pm$ into the port, to assure that $\mathcal{Y} > 0$, for both waves.

1st ABCD matrix: The general solution is compactly formulated as an ABCD matrix (i.e., Section 1.3.6, p. 62), starting from

$$\begin{bmatrix} \mathcal{P}(x) \\ \mathcal{V}(x) \end{bmatrix} = \begin{bmatrix} e^{-\kappa x} & e^{\kappa x} \\ \mathcal{Y} & -\mathcal{Y} e^{\kappa x} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

where $\alpha$, $\beta$ are the relative weights on the two unknown primitive solutions.

Solving Eq. 1.121 for $\alpha$ and $\beta$ and evaluating at $x = L$, the amplitudes, in terms of the pressure and velocity, at any point along the line at $L$, are

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -\frac{1}{2\mathcal{Y}} \begin{bmatrix} \mathcal{Y} (e^{-\kappa x} & e^{-\kappa x} \\ -\mathcal{Y} e^{-\kappa x} & e^{\kappa x} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ \mathcal{V}_L \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{\kappa L} & \mathcal{Z} e^{\kappa L} \\ 1 & -\mathcal{Z} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix},$$

where $\mathcal{Z} \equiv 1/\mathcal{Y}$. The sign of $\mathcal{Y}_L$ has been chanted so as to be positive looking into the port at $x = L$. Since in this case $\Delta_L = -2\mathcal{Y} e^{-\kappa L}$ is independent of $x$, evaluating it at $x = L$ has no effect. Substitution of Eq. 1.122 into Eq. 1.121 and evaluating the result at $x = 0$ gives the final ABCD matrix

$$\begin{bmatrix} \mathcal{P}_o \\ \mathcal{V}_o \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & e^{-\kappa L} \\ \mathcal{Y} & -\mathcal{Y} e^{-\kappa L} \end{bmatrix} \begin{bmatrix} e^{\kappa L} & \mathcal{Z} e^{\kappa L} \\ 1 & -\mathcal{Z} \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}$$

Multiplying these out gives the final transmission matrix as

$$\begin{bmatrix} \mathcal{P}_o \\ \mathcal{V}_o \end{bmatrix} = \begin{bmatrix} \cosh(\kappa L) & \mathcal{Z} \sinh(\kappa L) \\ \mathcal{Y} \sinh(\kappa L) & \cosh(\kappa L) \end{bmatrix} \begin{bmatrix} \mathcal{P}_L \\ -\mathcal{V}_L \end{bmatrix}$$

with $\kappa = s/c$, $\mathcal{Y} = 1/\mathcal{Z} = A_o/\rho_o c$ (Pipes, 1958). The two velocities are defined into their respective ports.

Exercise: Show that the above equation follows from Eq. 1.123. Sol: This follows from the definitions of $\cosh(x) = e^x + e^{-x}$ and $\sinh(x) = e^x - e^{-x}$.

Impedance matrix: Expressed Eq. 1.124 as an impedance matrix gives (algebra required)

$$\begin{bmatrix} \mathcal{P}_o \\ \mathcal{P}_L \end{bmatrix} = \frac{\mathcal{Z}}{\sinh(\kappa L)} \begin{bmatrix} \cosh(\kappa L) & 1 \\ 1 & \cosh(\kappa L) \end{bmatrix} \begin{bmatrix} \mathcal{V}_o \\ \mathcal{V}_L \end{bmatrix}.$$

The input admittance: Given the input admittance of the horn, it is possible to determine if it is uniform, without further analysis. Namely if the horn is uniform and infinite in length, the input impedance at $x = 0$ is

$$Y_{in}(0, s) \equiv \frac{\mathcal{V}(0, \omega)}{\mathcal{P}(0, \omega)} = \mathcal{Y},$$

since $\alpha = 1$ and $\beta = 0$. That is for an infinite uniform horn, there are no reflections.

When the horn is terminated with a fixed impedance $\mathcal{Z}_L$ at $x = L$, one may substitute pressure and velocity measurements into Eq. 1.122 to find $\alpha$ and $\beta$, and given these, one may calculate the reflectance at $x = L$ (see Eq. 1.119, 105)

$$\Gamma_L(s) \equiv \frac{\mathcal{P}^-}{\mathcal{P}^+}_{x=L} = \frac{\beta}{\alpha} = \frac{\mathcal{P}(L, \omega) - \mathcal{Z}_L \mathcal{V}(L, \omega)}{\mathcal{P}(L, \omega) + \mathcal{Z}_L \mathcal{V}(L, \omega)} = \frac{\mathcal{Z}_L - \mathcal{Z}}{\mathcal{Z}_L + \mathcal{Z}},$$

given sufficiently accurate measurements of the throat pressure $\mathcal{P}(0, \omega)$, velocity $\mathcal{V}(0, \omega)$, and the characteristic impedance of the input $\mathcal{Z} = \rho_o c/A(0)$. 
2D parabolic Horn

For 2D cylindrical waves the area function is \( A(x) = A_o x \), (horn radius \( r \propto \sqrt{x} \)) thus the Webster horn equation reduces to the cylindrical wave equation (Appendix 1.5.3.)

\[
P_{xx}(x, \omega) + \frac{1}{x} P_x(x, \omega) = \kappa^2(s) P(x, \omega)
\]

having Bessel function primitive solutions

\[
\mathcal{P}^+(x, \omega) = J_\nu(\omega x/c) - iY_\nu(\omega x/c) = H_\nu^+(j\kappa x)
\]

and

\[
\mathcal{P}^-(x, \omega) = J_\nu(\omega x/c) + iY_\nu(\omega x/c) = H_\nu^-(j\kappa x),
\]

where \( J_\nu \) and \( Y_\nu \) are the standard two types Bessel functions (like \( \cos() \) and \( \sin() \)), and \( H_\nu^\pm(x) \) are the two kinds of Hankel function, of the first (-) and second (+) kind (similar to \( e^{\pm j\kappa x} \)), all of order zero (indicated by the subscript) (Salmon, 1946a; Olson, 1947; Morse and Feshbach, 1953).

2D input admittance: Given a half-infinite section of parabolic horn, from Eq. ??

\[
Y_{in}^\pm(x, s) = \mp \frac{Y}{k} \frac{\partial}{\partial x} \ln H_\nu^\pm(-j\kappa x) = \mp j \frac{H_\nu^\pm}{H_\nu^0}.
\]

2D ABCD Transmission matrix: Based on Eq. 1.118 (p. 105)

\[
\begin{bmatrix} P_o \\ V_o \end{bmatrix} = \frac{-1}{\Delta L} \begin{bmatrix} 1 & P_{-L}^- \\ Y_{in}^+ & -Y_{in}^- P_{-L}^- \end{bmatrix}_o \begin{bmatrix} Y_{in}^- \\ Y_{in}^+ P_{L}^+ \end{bmatrix}_L \begin{bmatrix} P_L \\ -V_L \end{bmatrix},
\]

Verify the following

\[
\begin{bmatrix} P_o \\ V_o \end{bmatrix} = \frac{-1}{\Delta L} \begin{bmatrix} 1 & H_\nu^-(x - L) \\ Y_{in}^+ & -Y_{in}^- H_\nu^0(x - L) \end{bmatrix}_{x=0} \begin{bmatrix} Y_{in}^- \\ Y_{in}^+ H_\nu^0(x) -1 \end{bmatrix}_{x=L} \begin{bmatrix} P_L \\ -V_L \end{bmatrix},
\]

where the subscript on each matrix indicates the value of \( x \) at which it is evaluated.

Impedance matrix:

\[
\begin{bmatrix} P_o \\ P_L \end{bmatrix} = \frac{1}{C(s)} \begin{bmatrix} A(s) & 1s \\ 1 & D(s) \end{bmatrix} \begin{bmatrix} V_o \\ V_L \end{bmatrix},
\]

3D Conical Horn

For each horn we must find the natural normalization from the range variable to the normalize range variable \( x \). For the conical horn the radius is proportional to the range variable \( r \) thus

\[
A(r) = 4\pi \sin^2(\Theta/2) \ r^2 \ [m^2].
\]

The angle \( \Theta \) is a measure of the solid (cone) angle. When \( \Theta = \pi \) we have the case of the entire sphere so the solid angle is \( 4\pi \) [steradians] and the area is \( 4\pi r^2 \). The formula for the area may be simplified by defining \( A_\theta \equiv 4\pi \sin^2(\Theta/2) r_o^2 \ [m^2] \), resulting in the more convenient relation

\[
A(r) = A_\theta \ (r/r_o)^2 \ [m^2].
\]

\( \text{Note that} \)

\[
\frac{\partial \ln H_\nu^\pm(kx)}{\partial x} = -k \frac{H_\nu^\pm(kx)}{H_\nu^0(kx)}
\]
CHAPTER 1. INTRODUCTION

Figure 1.21: Real and imaginary parts of the radiation admittance $Y_{in}/Y$ (upper) and impedance (lower), as a function of frequency, for an infinite section of horn, as computed from Eq. 1.125. The real and imaginary parts of the impedance are very similar to the results of Olson (1947) shown in Fig. 5.1. The real and imaginary parts of the admittance are much simpler to characterize: the imaginary part is a mass (i.e., is $\propto 1/s$) while the real part $\rightarrow 1$ above the critical frequency (where the real and imaginary parts are equal) at $\approx 0.3$ [kHz].

This is a bit tricky because $A_\theta$ is not a constant since it depends on the place where the area was normalized, in this case $r_o$.

Using the conical horn area in Eq. 5.7 results in the spherical wave equation [Appendix 5.7, (Olson, 1947; Morse, 1948; Benade, 1988)]

$$P_{rr}(r, \omega) + \frac{2}{r} P_r(r, \omega) = \kappa^2 P(r, \omega)$$  \hspace{1cm} (1.129)

(one must remember the steradian scale factor $A_\theta$). Here $F(r) = \partial_r \ln A(r) = \frac{2}{r}$ (see Table 1.9).

**Primitive solutions:** We may now identify the two primitive d’Alembert solutions of Eq. 1.129 (see Appendix 1.5.3)

$$P^\pm(r, s) = P^\pm_\theta(r, s) e^{\mp \kappa(s) r} / r.$$  

These are also known as the zero order spherical Hankel functions (i.e., $h^\pm_0(\kappa r)$).

We wish to normalize these primitive solution such that they are 1 at the source locations $r = r_o$ and $r = r_L$ (i.e., $P^+(r_o, s) = 1$ and $P^-(r_L, s) = 1$). The normalized primitive solutions are

$$P^+(r, s) \equiv \frac{r_o}{r} e^{-\kappa(r-r_o)} \quad P^-(r, s) \equiv \frac{r_L}{r} e^{+\kappa(r-r_L)},$$

as may be easily verified by letting $r = r_o$ in $P^+(r, s)$ and $r = r_L$ in $P^-(r, s)$.

Figure 1.22: The equivalent circuit for the radiation impedance $Z_{rad}^+(s, r)$ for a conical is a parallel combination of the radiation resistance $Z(r) = \rho_0 c / A_\theta(r)$ and a mass given by $M(r) = \pm \rho_0 r / A_\theta$. Note that the inbound primitive wave has a negative mass! This means that the inbound wave is not the time-reverse version of the outbound wave. I take this to be due to the different curvature of the wave-fronts of the waves.
\textbf{Input admittance:} Given a half-infinite section of conical horn Eq. ?? gives

\begin{equation}
Y_{in}^\pm(r, s) = \mp \frac{Y(r)}{\kappa(s)} \left. \frac{\partial \ln P^\pm}{\partial r} \right|_{r=r_L} = Y(r) \left( 1 \pm \frac{c}{r s} \right) \Theta(t) \pm \frac{c \Gamma(t)}{r}, \tag{1.130}
\end{equation}

\begin{align*}
Y_{in}^+(r_o, s) &= -\frac{Y}{\kappa} \left. \frac{\partial \ln P^-}{\partial r} \right|_{r=r_o} = \frac{Y}{\kappa} \left. \frac{\partial (kr + \ln r)}{\partial r} \right|_{r=r_o} = Y(r) \left( 1 + \frac{c}{sr_o} \right) \\
Y_{in}^-(r_L, s) &= +\frac{Y}{\kappa} \left. \frac{\partial \ln P^+}{\partial r} \right|_{r=r_L} = \frac{Y}{\kappa} \left. \frac{\partial (kr - \ln r)}{\partial r} \right|_{r=r_L} = Y(r) \left( 1 - \frac{c}{sr_L} \right)
\end{align*}

where \( \kappa(s) = s/c, \Theta(t) \) and \( \Gamma(t) \) are the Dirac and Heaviside functions, respectively.

As shown in Fig. 1.22 there are two additive admittance terms, a real part equal to the characteristic conductance \( Y(r) \), and a mass, equal to \( \pm Z r/c = \pm \rho o r/A o \). For the reverse traveling primitive wave, these conventions result in negative mass, an interesting and perhaps surprising result. Since the admittances are summed, the impedances appear in parallel.

Figure 5.1 from Olson (1947, Fig. 5.3, p. 101) provides the real and imaginary impedance for each of the horns discussed here. The difference between Olson and the present work, is in our intent here to explain the physics behind the reactive term, in the time domain.

\textbf{3D ABCD Transmission matrix:} For clarity the relations are re-derive “from scratch”

\begin{equation}
\begin{bmatrix}
P(r) \\
V(r)
\end{bmatrix} = \begin{bmatrix}
r_o e^{-\kappa r_o} & r_o e^{+\kappa r_o} \\
Y^+(r) \frac{r_o}{r_L} e^{-\kappa r_o} & -Y^-(r) \frac{r_o}{r_L} e^{+\kappa r_o}
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}, \tag{1.131}
\end{equation}

Evaluate at the boundary \( r_L \)

\begin{align*}
\begin{bmatrix}
P(r_L) \\
V(r_L)
\end{bmatrix} &= \begin{bmatrix}
r_o e^{-\kappa r_o} & r_o e^{+\kappa r_o} \\
Y^+(r) \frac{r_o}{r_L} e^{-\kappa r_o} & -Y^-(r) \frac{r_o}{r_L} e^{+\kappa r_o}
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} \\
&= \begin{bmatrix}
P_L \\
V_L
\end{bmatrix} = -\begin{bmatrix}
-\frac{r_o}{r_L} e^{-\kappa r_o} & -1 \\
-Y^+_{rad}(r_L) \frac{r_o}{r_L} e^{-\kappa r_o} & Y^-_{rad}(r_L)
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}, \tag{1.132}
\end{align*}

and taking the inverse, gives the weights \( \alpha, \beta \) and the boundary (load) impedance\(^69\)

\begin{equation}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = -\frac{1}{\Delta_L} \begin{bmatrix}
Y^+_{rad}(r_L) \\
Y^-_{rad}(r_L)
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = \frac{P_L}{V_L}, \tag{1.134}
\end{equation}

where

\(-\Delta_L = Y^-_{rad}(r_L) + Y^+_{rad}(r_L) \frac{r_o}{r_L} e^{-\kappa r_o} = 2Y_L \frac{r_o}{r_L} e^{-\kappa r_o}.\)

Substituting Eq. 1.134 back into Eq. 1.131

\begin{equation}
\begin{bmatrix}
P(r) \\
V(r)
\end{bmatrix} = -\frac{1}{\Delta_L} \begin{bmatrix}
r_o e^{-\kappa r_o} & r_o e^{+\kappa r_o} \\
Y^+_{rad}(r) \frac{r_o}{r_L} e^{-\kappa r_o} & -Y^-_{rad}(r) \frac{r_o}{r_L} e^{+\kappa r_o}
\end{bmatrix} \begin{bmatrix}
Y^+_{rad}(r) \frac{r_o}{r_L} e^{-\kappa r_o} & -Y^-_{rad}(r) \frac{r_o}{r_L} e^{+\kappa r_o}
\end{bmatrix} \begin{bmatrix}
P_L \\
V_L
\end{bmatrix}. \tag{1.135}
\end{equation}

Evaluating this at the throat \( r = r_o \) we obtain the final ABCD matrix in factored form\(^70\)

\begin{equation}
\begin{bmatrix}
P_o \\
V_o
\end{bmatrix} = -\frac{1}{\Delta_L} \begin{bmatrix}
1 & \frac{r_o}{r_L} e^{+\kappa r_o} \\
Y^+_{rad}(r_o) & -Y^-_{rad}(r) \frac{r_o}{r_L} e^{+\kappa r_o}
\end{bmatrix} \begin{bmatrix}
Y^+_{rad}(r) \frac{r_o}{r_L} e^{-\kappa r_o} & -Y^-_{rad}(r) \frac{r_o}{r_L} e^{+\kappa r_o}
\end{bmatrix} \begin{bmatrix}
P_L \\
V_L
\end{bmatrix}. \tag{1.136}
\end{equation}

\(^69\) The three red signs have been changed to assign \( V_L \) into the port.

\(^70\) It is possible that this is useful in this format, say if \( \beta = 0 \).
At this point it is not difficult to compute $A(s), B(s), C(s), D(s)$.

For reference, the general case (Eq. 1.118) is\footnote{Notation changed to match that of Eq. 1.136.}

$$\begin{bmatrix} \mathcal{P}_o \\ \mathcal{V}_o \end{bmatrix} = \frac{-1}{\Delta_L} \begin{bmatrix} 1 & \mathcal{P}_L^- \\ -\mathcal{Y}^-(r_o) & \mathcal{Y}^-(r_L) \end{bmatrix} \begin{bmatrix} \mathcal{Y}^+(r_L) & -1 \\ \mathcal{Y}^+(r_L) \mathcal{P}_L^+ & -\mathcal{V}_L \end{bmatrix} \begin{bmatrix} \mathcal{P}_o \\ \mathcal{V}_o \end{bmatrix}. \quad (1.137)$$

**Impedance matrix:**

$$\begin{bmatrix} \mathcal{P}_o \\ \mathcal{P}_L \end{bmatrix} = \frac{1}{C(s)} \begin{bmatrix} A(s) & 1 \\ D(s) & \mathcal{V}_o \end{bmatrix} \begin{bmatrix} \mathcal{V}_o \\ \mathcal{V}_L \end{bmatrix}. \quad (1.138)$$

**Exponential Horn**

Starting from the Webster horn equation (Eq. 1.90, p. 96) having an exponential area of the form $A(x) = A_0^{2mx}$,

$$\frac{\partial^2 P(x, \omega)}{\partial x^2} + 2m \frac{\partial P(x, \omega)}{\partial x} = \kappa^2 P(x, \omega), \quad (1.139)$$

**Exercise:** Show that Eq. 5.7 follows from Eq. 1.90. \textbf{Sol:} Starting from Eq. 1.90 with $A(r) = A_0 e^{2mr}$

$$\frac{1}{A_0 e^{2mr}} \frac{\partial}{\partial r} \left( A_0 e^{2mr} \frac{\partial \varphi}{\partial r} \right) = \frac{1}{c_o^2} \frac{\partial^2 \varphi}{\partial t^2}$$

$$\varphi_{rr}(r, t) + 2m \varphi_r(r, t) = \frac{1}{c_o^2} \frac{\partial^2 \varphi}{\partial t^2} \leftrightarrow \kappa^2 \mathcal{P}(r, s)$$

which is the time domain version of Eq. 1.139.

Since Eq. 1.139 is an ordinary constant coefficient differential equation, it has a closed form solution (Olson, 1947; Salmon, 1946a,b; Morse, 1948; Beranek, 1954; Leach, 1996; Beranek and Mellow, 2012). By the substitution $P(x, \omega) = P(\kappa(s)) e^{-\kappa_\pm(s)x}$, one may solve for the characteristic roots $\kappa_\pm(s) = m \pm \sqrt{m^2 + \kappa^2}$. Thus

$$P^\pm(x) = e^{-mx} e^{\mp j\sqrt{m^2 + \kappa^2}x} = e^{-mx} e^{\mp j\sqrt{\omega^2 - \omega_c^2}x/c}, \quad (1.140)$$

which represent the horn’s right (+) and left (-) traveling pressure waves.

**Exercise:** Shown that $P^\pm(r, \omega)$ satisfy Eq. 1.90 (p. 96). \textbf{Sol:} Taking partials wrt $r$,

$$\partial_r P^\pm(r, \omega) = \left(-m \mp \sqrt{m^2 + \kappa^2}\right) P^\pm(r, \omega)$$

$$\partial_{rr} P^\pm(r, \omega) = \left(-m \mp \sqrt{m^2 + \kappa^2}\right)^2 P^\pm(r, \omega)$$

$$= \left(2m^2 + \kappa^2 \pm 2m \sqrt{m^2 + \kappa^2}\right) P^\pm(r, \omega)$$

Substituting the above partials into Eq. 1.139 gives

$$\left(2m^2 + \kappa^2 \pm 2m \sqrt{m^2 + \kappa^2}\right) + 2m \left(-m \mp \sqrt{m^2 + \kappa^2}\right) = \kappa^2,$$

which is an equality. Thus Eq. 1.140 satisfies Eq. 1.139.
Case of an arbitrary area function: Next consider the Fourier series (or Fourier transform) of the area function

\[ A(r) = \sum_k a_k e^{2mkr}. \]

It follows from the linearity of the wave equation that the general solution of Eq. 1.139 is

\[ P^\pm(r, s) = \sum_k a_k^\pm(\omega) e^{-mkr} e^{\mp \sqrt{m^2 + \kappa^2} r} \]

where we have combined \( P^\pm(\omega) \) and \( a_k \) into single terms \( a_k^\pm(\omega) \).

Dispersion diagram: The dispersion diagram corresponding to the horn is shown in Fig. 1.23. Note that the exponential horn has a dispersion diagram identical to the electron’s wave properties in a semiconductor. Simply by changing the flare function from conical to exponential, the horn impedance switches from a mass in parallel with a resistor, to a horn in cutoff, with conduction and stop band (an evanescent wave band region), as in a semiconductor.

The outbound time domain solution of the exponential (exp) horn is \( e^{-mx} E(t) \), with

\[ E(t) = \delta(t - x/c) + \frac{x}{c} J_1(\sqrt{t^2 - x^2/c^2}) U(t - x/c) \leftrightarrow e^{-\sqrt{m^2 + \kappa^2} x} \]

Exp-Horn ABCD Transmission matrix:

From Eq. 18

\[
\begin{bmatrix} P_o \\ V_o \end{bmatrix} = -\frac{1}{\Delta L} \begin{bmatrix} 1 & P^- \\ Y^+ & -Y^- P^- \end{bmatrix} \begin{bmatrix} Y^- & -1 \\ Y^+ P^+ & P^+ \end{bmatrix} L \begin{bmatrix} P_L \\ -V_L \end{bmatrix}. \tag{1.141}
\]

Input admittance: Given a half-infinite section of the exponential horn (Salmon, 1946b; Leach, 1996), from Eq. ??

\[
Y_{in}^\pm(x_o, s) = \frac{Y}{s} \left( \omega_c \mp \sqrt{\omega_c^2 + s^2} \right). \tag{1.142}
\]

where \( \omega_c \equiv mc \) is the horn cutoff frequency. At very high frequencies this approaches a real value of \( Y \). Below cutoff it is a purely reactive admittance.

\[^{\text{72}}\]See PowerWavesE paper for a more detailed expression.
Discussion: Since Eq. 1.139 contains no viscous or other loss terms, the solution is always loss-less. The propagation function’s roots $\kappa_\pm(s)$ are imaginary when $\omega > \omega_c$, but change to a purely real value below the cutoff frequency, i.e., $\omega < \omega_c$.

At all frequencies the wave propagation is dispersive, meaning that the speed is frequency dependent. Above the cutoff frequency there is normal wave propagation. However the impedance of the wave changes, making the horn an “ideal” acoustic transformer. As shown in Fig. 1.23, at very high frequencies ($\omega \geq \omega_c = mc$) the wave propagates without dispersion, but still with a decay, due to the exponential change in area.

When the diameter of the horn becomes greater than half a wavelength, higher order modes start to come into play, and the solution is no longer unique, as the quasi-static approximation totally breaks down. Any imperfection in the area will introduce cross modes, which can then propagate.

In regions where the diameter is less than a half wavelength, higher order modes will not propagate. This is best described by the quasi-static approximation. But the exact solution may be obtained by solving Eq. eq:WHE locally for the pressure.

Below cutoff requires further analysis, as the wave solution is still causal, but everywhere in phase. To see this we may take the inverse Laplace transform of $P(x, s)$ to obtain the explicit wave behavior in the time domain. It is very useful to look at this case in both the time and frequency domains, in order to fully appreciate what is happening.

1.5.5 Lec 38: Integral forms of $\nabla()$, $\nabla \cdot ()$ and $\nabla \times ()$

There are two definitions for each operator, the point or differential form, and the integral limit form. The integral form gives an intuitive view of what the operator does, and in the limit, converges to the differential form. These two definitions are discussed next.

The divergence and the curl each have fundamental theorems: Gauss’ Law (divergence theorem) and Stokes’ Law (curl theorem). Without the use of these very fundamental vector calculus theorems, Maxwell’s equations cannot be understood. The history of these important theorems is discussed in the caption of Fig. 1.11, p. 45.

Gradient: $E = -\nabla \phi(x, y, z)$ [V/m]

As briefly summarized on page 94, the differential definition of the gradient maps $\mathbb{R}^1 \rightarrow \mathbb{R}^3$. For example, the electric field strength is the gradient of the voltage

$$E(x) = -[\partial_x, \partial_y, \partial_z]^T \phi(x) = -\left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right]^T (x).$$

The negative sign is optional.

The gradient is the slope of the tangent plane of the potential $\phi(x)$ at $x$ pointing in the direction of the maximum slope. The gradient (i.e., $E(x)$) is $\perp$ of the constant potential contours of $\phi(x)$ (the iso-potential lines), since derivatives along any iso-contours must be zero.

As a specific example, consider the paraboloid $z = 1 - (x^2 + y^2)$ as the potential, with iso-potential circles of constant $z$ that have radius of zero at $z = 1$, and unit radius at $z = 0$. The negative gradient

$$E(x) = -\nabla z(x, y) = 2(x\hat{i} + y\hat{j} + 0\hat{k})$$

is $\perp$ to the circles of constant radius (constant $z$), thus points in the direction of the radius.

If one were free-fall skiing this surface, they would be the first one down the hill. Normally skiers try to stay close to the iso-clines (not in the direction of the gradient), so they can stay in control. If you ski an iso-cline, you must walk, since there is no pull due to gravity.
The integral definition of the gradient: The gradient may be defined in terms of the integral

$$\nabla \phi(x, y, z) \equiv \lim_{|S| \to 0} \left\{ \frac{\int_{S} \phi(x, y, z) \hat{n} d|S|}{|S||}} \right\},$$

(1.143)

over a closed surface \(\mathcal{S}\), having area \(|\mathcal{S}|\) and volume \(||\mathcal{S}|\|\), centered at \((x, y, z)\) (Greenberg, 1988, p. 773). Here \(\hat{n}\) is the unit vector perpendicular to the surface \(\mathcal{S}\)

$$\hat{n} = \frac{\nabla \phi}{||\nabla \phi||}.$$  

The dimensions of Eq. eq:GradInt are in the units of the potential times the area, divided by the volume, as needed for a gradient (e.g., \([\text{Volts}/\text{m}]\)).

The natural way to define the surface and volume is to place the surface on the iso-potential surfaces, forming either a cube or pill-box shaped volume. As the volume goes to zero, so must the area. One must avoid irregular volumes such that the area is finite as the volume goes to zero (Greenberg, 1988, footnote p. 762).

A well known example is the potential

$$\phi(x, y, z) = \frac{Q}{\epsilon_o \sqrt{x^2 + y^2 + z^2}} = \frac{Q}{\epsilon_o R} \quad [\text{Volts}]$$

around a point charge \(Q\) [SI Units of Coulombs]. The constant \(\epsilon_o\) is the permittivity [Farad/m²].

How does this work? To better understand what Eq. 1.143 means, consider a three-dimensional Taylor series expansion of the potential in \(x\) about the limit point \(x_o\). To compute the higher order terms (HOT) one needs the Hessian matrix

$$\mathcal{H}_{i,j} = \frac{\partial^2 \phi}{\partial x_i \partial x_j},$$

which will exist if the potential is analytic in \(x\) at \(x_o\).

For this definition to apply, \(x\) must approach \(x_o\) along \(\hat{n}\).

The natural form for the surface \(|\mathcal{S}|\) is to lie along the iso-potential surfaces as much as possible, so that the integral is a constant (the potential) times the area. The remainder of the surface must be perpendicular to these iso-potential surfaces, in the direction of the gradient, or maximum change of the potential. The secret to the integral definition is in taking the limit. As the volume \(||\mathcal{S}|\||\ shrinks to zero, the HOT terms are small, and the integral reduces to the first order term in the Taylor expansion, since the constant term integrates to zero. Such a construction is used in the proof of the Webster Horn equation (5.3; Fig. 5.2, p. 174).

Divergence: \(\nabla \cdot \mathbf{D} = \rho \quad [\text{Col/m}^3]\)

As briefly summarized on page 94, the differential definition of the gradient which maps \(\mathbb{R}^3 \mapsto \mathbb{R}^3\)

$$\nabla \cdot \mathbf{D} = \left[ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] = \rho(x, y, z)$$

The divergence is a direct measure of the flux (flow) of vector field it operates on \(\mathbf{D}\), coming from \(x\). A vector field is said to be incompressible if the divergence of that field is zero. It is therefore compressible

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73See further discussions on pages Greenberg (1988, pp. 778, 791, 809)

74\(H_{i,j} = \delta^2(x)\phi/\partial x_i \partial x_j\), which will exist if the potential is analytic in \(x\) at \(x_o\).
when the divergence is non-zero. Compared to air, water is considered to be incompressible. However at very low frequencies, air can also be considered as incompressible. Thus the definition of compressible depends on the wavelength in the medium, so the terms must be used with some awareness of the circumstances.

If we restrict ourselves to the electrical case, then at this point it is helpful to think of the physics of the electron, a negatively charged particle that is a single point in space. Of course this is nonsense to consider an electron to be a point, but given that it is so small (it is so small it is difficult to estimate its size). This size is called the Lorentz radius, and is estimated to be $2.810^{-15}$ [m]. One could summarize the Lorentz radius as follows: *Here lie many unsolved problems in physics.* More specifically, at dimensions of the Lorentz radius, what exactly is the structure of the electron?

Ignoring the difficulties, if one integrates the charge density of the electron over the Lorentz radius and places the total charge at a single point, then one may make a grossly simplified model of the electron. For example the Electric Displacement (flux density) around a point charge ($D = \epsilon_o E$)

$$D = -\epsilon_o \nabla \phi(R) = -Q \nabla \left\{ \frac{1}{R} \right\} = -Q \delta(R).$$

This is a formula taught in many classic texts, but one should remember how crude a model of an electron it is. But it does describe the electric flux in an easily remembered form. However, computationally, it is less nice, due to the delta function. A limitation of this model is that the electron has a magnetic dipole moment (aka, *spin*), which a simple point charge does not capture. When placed in a magnetic field, due to the dipole, the electron will align itself with the field.

### Divergence and Gauss’ Law

Like the gradient, the divergence of a vector field may be defined as the surface integral of a *compressible* vector field, as a limit as the volume enclosed by the surface goes to zero. As for the case of the gradient, for this definition to make sense, the surface $S$ must be a closed, defining volume $V$. The difference is that the surface integral is over the normal component of the a vector field being operated on. Specifically (Greenberg, 1988, p. 762-763)

$$\nabla \cdot D = \lim_{|S| \to 0} \left\{ \frac{\int_S D \cdot \hat{n} \, d|S|}{|S|} \right\} = \rho(x, y, z). \quad (1.144)$$

As with the case of the gradient we have defined the surface as $S$, its area as $|S|$ and the volume within as $|S|$. Here $\hat{n}$ is a unit vector normal to the surface $S$. The limit as the volume goes to zero, defines the total flux across the surface. Thus the surface integral is a measure of the total flux $\perp$ to the surface. It is helpful to compare this formula with that for the gradient Eq. 1.143.
1.5. STREAM 3B: VECTOR CALCULUS (9 LECTURES)

**Gauss’ Law:** The above definitions resulted in a major breakthrough in vector calculus, and the first fundamental theorem of vector calculus, Gauss’ Law

The surface integral over the normal component of the flux (i.e., the total flux) is equal to the volume integral over the divergence of the flux.

For the electrical case this is equivalent to the observation that the total flux across the surface is equal to the net charge enclosed by the surface. Since the volume integral over charge density \( \rho(x, y, z) \) is total charge enclosed \( Q_{\text{enc}} \) [Col]

\[
Q_{\text{enc}} = \iiint_V \nabla \cdot D \, dV = \int_S D \cdot \hat{n} \, dA. \tag{1.145}
\]

It trivially follows that when \( Q_{\text{enc}} = 0 \). That is when the surface integral over the normal component of \( D(x) \), and the volume integral the charge density \( \nabla \cdot D = \rho(x) \), are both zero.

Hopefully this result makes common sense, and can be grasped intuitively.

**Integral definition of the Curl:** \( \nabla \times H = C \) [Amps]

As briefly summarized on page 94 (p. 94), the differential definition of the Curl maps \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). For example, the curl of the magnetic field strength \( H(x) \) is equal to the total current \( C \)

\[
\nabla \times H \equiv \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\partial_x & \partial_y & \partial_z \\
H_x & H_y & H_z
\end{vmatrix} = C.
\]

As we shall see in Sect. 1.5.7 (p. 119), the curl and the divergence are both key when writing out Maxwell’s equation four equations. Without a full understanding of these two differential operators \( (\nabla \cdot (), \nabla \times ()) \), there is no hope of understanding Maxwell’s basic result, typically viewed as the most critical elements of mathematical physics, and the starting point for Einstein’s relativity theories. Some will say that quantum mechanics fall outside the realm of MEs, but this at least open to debate, if not hotly debated.

The curl is a measure of the rotation of a vector field. If this were water, it would correspond to the angular momentum of the water, such as water going down the drain, as in a whirl pool, or with wind, a tornado. A spinning top is another an excellent example, given a spinning solid body. While a top (aka gyro-scope) will fall over if not spinning, once it is spinning, it can stability stand on its pointed tip. These structure are stable due to conservation of angular momentum: Once something is spinning, it will continue to spin.

By way of example, when \( H = -y\hat{i} + x\hat{j} + 0\hat{k} \) then \( \nabla \times H = 2\hat{k} \), thus has a constant rotation; when \( H = 0\hat{i} + 0\hat{j} + z^2\hat{k} \) then \( \nabla \times H = 0 \) has a curl of zero, thus has zero rotation thus is said to be irrotational. There are rules that precisely govern when a vector field is rotational versus irrotational, and compressible versus incompressible. These classes are dictated by Helmholtz’s Theorem, the third fundamental theorem of vector calculus (Eq. 1.148, p. 117).

**Curl and Stokes Law:** As in the cases of the gradient and divergence, the curl also may be written in integral form, allowing for the physical interpretation of its meaning.

Surface integral definition of \( \nabla \times H = C \) where the current \( C \) is \( \perp \) to the rotation plane of \( H \). As shown in Fig. 1.25, Stokes law states that the open surface integral over the normal component of the curl of the magnetic field strength \( \langle \hat{n} \nabla \times H \rangle \) [Amps/m²] is equal to the line integral \( \int_B H \cdot d\ell \) along the boundary \( B \). Namely, Stokes’ Law is

\[
I_{\text{enc}} = \iint_S (\nabla \times H) \cdot \hat{n} \, dA = \int_B H \cdot d\ell \quad \text{[Amps]}, \tag{1.146}
\]

\(^{75}\)As best I know, the three fundamental theorems of vector calculus, Gauss’ Law, the Stokes’ Law and Helmholtz’s theorem (aka decomposition) are not ordered.

\(^{76}\)aka, Gauss’s Law
CHAPTER 1. INTRODUCTION

\[ \nabla \times \mathbf{H} \equiv \lim_{|S| \to 0} \left\{ \frac{\oint_S \hat{n} \times \mathbf{H} \, d|S|}{||S||} \right\} \]

\[ I_{\text{enc}} = \iint_S (\nabla \times \mathbf{H}) \cdot \hat{n} \, dA = \oint_B \mathbf{H} \cdot dl \, [\text{Amps}] \]

Figure 1.25: The integral definition of the curl is similar to that of the divergence (Greenberg, 1988, p. 774), except that 1) the integration is over the plane tangent to the surface, defined as the cross product of \( \hat{n} \), the normal to the surface, crossed with a current density of \( \hat{n} \times \mathbf{H} \, [\text{Amps/m}^2] \), and 2) the surface is open, giving it a boundary \( B \), along the open edge. As with the divergence, which lead to Gauss' Law, this definition leads to a second fundamental theorem of vector calculus: Stokes' Law (aka the Curl theorem). There is a serious problem here, since an open area does not have any enclosed volume (Greenberg, 1988, p. 814). Greenberg (1988, p. 824) try to resolve this, but perhaps fails.

namely

The line integral of \( \mathbf{H} \) along the open surface’s boundary \( B \) is equal to the total current enclosed \( I_{\text{enc}} \) [Amps].

Summing it up: As mentioned earlier (Fig. 1.11, p. 45), the history of the discovery and proof of this theorem was both complex and interesting.

Since integration is a linear process (sums of smaller elements), one may tile, or tessellate the surface, breaking it up into a sum over smaller surfaces, and their boundaries, the sum over which is equal to the integral over the original boundary. This is an important concept, which leads to the proof of Stokes' Law.

The integral formulations of Gauss’ and Stokes’ Laws use \( \hat{n} \cdot \mathbf{D} \) and \( \mathbf{H} \times \hat{n} \) in the integrands. The key distinction between the two laws naturally follows from the properties of the scalar (\( \mathbf{A} \cdot \mathbf{B} \)) and vector (\( \mathbf{A} \times \mathbf{B} \)) products, as discussed in Sect. 1.3.4, p. 55, and detailed in Fig. 1.12, p. 57. To fully appreciate the differences between Gauss’ and Stokes’ Laws, these two types of vector products must be mastered.

1.5.6 Lec 39 Second-order operators: Terminology

Besides the above first order vector derivatives, second order combinations exist, the most common being the scalar Laplacian \( \nabla \cdot \nabla () = \nabla^2 () \) (Table 1.8, p. 94; Appendix 5.7, p. 176).

There are other important second-order combinations of \( \nabla \), enough that we need a memory aid to remember them. Thus I define mnemonics DoC, DoG, CoG CoC and GoD as follows:

1. DoG: Divergence of the Gradient (\( \nabla \cdot \nabla = \nabla^2 \)), i.e., Laplacian
2. DoC: Divergence of the Curl (\( \nabla \cdot \nabla \times \)),
3. CoG: Curl of the Gradient (\( \nabla \times \nabla \)),
4. CoC: Curl of the curl (\( \nabla \times \nabla \times \)), and
5. GoD: Gradient of the Divergence (\( \nabla^2 \)), or the vector Laplacian.

Two of these, DoC and CoG, are special because they are always zero,

\[ \nabla \times \nabla \phi = 0 ; \quad \nabla \cdot \nabla \times \mathbf{A} = 0 , \]
a property that makes them useful in proving the Fundamental theorem of vector calculus (Helmholtz decomposition) (Eq. 1.148, p. 117). A third key vector identity CoC may be expanded as
\[ \nabla \times \nabla \times A = \nabla (\nabla \cdot A) - \nabla^2 A, \quad (1.147) \]
thus defining the vector Laplacian (GoD) \( \nabla^2 \).

When using second order differential operators one must be careful with the order of operations, which can be subtle in a few cases. Most of this is common sense. For example, don’t operate on a scalar field with \( \nabla \times \), and don’t operate on a vector field with \( \nabla \). The vector Laplacian GoD must not be thought of as \( \nabla (\nabla \cdot A) \), rather it acts as the Laplacian on each vector component \( \nabla^2 (A) = \nabla^2 A_x \hat{i} + \nabla^2 A_y \hat{j} + \nabla^2 A_z \hat{k} \).

**Helmholtz’s decomposition**

We may now restate everything defined above in terms of two types of vector fields that decompose every analytic vector field. The **irrotational fields** is define as one that is “curl free.” An **incompressible field** is one that is “diverge free.” According to Helmholtz’s decomposition, every analytic vector field may be decomposed into independent rotational and a compressible components. Another name for **Helmholtz decomposition** is the **Fundamental theorem of vector calculus** (FTVC). Gauss’ and Stokes’ Laws, along with Helmholtz’s decomposition, form the three key fundamental theorems of vector calculus. Images of Helmholtz and Kirchhoff are provided in Fig. 1.26, p. .118.

The term **magnetic solenoidal field** is one that is generated by a solenoidal coil, is an excellent approximation, uniform inside the coil. As a result, the curl of a solenoidal field is, to a good approximation, zero. I recommend you know this term (it is widely used), but suggest the preferred term **irrotational**, Strictly speaking the term “solenoidal field” only applies to a magnetic field produced by a solenoid, thus the term is specific to that case.

**The decomposition of differentiable vector fields:** This theorem is easily stated (and proved), but less easily appreciated. A physical description facilitates: **Every vector field may be split into two independent parts: rotation and dilation.** We have seen this same idea appear in vector algebra, where the vector and cross products of two vectors are perpendicular (Fig. 1.12), p. 57. Also think of linear and angular momentum, which are independent in that they represent different ways of absorbing and delivering kinetic energy, leading to independent degrees of freedom. Thus the idea of the linear and rotational parts being independent, is a common theme, rooted in geometry. In the same sense, a vector field may be split into a dilation and rotational parts, which are independent (but can interact under certain conditions). An object with mass can be moving along a path, and independently be rotating. The two modes of motion define different types of kinetic energy, transnational and rotational. In some real sense, Helmholtz decomposition quantifies these degrees of freedom, one DoF for translation and three DoFs for rotation. Each eigenmode of vibration can be viewed as a DoF.

Helmholtz’s decomposition is expressed as the linear sum of a scalar potential \( \phi(x, y, z) \) (think voltage) and a vector potential (think magnetic vector potential). Specifically
\[ E(x, s) = -\nabla \phi(x, s) + \nabla \times A(x, s), \quad (1.148) \]
where \( \phi \) is the scalar and \( A \) is the vector potential, as a function of the Laplace frequency \( s \). Of course this decomposition is general (not limited to the electro-magnetic case). It applies to linear fluid vector fields, which includes most liquids and air. When the rotational and dilation become coupled, this relation must break down.\(^{77}\)

To show how this relationship splits the vector fields \( E \) into its two parts, we need DoC and CoG, the two key vector identities that are always zero for analytic fields: the **curl of the gradient** (CoG)
\[ \nabla \times \nabla \phi(x) = 0, \quad (1.149) \]
\(^{77}\)The nonlinear Navier–Stokes equations may be an example.
and the **divergence of the curl** (DoC)

\[ \nabla \cdot (\nabla \times A) = 0. \tag{1.150} \]

The above identities are easily verified by working out a few specific examples, based on the definitions of the three operators, gradient, divergence and curl, or in terms of the operator’s integral definitions, defined in Sect. 1.5.5, p. 112. The identities have a physical meaning, as stated above: every vector field may be split into its transnational and rotational parts. If \( \mathbf{E} \) is the electric field \([\text{V/m}]\), \( \phi \) is the voltage and \( \mathbf{A} \) is the induced rotational part, induced by a current. We shall explore this in our discussion of Maxwell’s equations in Sect. 1.5.7 and Chapter 5.

By applying these two identities to Helmholtz’s decomposition, we can better appreciate the theorems significance. It is a form of proof actually, once you have satisfied yourself that the vector identities are true. In fact one can work backward using a physical argument, that rotational momentum (rotational energy) is independent of the transnational momentum (transnational energy). Once these forces are made clear, the meaning of the vector operations all take on a very well defined meaning, and the mathematical constructions, centered around Helmholtz’s theorem, begins to provide some common-sense meaning. One could conclude that the physics is simply related to the geometry via the scalar and vector product.

Specifically, if we take the divergence of Eq. 1.148, and use the DoG

\[ \nabla \cdot \mathbf{E} = \nabla \cdot \{ -\nabla \phi + \nabla \times \mathbf{A} \}^0 = -\nabla \cdot \nabla \phi = -\nabla^2 \phi, \]

since the DoG zeros the vector potential \( \mathbf{A}(x, y, z) \). If instead we use the CoG

\[ \nabla \times \mathbf{E} = \nabla \times \{ -\nabla \phi^0 + \nabla \times \mathbf{A} \} = \nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \]

since the CoG zeros the scalar field \( \phi(x, y, z) \), and followed with the used of GoD.

**The four categories of fluid flow:** The following is a summary of the four cases for fluid flow, as summarized in Fig. 1.27:

---

\(^{78}\)Helmholtz was the first person to apply mathematics in modeling the eye and the ear (Helmholtz, 1863a).
1.1 Compressible and rotational fluid (general case): $\nabla \phi \neq 0$, $\nabla \times \mathbf{w} \neq 0$. This is the case of wave propagation if a medium where viscosity cannot be ignored, as in the case of acoustics close to the boundaries, where viscosity contributes losses (Batchelor, 1967).

1.2 Incompressible, but rotational fluid (Lubrication theory): $\mathbf{v} = \nabla \times \mathbf{w} \neq 0$, $\nabla \cdot \mathbf{v} = 0$, $\nabla^2 \phi = 0$. In this case the flow is dominated by the walls, the viscosity and heat transfer introduce shear. This is typical of lubrication theory.

2.1 Fluid compressible irrotational flow (acoustics): $\mathbf{v} = \nabla \phi$, $\nabla \times \mathbf{w} = 0$. Here losses (viscosity and thermal diffusion) are small (assumed to be zero). One may define a velocity potential $\psi$, the gradient of which gives the air particle velocity. For an irrotational fluid ($\nabla \times \mathbf{v} = 0$) (Greenberg, 1988, p. 826), thus $\mathbf{v} = -\nabla \phi$. This is the case of the conservative field, where $\int \mathbf{v} \cdot \mathbf{n} \, dR$ only depends on the end points, and $\oint \mathbf{v} \cdot \mathbf{n} \, dR = 0$. When a fluid may be treated as having no viscosity, it is typically assumed to be irrotational, since it is the viscosity that introduces the shear (Greenberg, 1988, p. 814). A fluid’s angular velocity is $\Omega = \frac{1}{2} \nabla \times \mathbf{v} = 0$, thus irrotational fluids have zero angular velocity ($\Omega = 0$).

2.2 Incompressible and irrotational fluid (statics): $\nabla \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$ thus $\mathbf{v} = \nabla \phi$ and $\nabla^2 \phi = 0$. An example of such a case is water in a small space at low frequencies, where the wavelength is long compared to the size of the container, the fluid may be treated as incompressible. When $\nabla \times \mathbf{v} = 0$, the effects of viscosity may be ignored, as it is the viscosity that creates the shear leading to rotation. This is the case of modeling the cochlea, where losses are ignored and the quasi-static limit is justified.

In summary, each of the cases is some sort of approximation that best applies in the low frequency limit. This is why it is called quasi-static, meaning low, but not zero frequency, where the wavelength is large compared with the dimensions (e.g., diameter).

**Thanksgiving Holiday 11/19–11/27 2016**

1.5.7 Lec 40 Maxwell’s Equations: The unification of electricity and magnetism

Once you have mastered the three basic vector operations, the gradient, divergence and curl, you are ready to appreciate Maxwell’s equations. Like the vector operations, these equations may be written in integral or vector form. A very important difference is, that with Maxwell’s equations, we are dealing with well defined physical quantities. The scalar and vector fields take on meaning, and units. Thus to understand these important equations, one must master the names of the four fields $\mathbf{E, H, B, D}$.

**Field strengths $\mathbf{E, H}$:** As summarized in Fig. 1.29 there are two field strengths, the electric $\mathbf{E}$, with units of [Volts/meter] and the magnetic $\mathbf{H}$ having units of [Amps/meter]. Their the ratio $|\mathbf{E}|/|\mathbf{H}|$ is in [ohms].

<table>
<thead>
<tr>
<th>Field: $\mathbf{v}(x,t)$</th>
<th>Compressible</th>
<th>Incompressible</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla \cdot \mathbf{v} \neq 0$</td>
<td>$\mathbf{v} = \nabla \phi + \nabla \times \mathbf{w}$</td>
<td>$\mathbf{v} = \nabla \times \mathbf{w}$</td>
</tr>
<tr>
<td>Rotational $\nabla \times \mathbf{v} \neq 0$</td>
<td>$\nabla^2 \mathbf{v} = \frac{1}{\rho} \frac{\partial \mathbf{v}}{\partial t}$</td>
<td>Lubrication theory</td>
</tr>
<tr>
<td>Acoustics $\mathbf{v} = \nabla \psi$</td>
<td>$\nabla^2 \psi = 0$</td>
<td>Boundary layers</td>
</tr>
<tr>
<td>Incompressible $\nabla \times \mathbf{v} = 0$</td>
<td>$\nabla^2 \phi = 0$</td>
<td>Laplace’s Eq. ($c \to \infty$)</td>
</tr>
</tbody>
</table>

Figure 1.27: Figure showing the four possible classifications of scalar and vector potential fields, rotational/irrotational, compressible/incompressible. Rotational fields are generated by on the vector potential (e.g., $\mathbf{A}(x,t)$), while compressible fields are generated by a the scalar potentials (e.g., voltage $\phi(x,t)$, (velocity $\mathbf{v}$, pressure $p(x,t)$), temperature $T(x,t)$).
Figure 1.28: A solenoid is a uniform coil of wire. When a current is passed through the wire, a uniform magnetic field intensity \( H \) is created. From a properties point of view, this coil is indistinguishable from a permanent bar magnet, having north and south poles. Depending on the direction of the current, one end of a finite solenoidal coil is the north pole of the magnet, and the other end is the south pole. The uniform field inside the coil is called solenoidal, a confusing synonym for irrotational. Figure taken from Wikipedia.

To understand the meaning of \( E \), if two conducting plates are place 1 [m] apart, with 1 [V] across them, the electric field is \( E = 1 \) [V/m]. If a charge (i.e., and electron) is placed in an electric field, it feels a force \( F = qE \), where \( q \) is the magnitude of the charge [Col].

To understand the meaning of \( H \), consider the solenoid made of wire, as shown in Fig. 1.28, which carries a current of 1 [Amp]. The magnetic field \( H \) inside such a solenoid is uniform and is pointed along the long axis, with a direction that depends on the polarity of the applied voltage (i.e., direction of the current in the wire).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Units</th>
<th>Maxwell’s Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>EF: Electric Field strength</td>
<td>[Volts/m]</td>
<td>( \nabla \times E = -\partial_t B )</td>
</tr>
<tr>
<td>( D = \epsilon_0 E )</td>
<td>ED: Electric Displacement (flux density)</td>
<td>[Col/m²]</td>
<td>( \nabla \cdot D = \rho )</td>
</tr>
<tr>
<td>( H )</td>
<td>MF: Magnetic Field strength</td>
<td>[Amps/m]</td>
<td>( \nabla \times H = \partial_t D )</td>
</tr>
<tr>
<td>( B = \mu_0 H )</td>
<td>MI: Magnetic Induction (flux density)</td>
<td>[Webers/m²]</td>
<td>( \nabla \cdot B = 0 )</td>
</tr>
</tbody>
</table>

Figure 1.29: The variables of Maxwell’s equations have names (e.g., \( EF, MI \)) and units (in square brackets [SI Units]). The units are required to gain a full understanding of each of the four variable and their corresponding equation. For example, \( EF \) has units [Volts/m]. When you integrate \( E \) from \( x = a, b \), you obtain the voltage difference between those two points. The speed of light in vacuo is \( c = 3 \times 10^8 \) [m/s], and the characteristic resistance of light \( r_o = 377 = 1/\mu_0\epsilon_o \) [Ω] (i.e., ohms). The dot over a vector is shorthand for the partial with respect to time (i.e., \( \dot{B} = \partial B/\partial t \)).

Flux: Flux is a flow, such as the mass flux of water flowing in a pipe [kg/s], driven by a force (pressure drop) across the ends of the pipe, or the heat flux in a thermal conductor, having a temperature drop across it (i.e., a window or a wall). The flux is the same as the flow, be it current, mass or heat. In Maxwell’s equations there are also two fluxes, the electric flux \( D \), and the magnetic flux \( B \). The flux density units for \( D \) [Amps/m²] (flux in [Amps]) and the magnetic flux is \( B \) [Tesla] (flux in [Webers]).

Maxwell’s equations

As shown in Fig. 1.29, Maxwell’s equations consist of two curl equations, operating on the field strengths \( EF \) and \( MF \), and two divergence equations, operating of the field fluxes \( ED \) and \( MI \). Stokes’ Law may be applied to the curl equations and Gauss’ Law may be used on the divergence equations. This should be logically obvious.

Exercise: Explain how Stokes’ Law may be applied to \( \nabla \times E = -\dot{B} \), and explain what it means.

Sol: Integrating the left side of equation \( EF \) over an open surface results in an voltage (emf) induced in the loop closing the boundary \( B \) of the surface

\[
\phi_{\text{induced}} = \int \int_S (\nabla \times E) \cdot \hat{n} \, dA = \oint_B E \cdot dl \quad [\text{Volts}].
\]

The emf (electromagnetic force) is the same as the Thévenin source voltage induced by the rate of change of the flux. Integrating the right side of equation \( EF \) over the same open surface results in the source of this induced voltage, the rate of change of the flux [Webers]

\[
\phi_{\text{induced}} = -\frac{\partial}{\partial t} \int \int_S B \cdot \hat{n} \, dA = \dot{\psi} \quad [\text{Webers/s}].
\]
The area integral on the left is in \( [W/m^2] \) resulting in the total flux crossing normal to the surface \( \psi \) [W].

If we apply Gauss’ Law to the divergence equations, we find the total flux crossing the closed surface.

**Exercise:** Apply Gauss’ Law to equation ED and explain what it means in physical terms.

**Sol:** The area of the normal component of \( \mathbf{D} \) is equal to the volume integral over the charge density, thus Gauss’ Law says that the total charge within the volume \( Q_{\text{enc}} \), found by integrating the charge density \( \rho(\mathbf{x}) \) over the volume \( \mathcal{V} \) is equal to the normal component of the flux \( \mathbf{D} \) crossing the surface \( \mathcal{S} \)

\[
Q_{\text{enc}} = \iiint_{\mathcal{V}} \nabla \cdot \mathbf{D} \, d\mathbf{V} = \iint_{\mathcal{S}} \mathbf{D} \cdot \mathbf{n} \, d\mathbf{A}.
\]

**Summary:** Maxwell’s four equations relate the field strengths to the flux densities. There are two types of variables, field strengths \((\mathbf{E}, \mathbf{H})\) and flux densities \((\mathbf{D}, \mathbf{B})\). There are two classes, electric \((\mathbf{E}, \mathbf{D})\) and magnetic \((\mathbf{H}, \mathbf{B})\). One might naturally view this as a 2x2 matrix, with rows being electric and magnetic strengths, and columns being electric and magnetic and flux densities, defining a total of four variables:

<table>
<thead>
<tr>
<th>Strength</th>
<th>Flux</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric</td>
<td>( \mathbf{E} ) [V/m] ( \mathbf{D} ) [Col/m^2]</td>
</tr>
<tr>
<td>Magnetic</td>
<td>( \mathbf{H} ) [A/m] ( \mathbf{B} ) [W/m^2]</td>
</tr>
</tbody>
</table>

Applying Stokes’ curl law to the forces induces a Thévenin voltage (emf) or Norton current source. Applying Gauss’ divergence law to the flows, gives the total charge enclosed. The magnetic charge always zero since \( \nabla \cdot \mathbf{B} = 0 \), because magnetic mono-poles do not exist. However magnetic dipoles do exist, as in the example of the electron which contains a magnetic dipole.

**The wave equation:** When Maxwell’s equations are combined the vector wave equation results. There is the famous story that Maxwell introduce the displacement current \( \dot{\mathbf{D}} \) to make the equations symmetric, allowing him to predict the formula for the speed of light. The displacement current was not verified by the experiments of Faraday and Ampere.

Taking the curl of Eq. EF [Maxwell’s Electric Field strength equation for the \( \nabla \times \mathbf{E} \) (Fig. 1.29)], we obtain in the electric vector wave equation

\[
\nabla \times \nabla \times \mathbf{E} = -\nabla \times \mathbf{B} = -\mu_o \nabla \times \mathbf{H} = -\mu_o \epsilon_o \dot{\mathbf{E}} \tag{1.151}
\]

Using the GoD identity (Eq. 1.147) then gives

\[
\nabla^2 \mathbf{E} - \frac{1}{\epsilon_o} \nabla(\nabla \cdot \mathbf{D}) = \mu_o \dot{\epsilon}_o \dot{\mathbf{E}} \tag{1.152}
\]

In a charge free region \( \nabla \cdot \mathbf{D} = 0 \). Since \( 1/c^2 = \mu_o \dot{\epsilon}_o \), the **vector wave equation** results:

\[
\nabla^2 \mathbf{E} = \frac{1}{c^2} \ddot{\mathbf{E}}. \tag{1.153}
\]

It is important to distinguish the vector Laplacian from the scalar Laplacian. As was discussed in Table 1.8 (p. 94), Recall the d’Alembert solution of the scalar wave equation (Eq. 1.68, p. 79)

\[
\mathbf{E}(\mathbf{x}, t) = f(\mathbf{x} - ct) + g(\mathbf{x} + ct),
\]

where \( f, g \) are arbitrary vector fields.
In a like manner one may derive the wave equation in terms of $H$

$$\nabla^2 H = \frac{1}{c^2} \ddot{H}. \quad (1.154)$$

This equation does not have the restriction that there is no free charge, because Eq. MI is zero. Thus both $E, H$ obey the wave equation (thus they are locked together in space-time if we assume no free charge). For an excellent treatment of this topic, read Sommerfeld (1952).

1.5.8 Lec 41 The Quasi-static approximation

There are a number of assumptions and approximations that result in special cases, many of which are classic. These manipulations are all done at the differential equation level, by making assumptions that change the basic equations that are to be solved. These approximations distinct from assumptions made while solving a specific problem.

A few and important examples include

1. *In vacuo* waves (free-space scalar wave equation)
2. Expressing the vector wave equation in terms of scalar and vector potentials
3. Quasi-statics
   (a) scalar wave equation
   (b) Kirchhoff’s low-frequency lumped approximation (LRC networks)
   (c) Transmission line equations (Telephone and Telegraph equations)

One of the very first insights into wave propagation was due to Huygens (c1640) (Fig. 1.17).

**Quasi-statics and it implications:** The term *quasi-statics* (Postulate P9, p. 74) is an approximation, used to reduce a a partial differential equations to a scalar (one-dimensional) equation (Sommerfeld, 1952). Quasi-statics is a way of reducing a three dimensional problem to a one–dimensional problem. So that it is not miss-applied, it is important to understand the nature of this approximation, which goes to the heart of transmission line theory. The quasi-static approximation states that the wavelength $\lambda$ is greater than the dimensions of the size of the object $\Delta$ (e.g., $\lambda \gg \Delta$). The best known example, Kirchhoff’s current and voltage laws, KCL and KVL, almost follow from Maxwell’s equations given the quasi-static approximation (Ramo et al., 1965). These laws state that the sum of the currents at a node must be zero (KCL) and the some of the voltages around a loop must be zero (KCL).

These well known laws the analogue of Newton’s laws of mechanics. The sum of the forces at a point is the analogue of the sum of the voltages. Voltage $\phi$ is the force potential, since the electric field $E = -\nabla \phi$. The sum of the currents is the analogue of the vector sum of velocities at a point is zero.

The acoustic wave equation describes how the scalar field pressure $p(x, t)$, the vector force density potential $(f(x, t) = -\nabla p(x, t) [N/m^2])$, propagates in three dimensions. (The net force is the integral of the pressure gradient over an area.) If the wave propagation is restricted to a pipe (e.g., organ pipe), or to a string (e.g., an guitar or lute), the transverse directions may be ignored, due to the quasi-static approximation. What needs to be modeled by the equations is the wave propagation along the pipe (string). Thus we may approximate the restricted three-dimensional wave by a one-dimensional wave.

However if we wish to be more precise about this reduction in geometry ($\mathbb{R}^2 \to \mathbb{R}$), we need to consider the quasi-static approximation, as it makes assumptions about what is happening in the other directions, and quantifies the effect ($\lambda \gg \Delta$). Taking the case of wave propagation in a tube, say the ear canal, there is the main wave direction, down the tube. But there is also wave propagation in the transverse direction, perpendicular to the direction of propagation. As shown in Table 3.1 (p. 164), the key statement of the quasi-static approximation is that the wavelength in the transverse direction
is much larger that the radius of the pipe. This is equivalent to saying that the radial wave reaches the walls and is reflected back, in a time that is small compared to the distance propagated down the pipe. Clearly the speed of sound down the pipe and in the transverse direction is the same if the medium is homogeneous (i.e., air or water). Thus the sound reaches the walls and is returned to the center line in a time that the axial wave traveled about 1 diameter along the pipe. So if the distance traveled is several diameters, the radial parts of the wave have time to come to equilibrium. So the question one must ask is, what are the conditions of such an equilibrium. The most satisfying answer to this is to look at the internal forces on the air, due to the gradients in the pressure.

The pressure $p(x, y, z, t)$ is a potential, thus its gradient is a force density $f(x, y, z, t) = -\nabla p(x, y, z, t)$. What this equation tells us is that as the pressure wave approaches that of a plane wave, the radial (transverse) forces go to zero. If the tube has a curvature, or a change in area, then there will be local forces that create radial flow. But after traveling a few diameters, these forces will come to equilibrium and the wave will return to a plane wave. The internal stress caused by a change is area must settle out very quickly. There is a very important caveat however: it is only at low frequencies that the plane wave can dominate. At frequencies such that the wavelength is very small compared to the diameter, the distance traveled between reflections is much greater than a few diameters. Fortunately the frequencies where this happens are so high that they play no role in frequencies that we care about. This effect is referred to as cross-modes which imply some sort of radial standing waves. In fact such modes exist in the ear canal, but on the eardrum where the speed of sound is much slower that that of air. Because of the slower speed, the ear drum has cross-modes, and these may be seen in the ear canal pressure. Yet they seem to have a negligible effect on our ability to hear sound with good fidelity. The point here is that the cross modes are present, but we call upon the quasi-static approximation as a justification for ignoring them, to get closer to the first-order physics.

Quasi-statics and Quantum Mechanics

It is important to understand the meaning of Planck’s constant $\hbar$, which appears in the relations of both photons (light “particles”) and electrons (mass particles). If we could obtain a handle on what exactly Planck’s constant means, we might have a better understanding of quantum mechanics, and physics in general. By cataloging the dispersion relations (the relation between the wavelength $\lambda(\nu)$ and the frequency $\nu$), for between electrons and photons, this may be attainable.

Basic relations from quantum mechanics for photons and electrons include:

1. Photons (mass=0, velocity = $c$)

   (a) $c = \lambda \nu$: The speed of light $c$ is the product of its wavelengths $\lambda$ times its frequency $\nu$. This relationship is only for mono-chromatic (single frequency) light.

   (b) The speed of light is
   $\hspace{1cm} c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 0.3 \times 10^6 \ [m/s]$

   (c) The characteristic resistance of light $r_0 = \sqrt{\mu_0 / \varepsilon_0} = |E|/|H| = 377 \ \text{[ohms]}$ is defined as the magnitude of the ratio of the electric $E$ and magnetic $H$ field, of a plane wave in-vacuo.

2. Electrons (mass = $m_e$, velocity $V = 0$):

   (a) $E_e = m_e c^2 \approx 0.91 \cdot 10^{-30} \cdot 0.3^2 \cdot 10^{12} = 8.14 \times 10^{-20} \ [J]$ is the electron rest energy (velocity $V = 0$) of every electron, of mass $m_e = 9.1 \times 10^{-31} \ [kgm]$, where $c$ is the speed of light.
(b) \( p = h/\lambda \): The momentum \( p \) of an electron is given by Planck’s constant \( h \) divided by the wavelength of an electron \( \lambda \). It follows that the bandwidth of the photon is given by

\[
\nu = \frac{E_e}{h}
\]

and the wavelength of an electron is

\[
\lambda_e = \frac{h}{p_e}
\]

One might reason that QM obeys the quasi-static (long wavelength) approximation. If we compare the velocity of the electron \( V \) to the speed of light \( c \), then we see that

\[
c = E/p \gg V = E/p = \frac{mV^2}{V}
\]

**Conjecture on photon energy:**

Photons are seen as quantized because they are common generator by atoms, which produce light-particles having the difference in two energy (quantum, or eigen-states) levels. The relation \( E = h\nu \) does not inherently depend on \( \nu \) being a fixed frequency. Planck’s constant \( h \) is the EM energy density over frequency, and \( E(\nu_o) \) is the integral over frequency

\[
E(\nu_o) = h \int_{-\nu_o}^{\nu_o} d\nu = 2h\nu_o.
\]

When the photon is generated by an atom \( \nu_o \) is quantized by the energy level difference that corresponds to the frequency (energy level difference) of the photon jump.

### 1.5.9 Lec 42: Review for Final Exam

#### Summary

Mathematics began as a simple way of keeping track of how many things there were. But eventually physics and mathematics evolved together as tools to help us navigate our environment, not just physically around the globe, but how to solve daily problems such as food, water and waste management, understand the solar system and the stars, defend ourselves, use tools of war, etc.

Based on the historical record of the abacus, one can infer that people precisely understood the concept of counting, addition, subtraction and multiplication (recursive addition).

There is some evidence that the abacus, a simple counting tool, formalizing the addition of very large numbers, was introduced to the Chinese by the Romans, where it was used for trade. However this working knowledge of arithmetic did not to show up in written number systems. The Roman numerals were not useful for doing calculations done on the abacus. The final answer would then be expressed in terms of the Roman number system.

According to the known written record, the number zero (null) had no written symbol until the time of Brahmagupta (628 CE). One should not assume the concept of zero was not understood simply because there was no symbol for it in the Roman Numeral system. Negative numbers and zero would be obvious when using the abacus. Numbers between the integers would be represented as rational numbers \( \mathbb{Q} \) since any number may be approximated with arbitrary accuracy with rations numbers.

Mathematics is the science of formalizing a repetitive method into a set of rules, and then generalizing it as much as possible. Generalizing the multiplication and division algorithm, to different types of numbers, becomes increasingly more complex as we move from integers to rational numbers, irrational numbers, real and complex numbers and ultimately, vectors and matrices. How do you multiply two vectors, or multiply and divide one matrix by another? Is it subtraction as in the case of two numbers? Multiplying and dividing polynomials (by long division) generalizes these operations even
Linear algebra is further important generalization, fallout from the fundamental theorem of algebra, and essential for solving the generalizations of the number systems.

Many of the concepts about numbers naturally evolved from music, where the length of a string (along with its tension) determined the pitch (Stillwell, 2010, pp. 11, 16, 153, 261). Cutting the string’s length by half increased the frequency by a factor of 2. One forth of the length increases the frequency by a factor of 4. One octave is a factor of 2 and two octaves a factor of 4 while a half octave is \( \sqrt{2} \). The musical scale was soon factored into rational parts. This scale almost worked, but did not generalize (sometimes known as the Pythagorean comma\(^7\)), resulting in today’s well tempered scale, which is based on 12 equal geometric steps along one octave, or 1/12 octave (\( \sqrt[12]{2} \approx 1.05946 \approx 18/17 = 1 + 1/17 \)).

But the concept of a factor was clear. Every number may be written as either a sum, or a product (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based on a unique factoring of every integer. At this same time (c5000 BCE), the solution of a second degree polynomial was understood, which lead to a generalization of factoring, since the polynomial, a sum of terms, may be written in factored form. If you think about this a bit, it is sort of an amazing idea, that needed to be discovered (Stillwell, 2010, p. ). This concept lead to an important string of theorems on factoring polynomials, and how to numerically describe physical quantities. Newton was one of the first to master these tools with his proof that the orbits of the planets are ellipses, not circles. This lead him to expanding functions in terms of their derivatives and power series. Could these sums be factored? The solution to this problem led to calculus.

So mathematics, a product of the human mind, is a highly successful attempt to explain the physical world. All aspects of our lives were impacted by these tools. Mathematical knowledge is power. It allows one to think about complex problems in increasingly sophisticated ways. An equation is a mathematical sentence, expressing deep knowledge. Witnessed \( E = mc^2 \) and \( \nabla^2 \psi = \ddot{\psi} \).

Reading List: The above concepts come straight from mathematical physics, as developed in the 17\(^{th}\)–19\(^{th}\) centuries. Much of this was first developed in acoustics by Helmholtz, Stokes and Rayleigh, following in Green’s footsteps, as described by Lord Rayleigh (1896). When it comes to fully appreciating Green’s theorem and reciprocity, I have found Rayleigh (1896) to be a key reference. If you wish to repeat my reading experience, start with Brillouin (1953, 1960), followed by Sommerfeld (1952); Pipes (1958). Second tier reading contains many items Morse (1948); Sommerfeld (1949); Morse and Feshbach (1953); Ramo et al. (1965); Feynman (1970); Boas (1987). A third tier might include Helmholtz (1863a); Fry (1928); Lamb (1932); Bode (1945); Montgomery et al. (1948); Beranek (1954); Fagen (1975); Lighthill (1978); Hunt (1952); Olson (1947). It would be a mistake to ignore other massive physics writings by stalwart authors, J.C. Slater\(^8\) and Landau and Lifshitz,\(^9\) and their impressive series of mathematical physics books.

You must enter at a level that allows you to understand. Successful reading of these books critically depends on what you already know. A rudimentary (high school) level of math comprehension must be mastered first. Read in the order that helps you best understand the material.

Without a proper math vocabulary, mastery is hopeless. I suspect that one semester of college math can bring you up to speed. This book is my attempt to present this level of understanding.

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\(^7\)https://en.wikipedia.org/wiki/Pythagorean_comma
\(^8\)https://en.wikipedia.org/wiki/John_C._Slater
\(^9\)https://www.amazon.com/Mechanics-Third-Course-Theoretical-Physics/dp/0750628960
Appendix A

Notation

A.1 Number systems

The notation used in this book is defined in this appendix so that it may be quickly accessed. Where the definition is sketchy, page numbers are provided where these concepts are fully explained, along with many other important and useful definitions. For example a discussion of \( \mathbb{N} \) may be found on page 22. Math symbols such as \( \mathbb{N} \) may be found at the top of the index, since they are difficult to alphabetize.

A.1.1 Units

Strangely, or not, classical mathematics (as taught today in schools) does not contain the concept of units. It seems units have been abstracted away. This makes mathematics distinct from physics, where almost everything has units. Presumably this makes mathematics more general (i.e., abstract). But for the engineering mind, this is not ideal, as it necessarily means that important physical meaning has been surgically removed, by design. We shall stick to SI units when ever possible. Spatial coordinates are quoted in meters \([m]\), and time in seconds \([s]\). Angles in degrees have no units, whereas radians have units of inverse-seconds \([s^{-1}]\).

A.1.2 Symbols and functions

We use \( \ln \) as the log function base \( e \), \( \log \) as base 2, and \( \pi_k \) to indicate the \( k \)th prime (e.g., \( \pi_1 = 2, \pi_2 = 3 \)).

When working with Fourier \( F \) and Laplace \( L \) transforms, lower case symbols are in the time domain while upper case indicates the frequency domain, as \( f(t) \leftrightarrow F(\omega) \). An important exception are Maxwell’s equations, because they are so widely used as upper case bold letters (e.g., \( \mathbf{E}(\mathbf{x}, \omega) \)). It seems logical to change this to conform to lower case, with \( \mathbf{e}(\mathbf{x}, t) \leftrightarrow \mathbf{E}(\mathbf{x}, \omega) \) as the preferred notation.

A.1.3 Special symbols common to mathematical:

There are many pre-defined symbols in mathematics, too many to summarize here. We shall only use a small subset, defined here.

- A set is a collection of objects that have a common property. A set is defined using braces. For example, if set \( P = \{a, b, c\} \) such that \( a^2 + b^2 = c^2 \), then members of \( P \) obey the Pythagorean theorem. Thus we could say that \( 1, 1, \sqrt{2} \in P \). In this last case the three numbers \( 1, 1, \sqrt{2} \) are not a set, rather than are in (members of) the Pythagorean set \( P \).

- Number sets: \( \mathbb{N}, \mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{I}, \mathbb{R}, \mathbb{C} \) are briefly discussed below, and in more detail in Section 1.2.1 on p. 22.

\(^1\text{https://en.wikipedia.org/wiki/List_of_mathematical_symbols_by_subject#Definition_symbols}\)
• One can define sets of sets and subsets of sets, and this is prone (in my experience) to error. For example, what is the difference between the number 0 and the null set \( \varnothing = \{0\} \)? Is \( 0 \in \varnothing \)? Ask a mathematician. It is not a useful construction in the world of engineering.

• They symbol \( \perp \) is used in many different ways to indicate two things are perpendicular, orthogonal, or in disjoint sets. In set theory \( A \perp B \) is equivalent to \( A \cap B = \varnothing \). If two vectors \( \mathbf{E}, \mathbf{H} \) are perpendicular \( \mathbf{E} \perp \mathbf{H} \), then their inner product \( \mathbf{E} \cdot \mathbf{H} = 0 \) is zero. One must infer the meaning of \( \perp \) from its usage (the context).

A.1.4 Greek letters

The Greek letters used in this text include (at least) \( \alpha, \beta, \gamma, \delta, \kappa, \xi, \omega, \phi, \psi, \zeta \), and upper-case \( \Gamma, \Xi, \Phi, \Psi, \Delta, \Omega \). Unfortunately some common Greek letters have no upper-case symbol, such as \( \alpha, \zeta \).

Many of these are pre-associated in engineering and physics with a specific physical meaning. For example, \( \omega \) [rad] is the radian frequency \( 2\pi f \), \( \rho \) [kgm/\( m^3 \)] is commonly the density. \( \phi, \psi \) are commonly used to indicate angles of a triangle, and \( \zeta(s) \) is the Riemann zeta function. Many of these are so well established it makes no sense to define new terms, so we will adopt these common terms (and define them).

Likely you do not know all of these Greek letters, commonly used in mathematics. Some of them are pronounced in strange ways. The symbol \( \xi \) is pronounced “see,” \( \zeta \) is “zeta,” \( \beta \) is “beta,” and \( \chi \) is “kie” (rhymes with pie and sky). I will assume you know how to pronounce the others, which are more phonetic in English. One advantage of learning \LaTeX{} is that all of these math symbols are built in, and thus more easily learned, once you have adopted this powerful open-source math-oriented word-processing system (e.g., used to write this book).

Table A.1: Double-bold notation for the types of numbers. (#) is a page number. Symbol with an exponent denote the dimensionality. Thus \( \mathbb{R}^2 \) represents the real plane. An exponent of 0 denotes point, e.g., \( j \in \mathbb{C}^0 \).

<table>
<thead>
<tr>
<th>Symbol (p. #)</th>
<th>Genus</th>
<th>Examples</th>
<th>Counter Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{N} ) (22)</td>
<td>Counting</td>
<td>1, 2, 17, 3, 10(^{20})</td>
<td>0, -10, 5j</td>
</tr>
<tr>
<td>( \mathbb{P} ) (22)</td>
<td>Prime</td>
<td>2, 17, 3, 10(^{20})</td>
<td>0, 1, 4, 3(^2), 12, -5</td>
</tr>
<tr>
<td>( \mathbb{Z} ) (23)</td>
<td>Integer</td>
<td>-1, 0, 17, 5j, -10(^{20})</td>
<td>1/2, ( \pi ), ( \sqrt{5} )</td>
</tr>
<tr>
<td>( \mathbb{Q} ) (23)</td>
<td>Rational</td>
<td>2/1, 3/2, 1.5, 1.14</td>
<td>( \sqrt{2}, 3^{-1/3}, \pi )</td>
</tr>
<tr>
<td>( \mathbb{F} ) (23)</td>
<td>Fractional</td>
<td>1/2, 7/22</td>
<td>2/1, 1/( \sqrt{2} )</td>
</tr>
<tr>
<td>( \mathbb{I} ) (23)</td>
<td>Irrational</td>
<td>( \sqrt{2}, 3^{-1/3}, \pi, e )</td>
<td>Vectors</td>
</tr>
<tr>
<td>( \mathbb{R} ) (23)</td>
<td>Reals</td>
<td>( \sqrt{2}, 3^{-1/3}, \pi )</td>
<td>( 2\pi j )</td>
</tr>
<tr>
<td>( \mathbb{C} ) (159)</td>
<td>Complex</td>
<td>1, ( \sqrt{2}j, 3^{-j/3}, \pi j )</td>
<td>Vectors</td>
</tr>
</tbody>
</table>

A.1.5 Double-Bold notation

Table A.1 indicates the symbol followed by a page number indication where it is discussed, and the Genus (class) of the number type. For example, \( \mathbb{N} > 0 \) indicates the infinite set of counting numbers \{1, 2, 3, \( \cdots \)\}, not including zero. Starting from any counting number, you get the next one by adding 1. Counting numbers are also known as the Cardinal numbers.

We say that a number is in the set with the notation \( 3 \in \mathbb{N} \in \mathbb{R} \), which is read as “3 is in the set of counting numbers, which in turn is in the set of real numbers,” or in vernacular language “3 is a real counting number.”

Prime numbers (\( \mathbb{P} \in \mathbb{N} > 1 \)) are taken from the counting numbers, do not include 1, and cannot be factored.
A.2. COMPLEX VECTORS AND THE IMPEDANCE MATRIX

The signed integers $\mathbb{Z}$ include 0 and negative integers. Rational numbers $\mathbb{Q}$ are historically defined to include $\mathbb{Z}$, a somewhat inconvenient definition, since the more interesting class are the fractions $\mathbb{F}$, a subset of rationals $\mathbb{F} \in \mathbb{Q}$ that exclude the integers (i.e., $\mathbb{F} \perp \mathbb{Z}$). This is a useful definition because the rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are formed from the union of integers and fractions.

Irrational numbers $\mathbb{I}$ are very special: They are formed by taking a limit of fractionals, as the numerator and denominator $\to \infty$, and approach a limit point. It follows that irrational numbers must be approximated by fractionals.

The reals ($\mathbb{R}$) include complex numbers ($\mathbb{C}$) having a zero imaginary part (i.e., $\mathbb{R} \subset \mathbb{C}$).

The size of a set is denoted by taking the absolute value (e.g., $|\mathbb{N}|$). Normally in mathematics this symbol indicates the cardinality, so we are defining it differently from the standard notation.

Classification of numbers: From the above definitions there exists a natural heretical structure of numbers:

\begin{align*}
\mathbb{P} & \in \mathbb{N}, & \mathbb{Z} & : \{\mathbb{N}, 0, -\mathbb{N}\}, & \mathbb{F} & \perp \mathbb{Z}, & \mathbb{Q} & : \mathbb{Z} \cup \mathbb{F}, & \mathbb{R} & : \mathbb{Q} \cup \mathbb{I} \subset \mathbb{C} \\
1. & \text{The primes are a subset of the counting numbers: } \mathbb{P} \subset \mathbb{N}. \\
2. & \text{The signed integers } \mathbb{Z} \text{ are composed of } \pm \mathbb{N} \text{ and } 0, \text{ thus } \mathbb{N} \subset \mathbb{Z}. \\
3. & \text{The fractionals } \mathbb{F} \text{ do not include of the signed integers } \mathbb{Z}. \\
4. & \text{The rationals } \mathbb{Q} = \mathbb{Z} \cup \mathbb{F} \text{ are the union of the signed integers and fractionals} \\
5. & \text{Irrational numbers } \mathbb{I} \text{ have the special properties } \mathbb{I} \perp \mathbb{Q}. \\
6. & \text{The reals } \mathbb{R} : \mathbb{Q}, \mathbb{I} \text{ are the union of rationals and irrationals } \mathbb{I} \\
7. & \text{Reals } \mathbb{R} \text{ may be defined as a subset of those complex numbers } \mathbb{C} \text{ having zero imaginary part.}
\end{align*}

A.2 Complex vectors and the impedance matrix

Vectors as columns of ordered sets of scalars. When we write them out in text, we typically use row notation, with the transpose symbol:

\[ [a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \]

This is strictly to save space on the page. The above example is said to be a 3-dimensional vector, because it has three components. With rare exceptions, vectors are rows.

The transpose of a vector (a “so-called” row-vector) are typically used as weights. A vector dot product is defined between weights and vectors, resulting in a scalar. When the elements are complex ($\in \mathbb{C}$), the transpose is defined as the complex conjugate of the elements. In such complex cases the transpose conjugate is denoted with a $^\dagger$ rather than $^T$. For example

\[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = 1 + 2 + 3 = 6. \]

Vectors are also frequency written using a bold font (e.g., $a^T = [2j \quad -3j \quad 1] \in \mathbb{C}$.) For this case the dot product is

\[ a \cdot b = a^\dagger b = a_1^* b_1 + a_2^* b_2 + a_3^* b_3. \]

Such a construction is useful when $a$ and $b$ are related by an impedance matrix

\[ V(s) = Z(s)I(s). \]
For example, the impedance of a mass is $m s$ and a capacitor is $1/sC$. When given a system of equations (a mechanical or electrical circuit) one may define a matrix impedance.

**GIVE MORE EXAMPLES**

**Complex power:** In this special case, the complex power $P(s) \in \mathbb{R}(s)$ is defined, in the complex frequency domain ($s$), as

$$P(s) = I^1(s)V(s) = I^1(s)Z(s)I(s).$$

The complex power must be positive, with $\mathcal{L}^{-1} p(t) \leftrightarrow P$.

**GIVE EXAMPLES**

**Norm of a vector:** The dot product of a vector with itself is called the norm of $a$, designated as

$$||a|| = \sqrt{a^\dagger a} \geq 0$$

which is always non-negative.

**A.2.1 Vectors in $\mathbb{R}^3$**

The case of three-dimensions is special, allowing definitions that are not defined in more dimensions. A vector in $\mathbb{R}^3$ labels the point having the coordinates of that vector.

**Dot product in $\mathbb{R}^3$:** The dot $B \cdot C = ||B|| ||C|| \cos(\theta)$, and $\cos(\theta)$ is called the direction cosine between $B$ and $C$.

**Norm of a vector in $\mathbb{R}^3$:** The norm of a vector is the dot product of it with itself

$$||A|| = \sqrt{A \cdot A}$$

**Euclidean distance between two points in $\mathbb{R}^3$:** The dot product of the difference between two vectors $(A - B) \cdot (A - B)$ is the Euclidean distance between the points they define

$$||A - B|| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$  

**Cross product:** and cross product $A \times B = ||A|| ||B|| \sin(\theta)$ are defined between the two vectors $A$ and $B$.

The cross product gives the area of the trapezoid (diamond) outlined by the two vectors (Fig. 1.12).

**The triple product:** This is defined between three vectors as

$$A \cdot (B \times C) = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

also defined in Fig. 1.12. This may be indicated without the use of parentheses, since there can be no other meaningful interpretation. However for rigor, parentheses should be used.
A.3 Matrices

Unfortunately when working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements of the column-vector of weights multiplied from the right. For example

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2M} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_M
\end{bmatrix}
= w_1
\begin{bmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{N1}
\end{bmatrix}
+ w_2
\begin{bmatrix}
a_{12} \\
a_{22} \\
\vdots \\
a_{N2}
\end{bmatrix}
\ldots w_M
\begin{bmatrix}
a_{1M} \\
a_{2M} \\
\vdots \\
a_{NM}
\end{bmatrix},
\]

where the weights are \([w_1, w_2, \ldots, w_M]^T\).

Another way to view the matrix is a set of row vectors of weights, each of which are applied to the vector \([w_1, w_2, \ldots, W_M]^T\).

The determinant of a matrix is denoted either as \(\text{det } A\) or simply \(|A|\), as in the absolute value. The inverse of a square matrix is \(A^{-1}\) or \(\text{inv } A\). If \(|A| = 0\), the inverse does not exist. \(AA^{-1} = A^{-1}A\).

Matlab’s notional convention for a row-vector is \([a, b, c]\) and a column-vector is \([a; b; c]\). A prime on a vector takes the complex conjugate transpose. To suppress the conjugation, place a period before the prime. The : argument converts the array into a column vector, without conjugation. A tacit notation in Matlab is that vectors are columns and the index to a vector is a row vector. Matlab defines the notation 1 : 4 as the “row-vector” \([1, 2, 3, 4]\), which is unfortunate as it leads users to assume that the default vector is a row. This can lead to serious confusion later, as Matlab’s default vector is a column. I have not found the above convention explicitly stated, and it took me years to figure this out for myself.

When writing a complex number we shall adopt \(1_j\) to indicate \(\sqrt{-1}\). Matlab/Octave prefer this as well, as its explicit.

Units are SI; Angles in degrees \([\text{deg}]\) unless otherwise noted. The units for \(\pi\) are always in radians \([\text{rad}]\). Ex: \(\sin(\pi), e^{j\theta} e^{jn/2}\).

A.4 Periodic functions

Fourier series tells us that periodic functions are discrete in frequency, with frequencies given by \(nT_x\), where \(T_x\) is the sample period \((T_x = 1/2F_{\text{max}} \text{ and } F_{\text{min}} = F_{\text{max}}/N\text{F})\).

This concept is captured by the Fourier series, which is a frequency expansion of a periodic function. This concept is quite general. Periodic in frequency implies discrete in time. Periodic and discrete in time requires periodic and discrete in frequency (the case of the DFT). The modulo function \(mmodn\) is periodic with period \(n\).

A periodic function may be conveniently indicated using double-parentheses notation. This is sometimes known as modular arithmetic. For example

\[f((t))_T = f(t) = f(t \pm kT),\]

is periodic on \(t, T \in \mathbb{R}\) with a period of \(T\) and \(k \in \mathbb{Z}\). This notation is useful when dealing with Fourier series of periodic functions.

When a discrete valued (e.g., time) sequence is periodic we use square brackets

\[f[[n]]_N = f[n] = f[n \pm kN],\]

with \(n, k, N \in \mathbb{Z}\) and period \(N\). This notation will be used with discrete-time signals that are periodic, such as the case of the DFT.
A.5 Differential equations vs. Polynomials

A polynomial has degree \( N \) defined by the largest power. A quadratic equation is degree 2, and a cubic has degree 3. We shall indicate a polynomial by the notation

\[ P_N(z) = z^N + a_{N-1}z^{N-1} \cdots a_0. \]

Always normalize the polynomial so that \( a_N = 1 \). This will not change the roots, defined by Eq. 1.19 (p. 44). The coefficient on \( z^{N-1} \) is always the sum of the roots \( z_n (a_{N-1} = \Sigma z_n) \), and \( a_0 \) is always their product \( (a_0 = \prod z_n) \).

Differential equations have order (polynomials have degree). If a second order differential equation is Laplace transformed (Lec. 1.3.10, p. 70), one is left with a degree 2 polynomial. For example:

\begin{align*}
\frac{d^2}{dt^2}y(t) + b \frac{dy}{dt}y(t) + cy(t) &= \alpha \left( \frac{d}{dt}x(t) + \beta x(t) \right) \leftrightarrow \quad (A.1) \\
(s^2 + bs + c)Y(s) &= \alpha(s + \beta)X(s), \quad (A.2) \\
\frac{Y(s)}{X(s)} &= \alpha \frac{s + \beta}{s^2 + bs + c} \equiv H(s) \leftrightarrow h(t). & (A.3)
\end{align*}

Using the same argument as for polynomials, the lead coefficient must always be 1. The coefficient \( \alpha \in \mathbb{R} \) is called the gain. The complex variable \( s \) is the Laplace frequency. Discuss half-derivatives, etc.

The ratio of the output \( Y(s) \) over the input \( X(s) \) is called the system transfer function \( H(s) \). When \( H(s) \) is the ratio of two polynomials in \( s \), the transfer function is said to be bilinear, since it is linear in both the input and output. The roots of the numerator are called the zeros and those of the denominator, the poles. The inverse Laplace transform of the transfer function is called the system impulse response, which describes the system’s output signal \( y(t) \) for any given input signal \( x(t) \), via convolution (i.e., \( y(t) = h(t) \ast x(t) \)).
Appendix B

Linear algebra of 2x2 matrices

B.1 Notation

Definitions:

1. **Scalar**: A number, e.g. \{a, b, c, \alpha, \beta, \cdots\} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}\}

2. **Vector**: A quantity having direction as well as magnitude, often denoted by a bold-face letter with an arrow, \( \mathbf{x} \). In matrix notation, this is typically represented as a single row \([x_1, x_2, x_3, \ldots]\) or single column \([x_1, x_2, x_3, \ldots]^T\) (where \(T\) indicates the transpose). In this class we will typically use column vectors. The vector may also be written out using unit vector notation to indicate direction. For example: \( \mathbf{x}_{3 \times 1} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k} = [x_1, x_2, x_3]^T \), where \(\hat{i}, \hat{j}, \hat{k}\) are unit vectors in the \(x, y, z\) Cartesian directions (here the vector’s subscript \(3 \times 1\) indicates its dimensions). The type of notation used may depend on the engineering problem you are solving.

3. **Matrix**: \(A = \begin{bmatrix} a_1, a_2, a_3, \cdots, a_M \end{bmatrix}_{N \times M} = \{a_{n,m}\}_{N \times M}\), can be a non-square matrix if the number of elements in each of the vectors (\(N\)) is not equal to the number of vectors (\(M\)). When \(M = N\), the matrix is square. It may be inverted if its determinant \(|A| = \prod \lambda_k \neq 0\) (where \(\lambda_k\) are the eigenvalues).

We shall only work with \(2 \times 2\) and \(3 \times 3\) square matrices throughout this course.

4. **Linear system of equations**: \(Ax = b\) where \(x\) and \(b\) are vectors and matrix \(A\) is a square.

   (a) **Inverse**: The solution of this system of equations may be found by finding the inverse \(x = A^{-1}b\)

   (b) **Equivalence**: If two systems of equations \(A_0x = b_0\) and \(A_1x = b_1\) have the same solution (i.e., \(x = A_0^{-1}b_0 = A_1^{-1}b_1\)), they are said to be equivalent.

   (c) **Augmented matrix**: The first type of augmented matrix is defined by combining the matrix with the right-hand-side. For example, given the linear system of equations \(Ax = y\)

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} =
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix},
\]

then the augmented matrix is

\[A|y = \begin{bmatrix} a & b & y_1 \\
                        c & d & y_2 \end{bmatrix}\]

A second type of augmented matrix may be used for finding the inverse of a matrix (rather than solving a specific instance of linear equations \(Ax = b\)). In this case the augmented matrix is

\[A|I = \begin{bmatrix} a & b & 1 & 0 \\
                        c & d & 0 & 1 \end{bmatrix}\]
Performing Gaussian elimination on this matrix, until the left side becomes the identity matrix, yields \( A^{-1} \). This is because multiplying both sides by \( A^{-1} \) gives \( A^{-1}AA^{-1}I = I \).

5. **Permutation matrix** \((P)\): A matrix that is equivalent to the identity matrix, but with scrambled rows (or columns). Such a matrix has the properties \( \det(P) = \pm 1 \) and \( P^2 = I \). For the 2x2 case, there is only one permutation matrix:

\[
P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

A permutation matrix \( P \) swaps rows or columns of the matrix it operates on. For example, in the 2x2 case, pre-multiplication swaps the rows

\[
PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} a & \alpha \\ b & \beta \end{bmatrix},
\]

whereas post-multiplication swaps the columns

\[
AP = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ \beta & \alpha \end{bmatrix}
\]

For the 3x3 case there are \( 3 \cdot 2 = 6 \) such matrices, including the original 3x3 identity matrix (swap a row with the other 2, then swap the remaining two rows).

6. **Gaussian elimination (GE) operations** \( G_k \): There are 3 types of elementary row operations, which may be performed without fundamentally altering a system of equations (e.g. the resulting system of equations is equivalent). These operations are (1) swap rows (e.g. using a permutation matrix), (2) scale rows, or (3) perform addition/subtraction of two scaled rows. All such operations can be performed using matrices.

For lack of a better term, we’ll describe these as ‘Gaussian elimination’ or ‘GE’ matrices.\(^1\) We will categorize any matrix that performs only elementary row operations (but any number of them) as a ‘GE’ matrix. Therefore, cascade of GE matrices is also a GE matrix.

Consider the GE matrix

\[
G = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}
\]

(a) Pre-multiplication scales and adds the rows

\[
GA = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} a - \alpha & b - \beta \\ a & b \end{bmatrix}
\]

The result is a Gaussian elimination operation.

(b) Post-multiplication adds and scales columns.

\[
AG = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a - b & b \\ \alpha - \beta & \beta \end{bmatrix}
\]

Here the second column is subtracted from the first, and placed in the first. The second column is untouched. **This operation is not a Gaussian elimination.** Therefore, to

\(^1\)The term ‘elementary matrix’ may also be used to refer to a matrix that performs an elementary row operation. Typically, each elementary matrix differs from the identity matrix by one single row operation. A cascade of elementary matrices could be used to perform Gaussian elimination.
put Gaussian elimination operations in matrix form, we form a cascade of pre-multiply matrices.

Here \( \det(G) = 1, G^2 = I \), which won’t always be true if we scale by a number greater than 1. For instance, if \( G = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} \) (scale and add), then we have \( \det(G) = 1, G^n = \begin{bmatrix} 1 \\ n \cdot m & 1 \end{bmatrix} \).

### B.1.1 Gaussian elimination exercises

Find the solution to the following 3x3 matrix equation \( Ax = b \) by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Matlab.

\[
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
9 \\
8
\end{bmatrix}
\]

1. Show (i.e., verify) that the first GE matrix \( G_1 \), which zeros out all entries in the first column, is given by

\[
G_1 = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\]

Identify the elementary row operations that this matrix performs. Sol: Operate with GE matrix on \( A \)

\[
G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & -1 \\
0 & -2 & 4 \\
0 & -2 & 5
\end{bmatrix}
\]

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row.

2. Find a second GE matrix, \( G_2 \), to put \( G_1 A \) in upper triangular form. Identify the elementary row operations that this matrix performs. Sol:

\[
G_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

which scales the second row by -1 and adds it to the third row. Thus we have

\[
G_2G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & -1 \\
0 & -2 & 4 \\
0 & 0 & 1
\end{bmatrix}
\]

3. Find a third GE matrix, \( G_3 \), which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs. Sol:

\[
G_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which scales the second row by -1/2. Thus we have

\[
G_3G_2G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}
\]
We shall now apply Gaussian elimination to find the solution $x$. 

### B.2 Inverse of the $2\times2$ matrix

#### B.2.1 Derivation of the inverse of a $2\times2$ matrix

Summary: This is a lot of algebra, that is why it is essential you memorize the formula for the inverse.

1. Step 1: normalize the first column to 1.
2. Step 2: subtract the top equation from the lower.
3. Step 3: express result in terms of the determinate $\Delta = ad - bc$.

$$
\begin{bmatrix}
1 & \frac{b}{c} \\
1 & \frac{d}{c}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{c} y_1 \\
\frac{1}{c} y_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a} y_1 \\
\frac{1}{a} y_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a} y_1 \\
\frac{1}{a} y_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a} y_1 \\
\frac{1}{a} y_2
\end{bmatrix}
$$

4. Step 4: These steps give the solution for $x_2$: $x_2 = -\frac{c}{\Delta} y_1 + \frac{b}{\Delta} y_2$.

5. Step 5: Finally the top equation is solved for $x_1$: $x_1 = \frac{1}{a} y_1 - \frac{b}{a} x_2 \Rightarrow x_1 = \left[\frac{1}{a} + \frac{b}{a^2}\Delta\right] y_1 - \frac{b}{a^2} y_2$.

In matrix format, in terms of the determinate $\Delta = ab - cd$ becomes:

$$
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a} y_1 \\
\frac{1}{a} y_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a} y_1 \\
\frac{1}{a} y_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a} y_1 \\
\frac{1}{a} y_2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a} y_1 \\
\frac{1}{a} y_2
\end{bmatrix}
$$

**Summary**: This is a lot of algebra, that is why it is essential you memorize the formula for the inverse.
Appendix C

Eigenvector analysis

Here we show how to compute the eigenvalues and eigenvectors for the 2x2 Pell matrix

\[ A = \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix}. \]

The analysis applies to any matrix, but since we are concentrated on Pell’s equation, we shall use the Pell matrix, for \( N = 2 \). By using a specific matrix we can check all the equations below with Matlab, which I advise you to do.

The Matlab command \([E,D]=\text{eig}(A)\) returns the eigenvector matrix \( E \)

\[ E = [e_+, e_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774 \end{bmatrix}, \]

and the eigenvalue matrix \( \Lambda \) (Matlab’s \( D \))

\[ \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{bmatrix}. \]

The factor \( \sqrt{3} \) on \( E \) normalizes each eigenvector to 1 (i.e., The Matlab command \( \text{norm}([\sqrt{2}, 1]) \) gives \( \sqrt{3} \)).

In the following discussion we show how to determine \( E \) and \( D \) (i.e, \( \Lambda \)), given \( A \).

Calculating the eigenvalue matrix (\( \Lambda \)): The matrix equation for \( E \) is

\[ AE = E\Lambda. \]  \hfill (C.1)

Pre-multiplying by \( E^{-1} \) diagonalizes \( A \), given the eigenvalue matrix (\( D \) in Matlab)

\[ \Lambda = E^{-1}AE. \]  \hfill (C.2)

Post-multiplying by \( E^{-1} \) recovers \( A \)

\[ A = E\Lambda E^{-1}. \]  \hfill (C.3)

Matrix power formula: This last relation is the entire point of the eigenvector analysis, since it shows that any power of \( A \) may be computed from powers of the eigen values. Specifically

\[ A^n = E\Lambda^n E^{-1}. \]  \hfill (C.4)

For example, \( A^2 = AA = E\Lambda (E^{-1}E) \Lambda E^{-1} = E\Lambda^2 E^{-1}. \)

Equations C.1, C.2 and C.3 are the key to eigenvector analysis, and you need to memorize them. You will use them repeatedly throughout this course, and possibly for a long time after it is over.
APPENDIX C. EIGENVECTOR ANALYSIS

Showing that $A - \lambda \pm I_2$ is singular: If we restrict Eq. C.1 to a single eigenvector (one of $e_{\pm}$), along with the corresponding eigenvalue $\lambda_{\pm}$, we obtain a matrix equations

$$A e_{\pm} = E_{\pm} \lambda_{\pm} = \lambda_{\pm} E_{\pm}$$

Note the important swap in the order of $E_{\pm}$ and $\lambda_{\pm}$. Since $\lambda_{\pm}$ is a scalar, this is legal (and critically important), since this allows us to remove (factored out) $E_{\pm}$

$$(A - \lambda_{\pm} I_2) E_{\pm} = 0. \quad (C.5)$$

This means that the matrix $A - \lambda_{\pm} I_2$ must be singular, since when it operates on $E_{\pm}$, which is not zero, it gives zero. It immediately follows that its determinant is zero (i.e., $|(A - \Lambda)| = 0$). This equation is used to uniquely determine the eigenvalues $\lambda_{\pm}$. Note the important difference between $\lambda_{\pm}$ and $\Lambda$ (i.e., $|(A - \Lambda)| \neq 0$).

Calculating the eigenvalues $\lambda_{\pm}$: The eigenvalues $\lambda_{\pm}$ of $A$ may be determined from $|(A - \lambda_{\pm} I_2)| = 0$

$$\begin{vmatrix} 1 - \lambda_{\pm} & N \\ 1 & 1 - \lambda_{\pm} \end{vmatrix} = (1 - \lambda_{\pm})^2 - N^2 = 0.$$ 

For our case of $N = 2$, $\lambda_{\pm} = (1 \pm \sqrt{2}).^1$

Calculating the eigenvectors $e_{\pm}$: Once the eigenvalues have been determined, they are substitute them into Eq. C.5, which determines the eigenvectors $E = \begin{bmatrix} e_+ \\ e_- \end{bmatrix}$, by solving

$$(A - \lambda_{\pm}) e_{\pm} = \begin{bmatrix} 1 - \lambda_{\pm} & 2 \\ 1 & 1 - \lambda_{\pm} \end{bmatrix} E_{\pm} = 0,$$

where $1 - \lambda_{\pm} = 1 - (1 \pm \sqrt{2}) = \mp \sqrt{2}$.

Recall that Eq. C.5 is singular, because we are using an eigenvalue, and each eigenvector is pointing in a unique direction (This is why it is singular). You might respectively suggest that this equation has no solution. In some sense you would be correct. When we solve for $E_{\pm}$, the two equations defined by Eq. C.5 co-linear (the two equations describe parallel lines). This follows from the fact that there is only one eigenvector for each eigenvalue.

Expecting trouble, yet proceeding to solve for $E = \begin{bmatrix} e_1^+ \\ e_2^+ \end{bmatrix}$

$$\begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1^+ \\ e_2^+ \end{bmatrix} = 0.$$ 

This gives two identical equations $-\sqrt{2} e_1^+ + 2 e_2^+ = 0$ and $e_1^+ - \sqrt{2} e_2^+ = 0$. This is the price of an over-specified equation (the singular matrix is degenerate). The most we can determine is $e_+ = c [\sqrt{2}, 1]^T$, where $c$ is a constant. We can determine eigenvector direction, but not its magnitude.

Following exactly the same procedure for $\lambda_-$, the equation for $e_-$ is

$$\begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} e_1^- \\ e_2^- \end{bmatrix} = 0.$$ 

In this case the relation becomes $e_1^- + \sqrt{2} e_2^- = 0$, thus $E_+ = c [\sqrt{2}, 1]^T$ where $c$ is a constant.

Normalization of the eigenvectors: The constant $c$ may be determined by normalizing the eigenvectors to have unit length. Since we cannot determine the length, we set it to 1. In some sense the degeneracy is resolved by this normalization. Thus $c = 1/\sqrt{3}$, since

$$\left(\pm \sqrt{2}\right)^2 + 1^2 = 3 = 1/c^2.$$ 

---

^1It is a convention to order the eigenvalues from largest to smallest.
Summary: Thus far we have shown
\[ E = [e_+, e_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \]
and
\[ \Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix}. \]

Verify that \( \Lambda = E^{-1} AE \): To find the inverse of \( E \), 1) swap the diagonal values, 2) change the sign of the off diagonals, and 3) divide by the determinant \( \Delta = 2\sqrt{2}/\sqrt{3} \) (see Appendix B)
\[ E^{-1} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.6124 & 0.8660 \\ -0.6124 & 0.8660 \end{bmatrix}. \]

By definition for any matrix \( E^{-1}E = EE^{-1} = I_2 \). Taking the product gives
\[ E^{-1} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2. \]

We wish to show that \( \Lambda = E^{-1}AE \)
\[ \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \cdot \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]
which is best verified with Matlab.

Verify that \( A = E\Lambda E^{-1} \): We wish to show that
\[ \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \cdot \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix}, \]
which is best verified with Matlab (or Octave). All the above equations have been verified both with Matlab and Octave.

I suggest that you verify \( E\Lambda \neq \Lambda E \) and \( AE = EA \) with Matlab. Here is the Matlab program which does this:
\[
A = [1, 2; 1, 1]; \%define the matrix
[E, D] = eig(A); \%compute the eigenvector and eigenvalue matrices
A*E-E*D \%This should be \( \approx 0 \), within numerical error.
E*D-D*E \%This is not zero
\]

Symbolic analysis of \( E \) and \( \Lambda \): Using Matlab/Octave’s syms A B C D M L M=[A B;C D] [E,L]=eig(M) the eigenvectors \( E_\pm \) are
\[ e_\pm = \frac{1}{2\sqrt{C}} \left( \begin{array}{c} (A-D) \mp \sqrt{(A-D)^2 + 4BC} \\ 1 \end{array} \right) \] (C.6)
and eigenvalues are
\[ \lambda_\pm = \frac{1}{2} \left( \begin{array}{c} (A+D) - \sqrt{(A-D)^2 + 4BC} \\ (A+D) + \sqrt{(A-D)^2 + 4BC} \end{array} \right) \] (C.7)

When \( T \) is symmetric \( (A = D) \) these simplify to
\[ E = \begin{bmatrix} -\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix} \] (C.8)
Appendix D

Solution to Pell’s Equation (N=2)

Section 2.2.2 (p. 135) showed that the solution \([x_n, y_n]^T\) to Pell’s equation, for \(N = 2\), is given by powers of Eq. 1.8. To find an explicit formula for \([x_n, y_n]^T\), one must compute powers of

\[
A = 1_j \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{D.1}
\]

We wish to find the solution to Pell’s equation (Eq. 1.8), based on the recursive solution, Eq. 1.9 (p. 40). Thus we need is powers of \(A\), that is \(A^n\), which gives the a closed form expression for \([x_n, y_n]^T\). By the diagonalization of \(A\), its powers are simply the powers of its eigenvalues. This diagonalization is called an eigenvalue analysis, a very general method rooted in linear algebra. This type of analysis allows us to find the solution to most of the linear the equations we encounter.

From Matlab with \(N = 2\) the eigenvalues of Eq. D.1 are \(\lambda_\pm \approx [2.4142j, -0.4142j]\) (i.e., \(\lambda_\pm = 1_j(1 \pm \sqrt{2})\)). The final solution to Eq. D.1 is given in Eq. 2.9 (p. 135). The solution for \(N = 3\) is provided in Appendix D.1 (p. 195).

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 1_j^n A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1_j^n E \Lambda^n E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

The eigenvalue matrix \(D\) is diagonal with the eigenvalues sorted, largest first. The Matlab command \([E,D]=eig(A)\) is helpful to find \(D\) and \(E\) given any \(A\). As we saw above,

\[
\Lambda = 1_j \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 2.414j & 0 \\ 0 & -0.414j \end{bmatrix}.
\]

D.1 Pell equation for N=3

This summarizes the solution of Pell’s equation due to the Pythagoreans using matrix recursion, for the case of \(N=3\). The integer solutions are shown in on the right. Note that \(x_n/y_n \to \sqrt{3}\), in agreement with the Euclidean algorithm.\(^1\) It seem likely that \(\beta_0\) could be absorbed in the starting solution, and then be removed from the generating function, other than as the known factor \(\beta_0^n\)

Case of \(N = 3\): \([x_0, y_0]^T = [1, 0]^T\), \(\beta_0 = j/\sqrt{2}\); Pell-3: \(x_n^2 - 3y_n^2 = 1\); \(x_n/y_n \to \sqrt{3}\)

Try other trivial solutions such as \([-1, 0]^T\) and \([\pm j, 0]^T\). Perhaps this can provide a clue to the proper value of \(\beta_0\).

\(^1\)The matlab program for generating this solution is PellSol3.m.
\[
\begin{align*}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \beta_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
(1\beta_0)^2 - 3(1\beta_0)^2 &= 1 \\
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \beta_0^2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
(4\beta_0^2)^2 - 3(2\beta_0^2)^2 &= 1 \\
\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= \beta_0^3 \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\
(10\beta_0^3)^2 - 3(6\beta_0^3)^2 &= 1 \\
\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} &= \beta_0^4 \begin{bmatrix} 28 \\ 16 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} \\
(28\beta_0^4)^2 - 3(16\beta_0^4)^2 &= 1 \\
\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} &= \beta_0^5 \begin{bmatrix} 76 \\ 44 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 28 \\ 16 \end{bmatrix} \\
(76\beta_0^5)^2 - 3(44\beta_0^5)^2 &= 1
\end{align*}
\]
Appendix E

Laplace transforms

Given a Laplace transform (L) pair \( f(t) \leftrightarrow F(s) \), the frequency domain will always be upper-case [e.g. \( F(s) \)] and the time domain lower case [\( f(t) \)] and causal (i.e., \( f(t < 0) = 0 \)). The definition of the forward transform \( (f(t) \to F(s)) \) is

\[
F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \, dt,
\]

where \( s = \sigma + j\omega \) is the complex Laplace frequency in [radians] and \( t \) is time in [seconds].

The inverse Laplace transform \( (L^{-1}) \), \( F(s) \to f(t) \) is defined as

\[
f(t) = \frac{1}{2\pi j} \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s) e^{st} \, ds = \frac{1}{2\pi j} \oint_C F(s) e^{st} \, ds
\]

with \( \sigma_0 > 0 \in \mathbb{R} \) is a positive constant.

As discussed in the lecture notes (Section 1.4.7, p. 72) we must the Cauchy Residue Theorem (CRT), requiring closure of the contour \( C \) at \( \omega j \to \pm j\infty \)

\[
\oint_C = \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} + \int_{\infty}
\]

where the path represented by ‘\( \subset \)’ is a semicircle of infinite radius. For a causal, ‘stable’ (e.g. doesn’t “blow up” in time) signal, all of the poles of \( F(s) \) must be inside of the Laplace contour, in the left-half \( s \)-plane.

---

![Figure E.1:](image)

This three element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).

Hooke’s Law for a spring states that the force \( f(t) \) is proportional to the displacement \( x(t) \), i.e., \( f(t) = K x(t) \). The formula for a dash-pot is \( f(t) = R v(t) \), and Newton’s famous formula for mass is \( f(t) = d[Mv(t)]/dt \), which for constant \( M \) is \( f(t) = M dv/dt \).

The equation of motion for the mechanical oscillator in Fig. E.1 is given by Newton’s second law: the sum of the forces must balance to zero

\[
M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + K x(t) = f(t).
\]
These three constants, the mass $M$, resistance $R$ and stiffness $K$ are all real and positive. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

Newton’s second law (c1650) is the mechanical equivalent of Kirchhoff’s (c1850) voltage law (KCL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$).

Equation E.1 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

The key idea that every impedance must be complex analytic and $\geq 0$ for $\sigma > 0$ was first proposed by Otto Brune in his PhD at MIT, supervised by a student of Arnold Sommerfeld, Ernst Guilliman, an MIT ECE professor, who played a major role in the development of circuit theory. Brune’s primary (non-MIT) advisor was Cauer, who was also well trained in 19th century German mathematics.\(^\dagger\)

\(^\dagger\)It must be noted, that Mac VanValkenburg, from the University of IL., was arguably more influential in circuit theory, during the same period. Mac’s book are certainly more accessible, but perhaps less widely cited.
Appendix F

Transmission lines

In this problem, we will look at the transfer function of a two-port network, shown in Fig. F.1. We wish to model the dynamics of a freight-train having \( N \) such cars. The model of the train consists of masses connected by springs.

F.1 Transfer functions

The velocity transfer function for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity \( V_1 \) and each car responding with a velocity of \( V_n \). Then

\[
H(s) = \frac{V_N(s)}{V_1(s)}
\]

is the frequency domain ratio of the last car having velocity \( V_N \) to \( V_1 \), the velocity of the engine, at the left most spring (i.e., coupler).

\[\text{Figure F.1: Depiction of a train consisting of cars, treated as a mass } M \text{ and linkages, treated as springs of stiffness } K \text{ or compliance } C = 1/K. \text{ Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force } f_n(t) \text{ to the voltage } v_n(t).}\]

F.2 The transmission matrix

Consistent with the figure, break the model into cells each consisting of three elements: a series inductor representing half the mass \( (L = M/2) \), a shunt capacitor representing the spring \( (C = 1/K) \), and another series inductor representing half the mass \( (L = M/2) \). Each cell is symmetric, making the model a cascade of identical cells.

To do: Use the ABCD method to find the matrix representation of Fig. F.1.

At each node define the force \( f_n(t) \leftrightarrow F_n(\omega) \) and the velocity \( v_n(t) \leftrightarrow V_n(\omega) \) at junction \( n \).
1. Write the ABCD matrix \( T \) for a single cell, composed of series mass \( M/2 \), shunt compliance \( C \) and series mass \( M/2 \), that relates the input node 1 to node 2 where

\[
\begin{bmatrix}
F_1 \\
V_1
\end{bmatrix} = T \begin{bmatrix}
F_2(\omega) \\
-V_2(\omega)
\end{bmatrix}
\]

Note that here the mechanical force \( F \) is analogous to electrical voltage, and the mechanical velocity \( V \) is analogous to electrical current.

\[
T = \begin{bmatrix}
1 & sM/2 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
1/CM & 1
\end{bmatrix}
\]

\( \equiv \)

\[
\begin{bmatrix}
1 + s^2/c^2 & sM (1 + s^2/c^2) \\
sC & 1 + s^2/c^2
\end{bmatrix}
\]

\( \approx \)

\[
\begin{bmatrix}
1 + sM \\
1/sC
\end{bmatrix}
\]

2. Assuming that \( N = 2 \) and that \( F_2 = 0 \) (two mass problem), find the transfer function \( H(s) \equiv V_2/V_1 \). From the results of the \( T \) matrix you determined above, find

\[
H_{21}(s) = \left. \frac{V_2}{V_1} \right|_{F_2=0}
\]

From the lower equation we see that \( V_1 = sCF_2 - (s^2MC/2 + 1)V_2 \). Recall that \( F_2 = 0 \), thus

\[
\frac{V_2}{V_1} = -\frac{1}{s^2MC/2 + 1} = \left( \frac{c_+}{s - s_+} + \frac{c_-}{s - s_-} \right).
\]

with \( s_\pm = \pm j\sqrt{\frac{2}{MC}} \) and \( c_\pm = \pm j/\sqrt{2MC} \).

3. Find \( h_{21}(t) \), the inverse Laplace transform of \( H_{21}(s) \).

\[
h(t) = \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} \frac{e^{st}}{s^2MC/2 + 1} \frac{ds}{2\pi j} = c_+e^{-s_+t}u(t) + c_-e^{-s_-t}u(t).
\]

The integral follows from the CRT. The poles are at \( s_\pm = \pm j\sqrt{\frac{2}{MC}} \) and the residues are \( c_\pm = \pm j/\sqrt{2MC} \).

4. What is the input impedance \( Z_2 = F_2/V_2 \) if \( F_3 = -r_0V_3? \)

Starting from \( T \) calculated above, find \( Z_2 \)

\[
Z_2(s) = \frac{F_2}{V_2} = T \begin{bmatrix}
F_3 \\
-V_3
\end{bmatrix} = \begin{bmatrix}
1 + s^2CM/2/r_0 & -(sM/2)(2 + s^2CM/2)/V_2 \\
-sCr_0 & -(1 + s^2CM/2)/V_2
\end{bmatrix}
\]

5. Simplify the expression for \( Z_2 \) with \( N \rightarrow \infty \) by assuming that:
1) \( F_3 = -r_0V_3 \) (i.e., \( V_3 \) cancels), 2) \( s^2MC << 1 \)

\[
Z_2(s) = \frac{(1 + s^2CM/2)r_0 + (sM/2)(2 + s^2CM/2)}{sCr_0 + (1 + s^2CM/2)} \approx \frac{r_0 + sM}{sCr_0 + 1}
\]

Assumption 3 gives

\[
Z_2 = \frac{sM + \sqrt{MC}}{sCM + \sqrt{MC}} = r_0 \frac{1 + s\sqrt{MC}}{1 + s\sqrt{MC}}
\]

The constant \( r_0 \) is called the characteristic impedance. This shows that below the cutoff frequency (approximation 2), the system approximates a transmission line.
6. State the ABCD matrix relationship between the first and Nth node, in terms of of the cell matrix.
\[
\begin{bmatrix}
F_1 \\
V_1
\end{bmatrix} = T^N \begin{bmatrix}
F_N(\omega) \\
-V_N(\omega)
\end{bmatrix}
\]

7. Given a \( T \) (ABCD) transmission matrix, the eigenvalues are and vectors are given in Appendix C of the Notes (p. 143), repeated here.

**Eigenvalues:**
\[
\begin{bmatrix}
\lambda_+ \\
\lambda_-\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(A + D) - \sqrt{(A - D)^2 + 4BC} \\
(A + D) + \sqrt{(A - D)^2 + 4BC}
\end{bmatrix}
\]

Due to symmetry, \( A = D \), this simplifies to \( \lambda_{\pm} = A \mp \sqrt{BC} \) so that the eigen matrix is
\[
\Lambda = \begin{bmatrix}
A - \sqrt{BC} & 0 \\
0 & A + \sqrt{BC}
\end{bmatrix}
\]

**Eigenvectors:** The eigenvectors are
\[
\begin{bmatrix}
E_+ \\
E_-
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(A - D) \mp \sqrt{(A - D)^2 + 4BC} \\
1
\end{bmatrix}.
\]

The determinant of \( E \) is
\[
\Delta_E = -\frac{1}{2C} \sqrt{(A - D)^2 + 4BC}.
\]

In the symmetric case when \( A = D \)
\[
E = \begin{bmatrix}
-\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\
1 & 1
\end{bmatrix}, \quad E^{-1} = \frac{1}{2} \begin{bmatrix}
-\sqrt{\frac{C}{B}} & 1 \\
+\sqrt{\frac{C}{B}} & 1
\end{bmatrix},
\]
and
\[
\Delta_E = -2 \sqrt{\frac{B}{C}} = -2r_o.
\]

There are two important invariants for the transmission lines, the wave number \( k = \sqrt{BC} = s/c \), and the characteristic resistance \( r_o = \sqrt{B/C} \).

**Transfer functions of the transmission line:** The velocity transfer function \( H_{N,1} = \frac{V_N}{V_1} \) is given by
\[
\begin{bmatrix}
F_1 \\
V_1
\end{bmatrix} = T^N \begin{bmatrix}
F_N(\omega) \\
-V_N(\omega)
\end{bmatrix}
\]
along with the eigenvalue expansion
\[
T^N = E \Lambda^N E^{-1} = E \begin{bmatrix}
\lambda_+^N & 0 \\
0 & \lambda_-^N
\end{bmatrix} E^{-1}.
\]
Appendix G

Filter classification

Let \( z \equiv e^{sT} \) where \( T \) is the “sample period” at which data is taken (every \( T \) seconds). For example if \( T = 22.7 = 1/44100 \) seconds then the data is sampled at 44100 k [Hz]. This is how a CD player works with high quality music. Thus the unit-time delay operator \( z^{-1} \) as

\[ \delta(t - T) \leftrightarrow e^{-sT} \]

G.1 Filter properties

Given the function

\[ F(s) = \frac{(s + 1)(s - 1)}{(s + 2)}, \]

1. find the minimum phase \( M(s) \) and all-pass \( A(s) \) parts. The minimum phase part has all of its poles and zeros in the left half plane (LHP), while the all-pass part has its poles in the LHP and mirrored zeros in the RHP. Thus we place a removable pole zero pair symmetrically across from the RHP zero, and then write the expression as the product, that is \( F(s) = M(s) \cdot A(s) \):

\[ F(s) = \frac{(s + 1)(s - 1)}{(s + 2)} \cdot \frac{s + 1}{s + 1} = \frac{(s + 1)^2}{s + 2} \cdot \frac{s - 1}{s + 1} \]

Thus \( M(s) \equiv \frac{(s+1)^2}{s+2} \) and \( A(s) \equiv \frac{s-1}{s+1} \)

2. Find the magnitude of \( M(s) \) Take the real part of the log of \( M \) and then the anti-log. Thus

\[ |M| = e^{\Re \ln(M(s))} \]

3. Find the phase of \( M(s) \) In this case we use the imaginary part: \( \angle M = \Im \ln M(s) \)

4. Find the magnitude of \( A(s) \) 1, by definition.

5. Find the phase of \( A(s) \) \( \angle A = \Im \ln(A) \)

G.2 More questions

There are a number of question I would like to address in this report. Some of these are

1. Can we interpret the zeta function as a frequency domain quantity, and then inverse transform it into the time domain?
   The answer to this is yes, and the results are quite interesting.

2. Make a histogram of the entropy for the first million integers.
   This is a 5 minute job in Matlab. It goes something line this: \( K=1e5; N=1:K; F=zeros(K,10); \)
   \( \text{for } n=1:K; f=factor(n); F(n,1:length(f))=f; end; hist(F); \)
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