ECE 403 **Exam I – Version 1.1 March 7, 2017** **Spring 2017**

Univ. of Illinois **Tue, Feb 7, 2017; 7-10PM** **Prof. Allen**

Topics: Acoustics, Transforms, Two-port networks

Instructions:

- If you need more space for calculation, use the back of any page.
- NO Cell phones.
- NO Calculators
- Note that each problem has points assigned. More points means "harder."

You may open the exam at 7:00 PM; You must close it by 9:00 PM.

NAME: ________________________ NETID: ________________________

I will not share information during this exam,
and I will discretely report others if I observe
cheating: ________________________ (sign here)

#	value	score
1	9	
2	7	
3	25	
4	5	
5	10	
6	8	
7	18	
8	15	
$\sum$	98	

Terminology, constants & transforms

Definitions of common acoustic variables,[1] the mathematical symbols and the units, as used in the text (see p. 180-181).

Variable	Symbol	[Units]
Total pressure	$P = P_o + p(t)$	[N/m^2 = Pa]
temperature	$T = T_o + \tau(t)$	[K°] Kelvin
density	$\rho = \rho_o + \delta(t)$	[kg/m^3]
adiabatic law	p/δ^γ = Const.	
Constants (air)		
Atm Pressure	$P_o = 10^5$	[Pa]
Abs temperature	$T_o = 273$	[K°]
density	$\rho_o = 1.18$	[kg/m^3]
Boyle's Law	$P_o/(\rho_o T_o)$ = Const.	
sound speed	$c = 345 = \sqrt{\gamma P_o/\rho_o}$	[m/s]
specific impedance	$\rho_o c = 407$	[Rayls]
viscosity	$\mu = 1.86 \times 10^{-5}$	[Ns/m^2] [Poise]
thermal conductivity	$\kappa = 25.4 \times 10^3$	[N/sK]
specific heat cap @V$_o$	c_v	[J/kg]
specific heat cap @P$_o$	c_p	[J/kg]
Boltzman's const.	$k = 1.38 \times 10^{-23}$	J/molecule
ratio of specific heats	$\gamma = c_p/c_v = 1.4$	

Transform	Time → Frequency	Frequency → Time
Laplace Transform (LT)	$F(s) = \int_0^\infty f(t)e^{-st}dt$	$f(t) = \frac{1}{2\pi j}\oint_{C_L} F(s)e^{st}ds$ C_L is the Laplace contour
Fourier Transform (FT)	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$	$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$
Fourier Series (FS)	$F[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-jnt}dt$	$f(t) = \sum_{k=-\infty}^{\infty} F[k]e^{jnt}$
Z-Transform (ZT)	$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n}$	$f[n] = \frac{1}{2\pi j}\oint_{C_z} F(z)z^{n-1}dz$ C_z is the Z-transform contour
Discrete-Time FT (DTFT)	$F(\omega) = \sum_{-\infty}^{\infty} f[n]e^{-j\omega n}$	$f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)e^{j\omega n}d\omega$
Discrete FT (DFT)	$F[k]\sum_{n=0}^{N-1} x[n]e^{-\frac{2\pi j}{N}kn}$	$f[n] = \frac{1}{N}\sum_{k=0}^{N-1} F[k]e^{\frac{2\pi j}{N}kn}$

Discrete convolution: $f[n] \star g[n] = \sum_{k=-\infty}^{\infty} f[k]g[n-k]$
Continuous convolution: $f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$

[1] c_p, c_v: `https://en.wikipedia.org/wiki/Heat_capacity#Specific_heat_capacity`

1 Basic Acoustics (9 pts)

1.1 The speed of sound

The formula for the speed of sound is $c = \sqrt{\gamma P_0/\rho_0}$.

1. **(1 pt)** What is the significance of γP_0? [This combination of variables represents the adiabatic compressibility of air. The $\gamma = c_p/c_v$ results from holding the temperature constant during the cycle of the wave. Heat diffusion is slow compared to the cycle at acoustic frequencies, such that the thermal energy is trapped in the air.]

2. **(1 pt)** Does P_0 depend on temperature? **Yes** / **No** (circle one) [No. The barometric pressure depends on the weight of the air above us. While its density dependents on temperature ($\rho \propto T$), the total weight (i.e., mass) is constant, and is therefore independent of temperature. That leaves the question as to why the barometric pressure varies over time, but that is more about the humidity, winds, and many other variables.]

3. **(1 pt)** Does ρ_0 depend on temperature? **Yes** / **No** (circle one) [Yes, as discussed in class

$$\rho(T, P_0) = 1.275 \frac{273}{273 + C} \frac{P_0}{10^5},$$

where C is the temperature in degrees Celsius and $K = 273 + C$ is the absolute temperature [K°].]

4. **(1 pt)** What is the form of the dependence of the speed of sound on temperature? Give the formula (or a proportional relationship) for $c(T)$, and explain the dependence. [Since $c = \sqrt{\gamma P_0/\rho_0}$ and following the state equation for a gas, $\rho_0 \propto 1/T$, we may conclude that $c(T) \propto \sqrt{T}$, where T is in degrees Kelvin. Because the percentage change in degrees Kelvin is much smaller than in degrees C, this dependence is relatively small.]

1.2 The wave equation

Below is the 2x2 matrix equation that describes, in the frequency domain, the propagation of 1 dimensional sound waves in a tube having area $A(x)$, in terms of the pressure $P(x,s)$ and volume velocity $U(x,s)$: As shown in class the basic equations are:

$$\frac{d}{dx}\begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & \mathcal{Z}(x,s) \\ \mathcal{Y}(x,s) & 0 \end{bmatrix}\begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix}. \tag{1}$$

where P is the pressure, U is the volume velocity, $\mathcal{Z} = s\rho_0/A(x)$ and $\mathcal{Y} = sA(x)/\eta P_0$, with $A(x)$ the area of the tube as a function of position along the length of the tube x.

1. **(3 pts)** Assuming $A(x) = A_o$ is constant, rewrite these equations as a second order equation (the wave equation) solely in terms of the pressure P (remove U), $\mathcal{Z}$ and $\mathcal{Y}$. [If we let $P' \equiv \partial P/\partial x$ (i.e., the partial with respect to space) then

$$P' + ZU = 0 \tag{2}$$

and

$$U' + YP = 0. \tag{3}$$

Taking the partial wrt x of the first equation, and then using the second, gives

$$P'' + ZU' = P'' - \mathcal{Z}\mathcal{Y}P = 0. \tag{4}$$

]

2. **(2 pts)** Find the formula for the speed of sound in terms of $\mathcal{Z}$ and $\mathcal{Y}$. [Since the wave equation is

$$\frac{\partial^2 P}{\partial x^2} = \frac{s^2}{c^2}P \leftrightarrow \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} \tag{5}$$

by inspection we see that $\omega^2/c^2 = \mathcal{Z}\mathcal{Y}$, which results in the final formula for the speed of sound.]

2 deciBels [dB]

1. **(3 pts)** There are two different definitions of acoustic dB, one based on pressure

$$dB_p = 20\log_{10}(P/P_{ref})$$

and a second based on acoustic intensity

$$dB_I = 10\log_{10}(I/I_{ref})$$

where P is the pressure in Pascals and I is the acoustic intensity.

To do: Using the relationships $I \equiv |P|^2/\rho c$ and $I_{ref} \equiv |P_{ref}|^2/\rho c$, demonstrate that these formulas are equivalent. **Show your work.** [The first is given by the formula $|P_{ref}|^2/\rho c$ and the second is given by I_{ref}. Thus we need to show that numerically these two quantities are similar (almost identical). Since $(20 \times 10^{-6})^2/407 \approx 10^{-12}$, the two references are nearly, but not exactly, the same.]

2. **(1 pt)** What is the attenuator gain, expressed in dB, if the pressure is reduced by a factor of 2? [-6 dB corresponds to dividing the pressure by 2.]

3. **(1 pt)** How many millibels [mB] in 1 bel [B]? [1000 mB = 1 B since mB is a much smaller unit that the bel.]

4. **(1 pt)** Give the formula for the intensity in mB units. [Note that 1 [dB] = 0.1 [Bel]. The intensity in mB is $1000\log_{10}(I/I_{ref})$ where $I_{ref} = 10^{-12}$ W/m^2.]

5. **(1 pt)** Give the formula for the sound pressure level in cB (centibel) units. [The sound pressure level in cB is $200\log_{10}(P/P_{ref})$ where $P_{ref} = 20 \times 10^{-6}$ [Pa].]

3 Transforms

3.1 Name that transform

Given the description of the signal/system, name the transform(s) that would be used to analyze it:

1. **(1 pt)** The time response is an infinite sine wave, $\sin(t)$. [Fourier transform or fourier series]

2. **(1 pt)** The time response is zero for $t < 0$ and the frequency response is a function of the complex radian frequency s. [Laplace, since it is strictly causal, and frequency is complex.]

3. **(1 pt)** The time response is given at times $t = nT$ for $n = 0 \ldots \infty$, where $T = 1/F_s$ with $F_s = 44100$ kHz, and the frequency response is specified on the continuous unit circle. [Since the frequency resp is specified on the unit circle, then it must be the z transform. It is a causal sampled systems.]

4. **(1 pt)** A finite-duration louspeaker voltage signal, measured in the laboratory, passed through an A/D converter, and analyzed in `Matlab`. [DFT]

3.2 Fourier vs. Laplace transforms

1. **(2 pts)** When do you use a Laplace transform and when do you use the Fourier transform? *Hint: Discuss differences in the time-domain signals/responses.* [Use the Laplace on causal impulse responses, such as impedances, and the Fourier transform on "signals" that go on forever (or at least for a long time). The $\mathcal{LT}$ codes the transient (and steady state), while the $\mathcal{FT}$ only captures the steady-state response.]

2. **(1 pt)** Give an example where you can use both the FT and LT. [A causal function typically has a $\mathcal{FT}$, but not always. For example $e^{-at}u(t)$ has both, while $e^{t}u(t)$ has a $\mathcal{LT}$ but no $\mathcal{FT}$.]

3. **(1 pt)** Give an example where you cannot use the Laplace transform. [The Laplace transform is always analytic in some region of the s plane. Functions that only have transforms on the $j\omega$ axis do not have $\mathcal{LT}$s. A speech signal does not have a $\mathcal{LT}$, for example.]

4. **(2 pts)** Derive the Fourier transform for the step function $\tilde{u}(t-1)$. *Hint: You may use the FT pair sgn(t)* $\leftrightarrow \frac{2}{j\omega}$.

   [Since the integral does not converge, one must fake it by using the time-symmetric relationship $2\tilde{u}(t) = 1 - \operatorname{sgn}(t)$, delayed:

$$\widetilde{U}(\omega) \equiv \int_{-\infty}^{\infty} \tilde{u}(t-1)e^{-j\omega t}dt = \mathcal{F}\left\{\frac{1-\operatorname{sgn}(t-1)}{2}\right\} = \pi\tilde{\delta}(\omega) + \frac{e^{-j\omega}}{j\omega}$$

$$\neq \int_{1}^{\infty} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{1}^{\infty} = \frac{e^{-j\omega} - e^{-j\omega\infty}}{j\omega} = \frac{e^{-j\omega}}{j\omega} - \frac{e^{-j\omega\infty}}{j\omega}$$

   ]

5. **(2 pts)** Derive the Laplace transform for the step function $u(t-1)$. [e^{-s}/s]

6. **(1 pt)** Describe the similarities and differences between $u(t-1)$ and $\tilde{u}(t-1)$. [The FT has a delta function in addition to $1/j\omega$ term. The Fourier transform is not ideal for analysis of 1-sided signals.]

3.3 Laplace transforms

1. Time derivative and integral properties of the Laplace transform: Assuming $f(t) \leftrightarrow F(s)$,

   (a) **(1 pt)** What is the LT of df/dt? [$sF(s)$. This is shown using integration-by-parts, as follows:
   $$d[f(t)e^{-st}] = e^{-st}\frac{df}{dt}dt - se^{-st}f(t)dt$$
   Integrate this from 0^- to ∞, giving
   $$f(t)e^{-st}|_{0^-}^{\infty} = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}dt - s\int_{0^-}^{\infty} e^{-st}f(t)dt$$
   Rearranging these and evaluating the limits gives the desired result
   $$\int_{0^-}^{\infty}\frac{df}{dt}\, e^{-st}dt = f(0^-) + sF(s),$$
   where $f(0^-) = 0$.]

   (b) **(1 pt)** What is the LT of $\int_{0^-}^{t} f(t)dt$? [$F(s)/s$]

2. Find the Laplace transforms of the following functions (show your work where indicated)

   (a) **(1 pt)** $\int_{-\infty}^{t}\delta(t)dt$ [This is simply $u(t)$, so the LT is $1/s$.]

   (b) **(1 pt)** $\frac{t^2}{2}u(t)$ [This is $u(t)$ integrated twice in time, so the LT is $1/s^3$.]

   (c) **(2 pts)** $h(t) = 3e^{-t/\tau}u(t)$ (show your work; explicitly evaluate the LT integral) [$h(t) \leftrightarrow H(s) = \frac{3}{s+1/\tau}$, which has a pole at $s = -1/\tau$.]

3. **(2 pts)** If $u(t)/\sqrt{\pi t} \leftrightarrow 1/\sqrt{s} = F(s)$, What is $g(t) \leftrightarrow \sqrt{s}$? [Since $d/dt \leftrightarrow s$ then $\sqrt{s} = s/\sqrt{s}$, thus $\frac{d}{dt}\frac{u(t)}{\sqrt{\pi t}} \leftrightarrow \sqrt{s}$. There appears to be a difficulty here, since $\int \frac{\delta(t)}{\sqrt{t}}dt$ is singular at $t = 0$. This problem may be resolved by inserting a small delay T_o, so that the delta function does not resolve at $t = 0$. If we let $G(s) = e^{-sT_o}\sqrt{s} \leftrightarrow \frac{d}{dt}\frac{u(t)}{\sqrt{\pi(t-T_o)}} = \frac{\delta(t)}{\sqrt{\pi(t-T_o)}} + u(t)\frac{d}{dt}\frac{1}{\sqrt{\pi(t-T_o)}}$.]

4. **(2 pts)** Integrate $I = \int_C \frac{1}{s}ds$ around the unit circle centered on $s = 0$.

   [Let C be the unit circle, then $s = e^{j\theta}$, so
   $$I = \int_0^{2\pi}\frac{de^{j\theta}}{e^{j\theta}} = \int_{\theta=0}^{2\pi}\frac{je^{j\theta}}{e^{j\theta}}d\theta = j\int_{\theta=0}^{2\pi}d\theta = j\,\theta|_{\theta=0}^{2\pi} = 2\pi j$$
   ]

5. **(2 pts)** Integrate $\int_C \frac{1}{s}ds$ around the unit circle centered on $s = 0.5$ (i.e., $\sigma = 0.5$, $\omega = 0$), and $s = -2$. [The first is $2\pi j$ and the second is 0. This is explained by the *Cauchy Residue Theorem.*]

4 Convolution

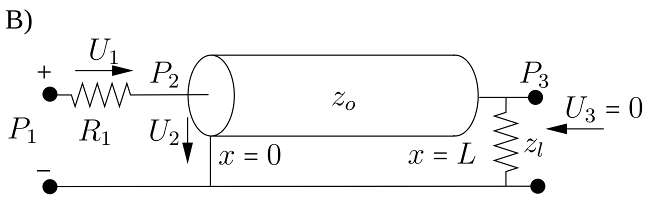
Given two "causal" sequences $a_n = [\cdots, \therefore, 0, 1, 0, -1, 0, \cdots]$ and $b_n = [\cdots, \therefore, 1, -1, 0, 0, \cdots]$. Here the rising dots $\therefore$ define $t = 0$, before and at which time the signal is zero. Assume that following $[0, \cdots]$ the signal remains zero.

1. **(2 pts)** Find causal sequence $c \equiv a \star b$ by direction convolution [Time reverse either a or b and slide it against the other, forming the output sequence $c_n = [0, 1, -1, -1, 1, 0, 0, \cdots]$]

2. **(2 pts)** Form the polynomials $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$, and find $C(z) = A(z) \cdot B(z)$ [$C(z) = (z - z^3)(1 - z) = z - z^2 - z^3 + z^4$ which has a coef vector $[0, 1, -1, -1, 1, 0, \cdots]$]

3. **(1 pt)** What can you say about the sequence c_n and the coefficients of $C(z)$? [They are the same.]

5 Acoustic transmission line



This circuit represents an acoustic tube transmission line (TL), terminated at the far end $x = L$ with impedance $z_l(s)$. Every TL has four parameters: a length L [m], wave speed $c = 343$ [m/s], cross-sectional area A_o [m^2] and *characteristic impedance* z_o.

To Do:

1. **(2 pts)** What is the acoustic impedance, z_o observed by a plane wave in the tube (area A_o)? What are its units? [The *specific acoustic impedance* of a plane wave is $\rho c = 407$ [Rayls]. The *acoustic impedance* of a plane wave in a tube of area A_0 is $\rho c / A_0$ [Rayls/m^2].]

2. **(3 pts)** Express the pressure and velocity as sums of the forward and backward traveling waves ($P = P^+ + P^-$, $U = U^+ - U^-$). Start with the definition of the impedance

$$Z(x, s) \equiv \frac{P(x, s)}{U(x, s)},$$

and find the relationship between $Z(x, s)$ and the 'reflectance' $\Gamma(x, s)$ (the frequency-dependent reflection coefficient), at any point x along the transmission line. Note that the characteristic impedance is also given as $z_o = P^+/U^+ = P^-/U^-$, namely the ratio of incident (reflected) pressure over volume velocity. [Starting from the definition of the impedance (at any point x),

$$Z(x, s) = \frac{P(x)}{U(x)} = \frac{P^+ + P^-}{U^+ - U^-} = \frac{P^+}{U^+}\left(\frac{1+\Gamma}{1-\Gamma}\right) = z_o \frac{1+\Gamma}{1-\Gamma}$$

]

3. **(2 pts)** Assuming $z_l = z_o$, find the input impedance $Z_{in}(s)$. [The quick answer is that since $\Gamma(s) = 0$, the input impedance is

$$Z_{in} = R_1 + z_0.$$

]

4. **(3 pts)** If the load impedance $z_l = z_o$, determine the transfer function $H_{31}(s)$. *Hint: Consider the forward and reverse traveling pressures, $P = P^+ + P^-$. How is P_3^+ related to P_2^+? If there are no reflections from z_l, what can you say about P_3^- and P_2^-?* [P_3^- and P_2^- are equal to 0 because there are no reflections. Thus, $P_3/P_2 = P_3^+/P_2^+$, where the forward pressures are related by the delay along the tube. In this case the wave is a delay of T. Since $\delta(t-T) \leftrightarrow e^{-sT}$,

$$H_{32}(s) = \frac{P_3}{P_2} = e^{-sT}.$$

and

$$H_{31}(s) = \frac{P_3}{P_1} = \frac{z_o}{R_1 + z_0} e^{-sT}.$$

]

6 Lumped-element impedance models

6.1 Impedance elements

1. Find the impedance, $Z(s) = F(s)/V(s)$, where $F(s)$ is the force and $V(s)$ is the velocity (flow), by taking the Laplace transform (LT) of each of the following three force relations.

   (a) **(1 pt)** Hooke's Law $f(t) = Kx(t)$ (note that $v(t) = dx/dt$), where K is the stiffness ($K = 1/C$, where C is the compliance, analogous to capacitance for electrical circuits). [Taking the $\mathcal{LT}$ gives

   $$F(s) = KX(s) = KV(s)/s,$$

   since

   $$v(t) = \frac{d}{dt}x(t) \leftrightarrow sX(s).$$

   Thus the impedance of the spring is

   $$Z_s(s) = \frac{K}{s},$$

   which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as $C = 1/K$.]

   (b) **(1 pt)** Resistance $f(t) = Rv(t)$ (e.g. an electrical resistor or mechanical dash-pot). [From the $\mathcal{LT}$ this becomes

   $$F(s) = RV(s)$$

and the impedance of the dash-pot is then

$$Z_r = R,$$

analogous to that of an electrical resistor.]

(c) **(1 pt)** Newton's Law for Mass $f(t) = M dv(t)/dt$ (note mechanical M is analogous to inductance L in an electrical circuit). [Taking the $\mathcal{LT}$ gives

$$F(s) = sMV(s)$$

thus

$$Z_m(s) = sM,$$

analogous to an electrical inductor.]

6.2 The Helmholtz Resonator

A bottle has a neck of area A_{neck} and length l. It is connected to the body of the bottle "barrel" of volume V_{barrel}. Treat the barrel as a short piece of transmission line, closed at one end, which looks like a compliance $C = V_{barrel}/\gamma P_0$, and the neck as a mass $M = \rho_0 l/A_{neck}$. These two impedances are in series, since they both see the same volume velocity (flow).

1. **(2 pts)** What is the input impedance $Z(s)$ of the bottle in terms of M and C?
2. **(1 pt)** What is true about $Z(s)$ at the bottle's resonant frequency?
3. **(1 pt)** Find the bottle's resonant frequency, in terms of M and C. *Hint: Set* $s = j2\pi f$. [Solving for the resonant frequency, $0 = s_0 M + 1/(s_0 C)$, gives

$$s_0 = j\omega_0 = \sqrt{\frac{-1}{MC}} \quad \text{which gives} \quad \omega_0 = \sqrt{\frac{1}{MC}}$$

]

4. **(1 pt)** Write out the formula for the resonant frequency in terms of the physical dimensions of the bottle, and the speed of sound c. [The formula for the Helmholtz resonator was derived in class, where it was shown to be

$$f_0 = \frac{c}{2\pi}\sqrt{\frac{A}{Vl}}, \tag{6}$$

where A, l are the area and length of the neck and V is the volume of the bottle.]

7 ABCD matrix method

7.1 Transfer functions of a transmission line

In this problem, we will look at the transfer function of the two-port network shown in Fig. 1.

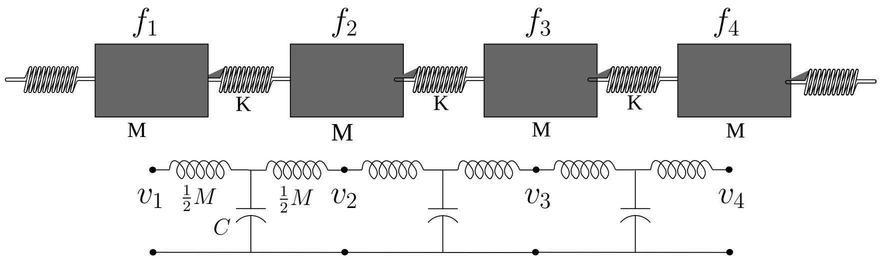


Figure 1: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $v_n(t)$.*

The velocity *transfer function* for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity V_1 and each car responding with a velocity of V_n. Then

$$H(s) = \frac{V_N(s)}{V_1(s)}$$

is the frequency domain ratio of the last car having velocity V_N to V_1, the velocity of the engine, at the left most spring (i.e., coupler).

To do:

1. **(3pts)** To start let $N = 2$. Write the ABCD matrix $\boldsymbol{T}$ for a single cell, composed of series mass $M/2$, shunt compliance C and series mass $M/2$, that relates the force (analogous to voltage) and velocity (analogous to current) at node 1 to node 2, where

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T}_{12} \begin{bmatrix} F_2(\omega) \\ -V_2(\omega) \end{bmatrix} \tag{7}$$

[

$$\boldsymbol{T}_{12} = \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + s^2MC/2 & (sM/2)(2 + s^2MC/2) \\ sC & 1 + s^2MC/2 \end{bmatrix} \tag{8}$$

]

2. **(2 pts)** Find the velocity transfer function assuming an unloaded input ($F_2 = 0$).

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}. \tag{9}$$

[From the lower equation of (Eq.7) we see that $V_1 = sCF_2 - (s^2MC/2 + 1)V_2$.]

3. **(3 pts)** Express $H_{21}(s)$ as a partial fraction expansion (e.g., put it in residue form). You can set $M = C = 1$. [Expanding in a partial fraction expansion

$$\left.\frac{V_2}{V_1}\right|_{F_2=0} = \frac{-1}{s^2MC/2 + 1} = \left(\frac{c_+}{s - s_+} + \frac{c_-}{s - s_-}\right).$$

with $s_\pm = \pm \jmath\sqrt{\frac{2}{MC}}$ and $c_\pm = \pm \jmath/\sqrt{2MC}$.]

4. **(1 pt)** Find $h_{21}(t)$, the inverse Laplace transform of $H_{21}(s)$. (Note the Laplace transform pair $re^{pt}u(t) \leftrightarrow \frac{r}{s-p}$) [

$$h(t) = \oint_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s^2 MC/2 + 1} \frac{ds}{2\pi j} = c_+ e^{-s_+ t} u(t) + c_- e^{-s_- t} u(t).$$

The poles are at $s_\pm = \pm \jmath \sqrt{\frac{2}{MC}}$ and the residues are $c_\pm = \pm \jmath / \sqrt{2MC}$. Its always a good idea to verify this result using Matlab/Octave `residue` command.]

5. **(2 pts)** What is the input impedance $Z_{in} = F_1/V_1$ if $F_2 = -r_0 V_2$? [

Starting from the equation for $T(s)$

$$Z_{in}(s) = \frac{F_1}{V_1} = T \begin{bmatrix} F_2 \\ -V_2 \end{bmatrix} = \frac{-(1 + s^2 CM/2) r_0 \cancel{V_2} - (sM/2)(2 + s^2 CM/2) \cancel{V_2}}{-sC r_0 \cancel{V_2} - (1 + s^2 CM/2) \cancel{V_2}}.$$

Note that V_2 cancels.]

6. **(2 pts)** State the ABCD matrix relationship between the first and Nth node, in terms of the cell matrix $\boldsymbol{T}_{12}$. [

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T}_{12}^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}$$

]

7.2 Model of a loudspeaker

The attached figure shows the equivalent circuit for an electro-dynamic earphone.

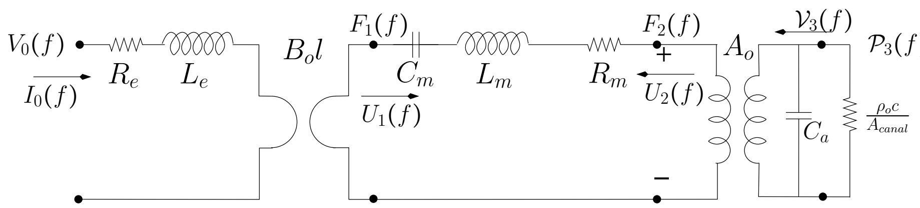


Figure 2: *There are three sections for this equivalent circuit for an earphone: the electrical input (left), the mechanical response (center), and the acoustic output (right). The electrical input in in terms of the voltage $V(f)$ and current $I(f)$. There are two elements, the coil resistance R and its inductance L. The center section corresponds to the mechanical components, with a compliance C (spring), mass M and mechanical damping r. The mechanical force $F_2(f)$ and a velocity $U_2(f)$ are the input to the transformer which converts the force into a pressure. The diaphragm has an area A, which results in a pressure $\mathcal{P}_2(f) = F_1/A_0$, and a volume velocity $\mathcal{V}_2(f) = A_0 U(f)$ at the right.*

1. **(3 pts)** Write out the transmission matrix for the loudspeaker between the input (V, I) and the mechanical output (F_2, U_2, to the left of the transformer). Namely, find $\boldsymbol{T}$ defined as

$$\begin{bmatrix} V \\ I \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} F_2 \\ -U_2 \end{bmatrix}.$$

Your answer should be expressed as a product of 2x2 matrices. You do **not** need to multiply them together. [

$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 1 & R + sL \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & B_o l \\ 1/B_o l & 0 \end{bmatrix} \begin{bmatrix} 1 & r + sM + 1/sC \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_2 \\ -U_2 \end{bmatrix}$$

]

2. **(2 pts)** The measured loudspeaker input impedance

$$Z_{in} = \frac{V}{I} = Z_e + Z_{mot}.$$

Describe Z_e and Z_{mot}.

8 Filter classes

In the following let $s = \sigma + i\omega$ be the Laplace (complex) frequency. *Filters* are causal functions (one sided in time) that modify a *signal* (any function of time) into another signal. For example if $h(t)$ is a filter, and $x(t)$ a signal then

$$y(t) = h(t) \star x(t) \equiv \int_{\infty}^{t} h(t-\tau)s(\tau)d\tau \leftrightarrow H(s)G(s)$$

where $\star$ defines convolution.

1. Define each of the following filter types (e.g. describe the characteristics of the filter's impulse response $h(t)$ or frequency response $H(s)$, or describe the pole-zero locations, where applicable).

   (a) **(1 pt)** Causal [The output is zero for $t < 0$.]

   (b) **(1 pt)** Stable [All poles must be in the left half plane, or on the $j\omega$ axis.]

   (c) **(1 pt)** All-pass [The magnitude of an all-pass filter has magnitude of 1. This filter only changes the phase. The zeros are symmetrically place in the right-half plane across from the poles, in the left half plane.]

   (d) **(1 pt)** Minimum phase [The poles and zeros of this filter are in the left half plane.]

   (e) **(1 pt)** Positive real [The real part of the frequency response is positive.]

2. **(1 pt)** Can an all-pass filter be minimum phase? Why or why not? [In some sense all-pass is the opposite of minimum phase. Any causal filter may be factored into the product of an all-pass ($A(\omega)$) and a minimum phase $M(\omega)$ filter. Namely given any causal transfer function $H(\omega) = A(\omega)M(\omega)$, where $|A| = 1$ is only a frequency dependent delay, and $M(\omega)$ is a filter with the smallest phase possible given $|M(\omega)|$. The real and imaginary parts of a minimum phase filter are Hilbert transforms of each other.]

3. **(1 pt)** Prove that $\delta(t-5)$ is all-pass. [$|e^{-5s}| = 1$]

4. **(1 pt)** Is $\delta(t+5)$ all-pass? [This response is not causal, since it is a time advance of 5 [s]. It has the $\mathcal{LT}$ e^{5s}. But it still is still all-pass, since $|e^{5s}| = 1$. Now one might argue that it cannot have a Laplace transform if its not causal. But one sided functions can be generalized to have a $\mathcal{LT}$.]

5. **(2 pts)** Is e^{-t} all-pass or minimum phase? Justify your answer. [if you assume its causal, then yes it has a single pole at $s = -1$. If the function exists for all time, then it doesn't have either a $\mathcal{LT}$ or $\mathcal{FT}$ transform.]

6. **(2 pts)** When is $F(s) = \frac{s-a}{s+b}$ all-pass? [In the trivial case, if b = -a, then F(s) =(s-a)/(s-a)=1 is all-pass. Otherwise, we require that b is equal to the complex conjugate of a. If b=a*, then the real part of the pole will be opposite the real part of the zero, so it falls on the other side of the jw axis. We need to take the complex conjugate so that the imaginary parts of the pole and zero will be the same.]

7. **(2 pts)** When is $F(s) = \frac{s-a}{s+b}$ minimum phase? [Only if the pole and zero are in the LHP. i.e., $a, -b$ both have negative real parts.]

8. **(1 pt)** Does $F(s) = \frac{s+j}{s-j}$ have a real impulse response? [No: $f(t) = \frac{d}{dt} e^{jt}u(t) + je^{jt}u(t)$]