ECE 403 **ExamII – Version 1.00 April 17, 2017** **Spring 2017**

Univ. of Illinois **Tue, Feb 7, 2017; 7-10PM** **Prof. Allen**

Topics: Acoustics, Transforms, Two-port networks

Instructions:

- If you need more space for calculation, use the back of any page.
- NO Cell phones.
- NO Calculators
- Note that each problem has points assigned. More points means "harder."

You may open the exam at 7:00 PM; You must close it by 9:00 PM.

NAME: ______________________ NETID: ______________________

I will not share information during this exam,
and I will discretely report others if I observe
cheating: ______________________ (sign here)

#	value	score
1	9	
2	7	
3	25	
4	5	
5	10	
6	8	
7	18	
8	15	
Σ	98	

Terminology and acoustic constants

Definitions of common acoustic variables,[1] the mathematical symbols and the units, as used in the text (see p. 180-181).

Variable name	Symbol	[Units]
pressure	$P = P_o + p(t)$	[N/m^2 = Pa]
density	$\rho = \rho_o + \delta(t)$	[kg/m^3]
temperature	$T_o + \tau(t)$	[K°] Kelvin
Constants		
Atm Pressure	$P_o = 10^5$	[Pa]
Abs temperature	$T_o = 273$	[K°]
sound speed	$c = 345$	[m/s]
density	$\rho_o = 1.18$	[kg/m^3]
specific impedance	$\rho c = 407$	[Rayls]
viscosity	$\mu = 1.86 \times 10^{-5}$	[Ns/m^2] Poise
Boyle's Law	$P_o = \rho_o T_o$ Const.	[Pa]
adiabatic law	$p = \delta^\gamma$ Const.	[Pa]
thermal conductivity	$\kappa = 25.4 \times 10^3$	[N/sK]
specific heat cap @V$_o$	c_v	[J/kg]
specific heat cap @P$_o$	c_p	[J/kg]
Boltzman's const.	$k = 1.38 \times 10^{-23}$	J/molecule
ratio of specific heats	$\gamma = c_p/c_v = 1.4$	

[1] C_p, C_v: https://en.wikipedia.org/wiki/Heat_capacity#Specific_heat_capacity

1 Wave equation

1.1 History of the wave equation

1. What year did d'Alembert derive his solution to the wave equation? [d'Alembert first proved this in 1747.]

2. What is the form of D'Alembert's solution? [$f(t-x/c)+g(t+x/c)$]

3. Who was the first person to calculate the speed of sound, and what was the result? [Newton did this in 1648. His formula was in error due to the dynamic stiffness of air, which is γP_o. His result was too small by the factor of $\sqrt{1.4}$.]

1.2 The Webster wave equation:

1. Write out the time-domain *Webster Horn equation.* Hint:

   (a) The first equation is Newton's second law (conservation of momentum density), i.e., the negative gradient in the pressure (force density) equals the mass-density times the rate of change of particle velocity.
   [The force density is $-\nabla p(x,t) = \rho_o \frac{\partial \mathbf{u}(x,t)}{\partial t}$.]

   (b) For static displacement (compression) the stiffness is the static pressure $P_o = 10^4$ [Pa], i.e., 1 [atm]). But when the heat cannot diffuse in one cycle of the sound wave, the heat is trapped. and the adiabatic condition changes the stiffness of air. [Thus the adiabatic (dynamic) stiffness is γP_o.]

   (c) The second equation is Hook's law, the equation for a spring, determined by the stiffness of air. [Conservation of mass requires $-\nabla \cdot \mathbf{u}(x,t) = \frac{1}{\gamma P_o}\frac{\partial p(x,t)}{\partial t}$.]

   [In matrix form (as I showed in class, many times):

$$\frac{\partial}{\partial x}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix} = -\begin{bmatrix} 0 & \frac{\rho_o}{A(x)} \\ \frac{A(x)}{\gamma P_o} & 0 \end{bmatrix} \frac{\partial}{\partial t}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix}, \tag{1}$$

   where $\nu(x,t) = A(x)u(x,t)$ is the *volume velocity*, more generally defined as the integral over the normal component of the particle velocity $\mathbf{u}(x,t)$, over the iso-pressure cross-sectional area $A(x)$ of the tube.]

2. Transform the Horn equation to the frequency domain. [

$$\frac{d}{dx}\begin{bmatrix} P(x,\omega) \\ V(x,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & Z_s(x,s) \\ Y_s(x,s) & 0 \end{bmatrix}\begin{bmatrix} P(x,\omega) \\ V(x,\omega). \end{bmatrix} \tag{2}$$

   Here we use the complex *Laplace frequency* s when referring to the *per-unit impedance*

$$Z_s(s,x) \equiv s\frac{\rho_o}{A(x)} = sM(x) \tag{3}$$

   and *per-unit admittance*

$$Y_s(s,x) \equiv s\frac{A(x)}{\gamma P_o} = sC(x), \tag{4}$$

where $M(x) = \rho_o/A(x)$ is the horn's *per-unit-length mass*, and $C(x) = A(x)/\gamma P_o$ *per-unit-length compliance*, to remind ourselves that these functions must be *causal*, and except at their poles, *analytic* in s.]

3. Assuming a *conical horn*, having area $A(x) = A_o(x/x_o)^2$ with $A_o \le 4\pi$, rewrite these equations as a second order equation solely in terms of the pressure P (remove U), and thereby find the frequency domain solutions $P_\pm(x,s)$ for the conical horn equation.

   Hint: The horn equation in pressure is

$$\frac{1}{A(x)}\frac{\partial}{\partial x}A(x)\frac{\partial}{\partial x}p(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}p(x,t).$$

[Starting from the basic definitions with $A(x) = A_0 e^{2mx}$ along with the basic equation for a horn, explicitly write out the two equations

$$\frac{dP}{dx} + s\frac{\rho_o}{A_0}e^{-2mx}V = 0 \tag{5}$$

$$\frac{dV}{dx} + s\frac{A_0 e^{2mx}}{\gamma P_o}P = 0 \tag{6}$$

Next solve for the pressure (remove V):

$$\frac{d^2P}{dx^2} + s\frac{\rho_o}{A_0}\left(e^{-2mx}\frac{dV}{dx} + V\frac{d}{dx}e^{-2mx}\right) = 0 \tag{7}$$

or

$$\frac{d^2P}{dx^2} + s\frac{\rho_o}{A(x)}\left(\frac{dV}{dx} - 2mV\right) = 0. \tag{8}$$

Going back to the basic equations, again removing V

$$\frac{d^2P}{dx^2} + s\frac{\rho_o}{A(x)}\left(-s\frac{A(x)}{\gamma P_o}P + 2m\frac{A(x)}{s\rho_o}\frac{dP}{dx}\right) = 0 \tag{9}$$

which simplifies to the requested result

$$\frac{d^2P}{dx^2} + 2m\frac{dP}{dx} - s^2\frac{\rho_o}{\gamma P_o}P = 0. \tag{10}$$

since $-s^2 = \omega^2$ and $\gamma P_o = \rho_o c^2$.

] [If we let $P_x \equiv \partial P/\partial x$ (i.e., the partial with respect to space) then

$$P_x + \mathcal{Z}V = 0 \tag{11}$$

and

$$V_x + \mathcal{Y}P = 0. \tag{12}$$

Taking the partial wrt x of the first equation, and then using the second, gives

$$P_{xx} + \mathcal{Z}_x U + \mathcal{Z}U_x = P_{xx} - \frac{\mathcal{Z}_x}{\mathcal{Z}}P_x - \mathcal{Z}\mathcal{Y}P = 0. \tag{13}$$

Using the relation

$$\mathcal{Z}_x/\mathcal{Z} = \frac{d}{dx}\ln\mathcal{Z} = -\frac{d}{dx}\ln A(x), \tag{14}$$

with $A = A_o(x/x_o)^2$, we find

$$P_{xx} + \frac{2}{x}P = \frac{s^2}{c^2}P. \tag{15}$$

Just as it was important to replace real frequency ω with the Laplace frequency s, since the roots are typically complex, using the same reasoning, we replace the "real wave number" $k = \omega/c$ with a *complex wave number* as $\kappa(s)$. Only in the case of non-dispersive waves (e.g., plane waves) is $\kappa(s) = s/c$.

Since the wave equation is

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 P}{\partial x^2} \tag{16}$$

by inspection we see that $\omega^2/c^2 = \mathcal{Z}\mathcal{Y}$, which results in the final formula for the speed of sound.]

4. Assuming an exponential area function

$$A(x) = A_0 e^{2mx}$$

(m is a positive constant, called the *horn flair parameter*), derive the exponential horn equation for the pressure.

$$\frac{\partial^2 p(x,t)}{\partial x^2} + 2m\frac{\partial p(x,t)}{\partial x} = \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial^2 t} \tag{17}$$

5. In general the soluton to a wave equation is of the form

$$p(x,t) = P^+(\kappa)e^{\kappa(s)x}e^{st} + P^-(\kappa)e^{-\kappa(s)x}e^{st}$$

where $s = \sigma + \jmath\omega$ is the Laplace frequency and $\kappa(s)$ is the complex "wave number." What is $\kappa(s)$ for the

(a) Conical horn? [$\kappa(s) = \pm s/c$.]

(b) What is the significance of $\kappa(s)$? [It is called the dispersion relation of the differential equation, that relates the wavelength to the frequency.]

(c) Why is it a function of s? [In general the wavelength is a function of frequency. This is a generalization of the plane wave relation $\lambda f = c$, or $\kappa(s) = s/c$.]

(d) What is the role of $P^{\pm}(\kappa, s)$? [These are aplitudes of the forward and reverse traveling waves, that are to be determined by applying boundary conditions to the solution.]

1.3 Reflectance:

1. Find the formula for the reflectance $R(s)$ in terms of the load impedance $Z_L(s)$ and the characteristic impedance z_0 if: [Solve above for $R(s)$ in terms of Z_L gives: same as problem 5?

$$\Gamma(s,L) = \frac{Z_L - z_0}{Z_L + z_0} \tag{18}$$

]

Same as problem 5?

(a) $Z_L(x,s) = r$ [Nt-s/m^5]

(b) $Z_L(x,s) = 1/sC$ [Nt-s/m^5]

(c) $Z_L(x,s) = r||sM$ [Nt-s/m^5] [Let $Z_L = \frac{rsM}{r+sM}$ then

$$R = \frac{rsM - z_0(r+sM)}{rsM + z_0(r+sM)} = \frac{(r-z_0)Ms - z_0 r}{(r+z_0)Ms + z_0 r}. \tag{19}$$

]

(d) Two transmission lines are in cascade, the first one having an area of 1 [cm^2] and a second having an area of 2 [cm^2], with lengths L_1 and L_2 respectively, terminated with a resistor $r = \rho c/A$, where $A = 2 \times 10^{-4}$ [m^2]. Find $R(x=0,s)$. [Since the second line is terminated in its own impedance, it is just a resistor at its input, which makes the problem very simple. As a result

$$R(s) = \frac{1/1 - 1/2}{1/1 + 1/2} e^{-s2L_1/c} = 1/3e^{-s2L_1/c}, \tag{20}$$

where L_1 is the length of the first TL. Note that if the line were not matched at the end, the story would be very different.]

(e) What is the inverse Laplace transform of

  i. $H(s) = 1/(s+1)$? Find $h(t)$. [$h(t) = e^{-t}U(t)$]

  ii. $R(s) = \frac{Z-z_0}{Z+z_0}$ where $L=1$, $Z=1$ and $z_0 = 2$? Find $r(t)$ at the input. [$r(t) = (1-2)/(1+2)\delta(t-2L/c) = -\delta(t-2)/3$]

  iii. $H(s) = s/(s+1)$? [H(s) = 1-1/(s+1) $\leftrightarrow h(t) = \frac{d}{dt}e^{-t}U(t) = \delta(t) - e^{-t}U(t)$]

2 Nyquist Thm on Thermal noise

A stub of transmission line having characteristic impedance z_0 is terminated in each end with this Thevenin model, with $R = z_0$. Then at $t=0$, the resistance is short or open circuited. This setup is shown in the figure.

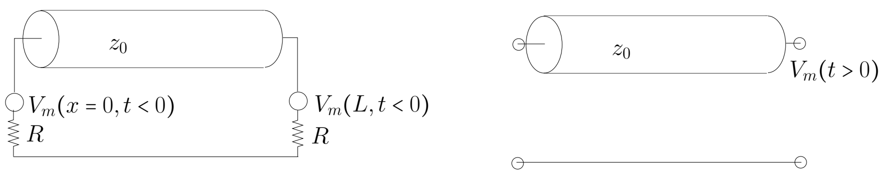

Figure 1: *On the left an ideal lossless transmission line, driven by two resistors. Each resistor has a noise source in series with it, that generates a voltage equal to the thermal noise in the resistor. At time $t=0$ the resistors are switched out, leaving the voltage in the transmission line frozen in place.*

The transmission line stores the voltage as a function of x at $t=0$ once the switch is opened removing the resistors (and also the Thevenin source). At that point voltage $V_m(L,t)$ becomes frozen. Discuss what happens after $t=0$. [The noise traveling in the TL is frozen as it is reflected from the two ends, with no loss. As a result this frozen noise is periodic with a period of the round-trip time of the line, which is $T = 2L/c$. Every periodic signal may be expanded in a Fourier series, which has energy at the frequencies $\omega_k = k/T = kc/2L$. The

Fourier coefficients will not be equal due to the random nature of the noise. On average, each spectral line will have a power of kT, but since the noise is frozen, the long term spectral values will not apply. Each time the experiment is repeated, there will be a randomness in the amplitude of each line.]

1. What is the fundamental period of the noise? [The period is the round trip delay $T = 2L/c \approx 58$ [ms].]

2. Every periodic signal has a *Fourier series*. If the period is T [s], what are the Fourier series frequencies? [The fourier frequencies are $\omega_k = 2\pi k/T$.]

3. Why aren't the values of each spectral peak the same? [While the long-term average power spectrum of each line is kT, a frozen piece of noise will not be equal to the long term average. Rather it will be a snap-shot of the power at that time.

   The average harmonic amplitude may be computed using Parseval's formula, since the power in the noise is independent of how it is calculated, either in the time domain, or in the frequency domain. Parseval's Thm is

   $$\frac{1}{T}\int_0^T |w(t)|^2 dt = \sum_{k=-\infty}^{\infty} |W(2\pi k/T)|^2.$$

   On the average the spectral level should be $|W(\omega)|$, but because of the finite sample of the noise, for any given frozen sample, the spectrum will not be constant. This is very important point that many students (and graduates) seem to miss. It is the ensemble that is constant, not the FT of a fixed sample. No matter now long the noise sample is, the spectral samples $|W(\omega_k)|$ will never become the same, except in the average.

   When you use the FFT, it is necessarily over a finite bandwidth, determined by the sampling rate you chose. Thus the above formula is not the one to use for the case of a band-limited signal (its not wrong, just not the most relevant, given the more detailed assumption of a band-limited and sampled noise. Thus the formula to use is Parseval's Thm for the DFT

   $$\sum_{n=0}^{N-1} x[n]^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(e^{j2\pi k/(N-1)})|^2.$$

   Now the problem with this formula is that the normalization is wrong, as it will not give you the true "spectral level" you would actually measure with a volt meter placed at the output of a filter around one of the spectral lines (i.e., what you would see coming out of a spectral analyzer, calibrated in volts/$\sqrt{Hz}$.

   In other words, the solution to this problem is a nightmare of details, and the books (at least that I read) don't tell you any of this. I prefer to proceed with a numerical solution, for speed and accuracy. Namely, make a narrow band filter around one of the spectral lines, and measure the level of the line. Do this for many noise samples, and take an average. That would be a good problem. Maybe next time.]

3 Hilbert transform

In all parts of this problem $h(t) \leftrightarrow H(s)$ and $H(\omega) = H(s)|_{s=j\omega}$

Analyze the real impulse response

$$h(t) = e^{-t/\tau_0} u(t),$$

with $\tau_0 = 10$ [ms], in terms of its Hilbert transform (integral) relations.

1. Find $H(s)$, the Laplace transform of $h(t)$. [Let $a = 1/\tau_0$. Then $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$.]

2. Where are the poles of $H(s) \leftrightarrow e^{-t/\tau_0}u(t)$? [Let $a = 1/\tau_0$. Since $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$ $H(s)$ has a simple pole at $s = -1/\tau_0$.]

3. Find the real and imaginary parts of $H(\omega) \equiv H(s)|_{s=j\omega}$. [First rationalize the denominator:

$$H(s) = \frac{s^* + a}{(s+a)(s^*+a)} = \frac{\sigma - j\omega + a}{\sigma^2 + \omega^2 + a^2}.$$

Next take the real $\Re$ and imaginary $\Im$ parts, and evaluate s on the ω axis (set $\sigma = 0$):

$$\Re H(\omega) = a/(a^2 + \omega^2), \qquad \Im H(\omega) = -\omega/(a^2 + \omega^2).$$

The real part is constant below the cutoff (resonant) frequency $\omega = a$, and goes at -12 dB/oct above the cutoff. The imaginary part is bandpass with ±6 dB/Oct above and below the resonance frequency.]

4. Write out the symmetric $h_e(t)$ and antisymmetric $h_o(t)$ functions.

   [$2h_e(t) = h(t) + h(-t) \equiv e^{-at}u(t) + e^{at}u(-t)$, while $2h_o(t) = h(t) - h(-t) \equiv e^{-at}u(t) - e^{at}u(-t)$. It trivially follows that $h(t) = h_e(t) + h_o(t)$.]

5. Find the Fourier transforms of $h_e(t) \leftrightarrow H_e(\omega)$ and $h_o(t) \leftrightarrow H_o(\omega)$.

   [Since $h(-t) \leftrightarrow H^*(\omega)$, a symmetric time function is real in the frequency domain,

$$2h_e(t) = h(t) + h(-t) \leftrightarrow H(\omega) + H^*(\omega) = 2\Re H(\omega),$$

thus $h_e(t) \leftrightarrow \Re H(\omega)$. In a similar fashion, an antisymmetric time function is pure imaginary

$$h_o(t) \leftrightarrow j\Im H(\omega).$$

Again with $a \equiv 1/\tau_0$:

$$H_e(\omega) = \Re H(\omega) = \frac{a}{\omega^2 + a^2},$$

$$H_o(\omega) = j\Im H(\omega) = \frac{-j\omega}{\omega^2 + a^2},$$

thus

$$H(\omega) = H_e(\omega) + H_o(\omega) \leftrightarrow h(t) = h_e(t) + h_o(t).$$

The inverse Fourier transform of $H_o(\omega)$ is zero at $t = 0$, which makes it very different from the inverse Laplace transform, which is not defined at $t = 0$. What is the inverse FT of $H_e(\omega)$? Be sure to discuss what happens at $t = 0$.]

6. Find the Hilbert (integral) relations between $H_r \equiv \Re H(\omega)$ (real part) and $H_i \equiv \Im H(\omega)$ (imag part) of $H(\omega)$.

   [It follows from the above results that

$$jH_i(\omega) = \frac{1}{j\pi}\int \frac{H_r(\omega')}{\omega - \omega'}d\omega' \tag{21}$$

   which for the case at hand is

$$\frac{\omega}{\omega^2 + a^2} = \frac{1}{\pi}\int \frac{a\,d\omega}{(\omega' - \omega)(\omega'^2 + a^2)}. \tag{22}$$

   A *second* derivation of the requested integrals may be found from

$$h(t) = h(t)u(t), \tag{23}$$

   (note this is not exactly true at $t = 0$) which after a FT, results in

$$H(\omega) = \frac{1}{2\pi}H(\omega) \star \left(\pi\delta(\omega) + \frac{1}{j\omega}\right), \tag{24}$$

   which may be rewritten as

$$H_r(\omega) = \frac{1}{2}H_r(\omega) + H_i(\omega) \star \frac{1}{2\pi\omega}. \tag{25}$$

   The final relations are [Papoulis (1977), *Signal Analysis*, McGraw Hill, page 251]

$$H_r(\omega) = \frac{1}{\pi}\int \frac{H_i(\omega')}{\omega - \omega'}d\omega' \quad \text{and} \quad H_i(\omega) = -\frac{1}{\pi}\int \frac{H_r(\omega')}{\omega - \omega'}d\omega' \tag{26}$$

   ]

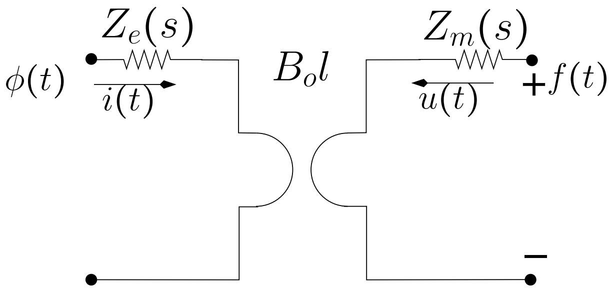

Figure 2: *The Hunt model is composed of an electrical impedance $Z_e(s)$, a Gyrator having parameter $T = lB_o$, and a mechanical impedance $Z_m(s)$. The electrical side is driven by a voltage $\phi(t)$ and a current $i(t)$, while the mechanical side produces a force $f(t)$ and a velocity $v(t)$.*

4 Transducer Thevenin Parameters

4.1 To do:

1. Define and then find the ABCD (Transmission) $\mathbf{T}(s)$ matrix. [The definition of the Transmission matrix is

$$\begin{bmatrix}\Phi(\omega)\\ I(\omega)\end{bmatrix} = \begin{bmatrix}1 & Z_e\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & lB_o\\ 1/lB_o & 0\end{bmatrix}\begin{bmatrix}1 & Z_m\\ 0 & 1\end{bmatrix}\begin{bmatrix}F(\omega)\\ U(\omega)\end{bmatrix} = \frac{1}{T}\begin{bmatrix}Z_e & \Delta_Z\\ 1 & Z_m\end{bmatrix}\begin{bmatrix}F(\omega)\\ -U(\omega)\end{bmatrix}$$

   where Δ_Z is the determinent of the impedance $\mathbf{Z}(s)$ matrix.]

2. Define and then find the impedance matrix $\mathbf{Z}(s)$. [This may be found via the table of relations between $\mathbf{T}$ and $\mathbf{Z}$ (VanValkenberg62). The answer is

$$\begin{bmatrix}\Phi(\omega)\\F(\omega)\end{bmatrix}=\begin{bmatrix}Z_e & -T_o\\T_o & Z_m\end{bmatrix}\begin{bmatrix}I(\omega)\\U(\omega)\end{bmatrix}$$

where $T_o = lB_o$, with l [m] the length of the wire and B_0 [Web] the strength of the magnet [Wb/m^2].]

3. Compute Δ_Z and Δ_T. What is the significance of each of these determinants? [$\Delta_Z = Z_e(s)Z_m(s)+T_o^2$ and $\Delta_T = -1$. If $\Delta_T = -1$ the system is 'anti-reciprocal,' and if $\Delta_T = +1$ the system is 'reciprocal.']

4. Define, and then find the Thevenin equivalent force $F_{\text{thev}}(s)$ [N], given an input voltage Φ_o. [The Thevenin equivalent force is 'open circuit' force (analogous to voltage). Therefore we may read it off the top row of the ABCD matrix equation:

$$\begin{bmatrix}\Phi_o\\I(\omega)\end{bmatrix}=\begin{bmatrix}\mathcal{A} & \mathcal{B}\\\mathcal{C} & \mathcal{D}\end{bmatrix}\begin{bmatrix}F(\omega)\\-U(\omega)\end{bmatrix}=\frac{1}{T}\begin{bmatrix}Z_e & \Delta_Z\\1 & Z_m\end{bmatrix}\begin{bmatrix}F(\omega)\\-U(\omega)\end{bmatrix}$$

giving the equation

$$\Phi_o = \mathcal{A}F - \mathcal{B}U|_{U=0} = \mathcal{A}F$$

therefore

$$F_{thev}(s) = \frac{\Phi_o T_o}{Z_e(s)}$$

]

5. Define, and then find the Thevenin equivalent mechanical impedance in mechanical Ohms, $Z_{\text{thev}}(s)$ [Ω]. [The Thevenin impedance is the equivalen impedance looking into the right side of the loudspeaker model, with the voltage source shorted ($\Phi(\omega) = 0$). First, we must reverse the terminals of the ABCD matrix, so we can perform our usual left-to-right analysis:

$$\begin{bmatrix}F(\omega)\\U(\omega)\end{bmatrix}=\frac{1}{\Delta_T}\begin{bmatrix}\mathcal{D} & \mathcal{B}\\\mathcal{C} & \mathcal{A}\end{bmatrix}\begin{bmatrix}\Phi(\omega)\\-I(\omega)\end{bmatrix}$$

Setting $\Phi = 0$ and taking the ratio $Z = F/U$, we find the Thevenin impedance:

$$Z_{thev}(s) = \frac{F}{U}\Big|_{\Phi=0} = \frac{\mathcal{B}}{\mathcal{A}} = \frac{Z_e(s)Z_m(s)+T_o^2}{Z_e(s)}$$

]

5 Simulation of the middle ear

As shown in the figure, the free field sound pressure, defined as $P_0(\omega)$ acts as a source in series with the radiation resistance R_{rad}. The total radiation impedance $Z_{rad}(s, A_{rad})$ is a combination of the resistance and a reactive component L_{rad}, which represents the local stored field. The two impedances are in parallel

$$Z_{rad}(s) = sL_{rad}R_{rad}/(sL_{rad}+R_{rad}) = 1/Y_{rad}(s).$$

where $s = \sigma + j\omega$ is the Laplace complex frequency variable. The *radiation admittance for a sphere* is

$$Y_{rad} = 1/Z_{rad} = \frac{A_{rad}}{s r_c \rho_o} + \frac{A_{rad}}{\rho_o c} = \frac{1}{sL_{rad}} + \frac{1}{R_{rad}},$$

where r_c is the radius of the sphere and A_{rad} is the *effective area* of the radiation.

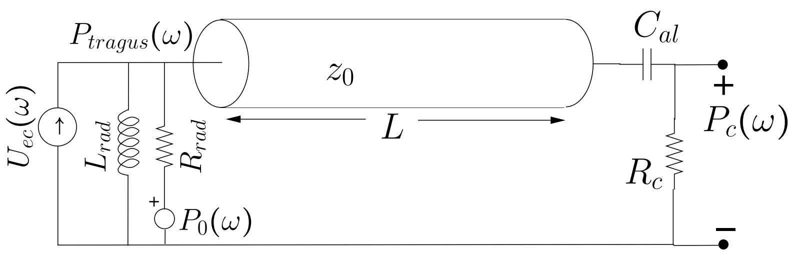

Figure 3: *Model of the ear canal, terminated by the radiation impedance $Z_{rad}(s)$ at the tragus ($x = 0$), and by the eardrum and cochlea at $x = L$.*

On the left we terminate the ear in a radiation impedance

$$Z_{rad}(s) = \frac{sL_{rad}R_{rad}}{sL_{rad} + R_{rad}}.$$

At the cochlear end we terminate the line with an impedance

$$Z_c = R_c + 1/(sC_{al}),$$

where R_c is the cochlear impedance and C_{al} is the stiffness of the annular ligament, which is the ligament that holds the stapes in the oval window. The cochlear resistance (R_c) is assumed to be twice the characteristic impedance of the ear canal.

Each impedance may be converted into a reflection coefficient, defined as

$$\Gamma(s) \equiv \frac{U_-}{U_+} = \frac{Z(s) - z_0}{Z(s) + z_0},$$

where $z_0 = \rho_o c/A_c$ is the characteristic impedance of the ear canal, having area A_c.

To Do:

1. Find z_0, the characteristic impedance of the ear canal. [The radius of the canal is $r_c = d_c/2 = 0.0075/2$ [m]. The canal area is $\pi r_c^2 = 4.42 \times 10^{-5}$, so $z_0 = \rho_o c/A = 407/\pi r_c^2$ or 9.2 [MΩ] (read this as "MKS Acoustic meg-ohms")

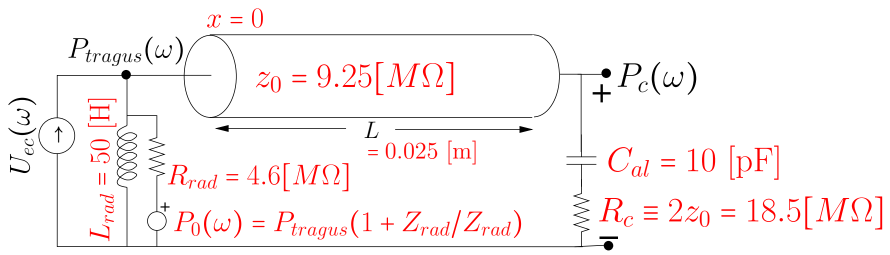

Figure 4: *Left: Model of the ear canal, terminated by the radiation impedance $Z_{rad}(s)$ at the tragus ($x = 0$), and by the eardrum and cochlea at $x = L$.*

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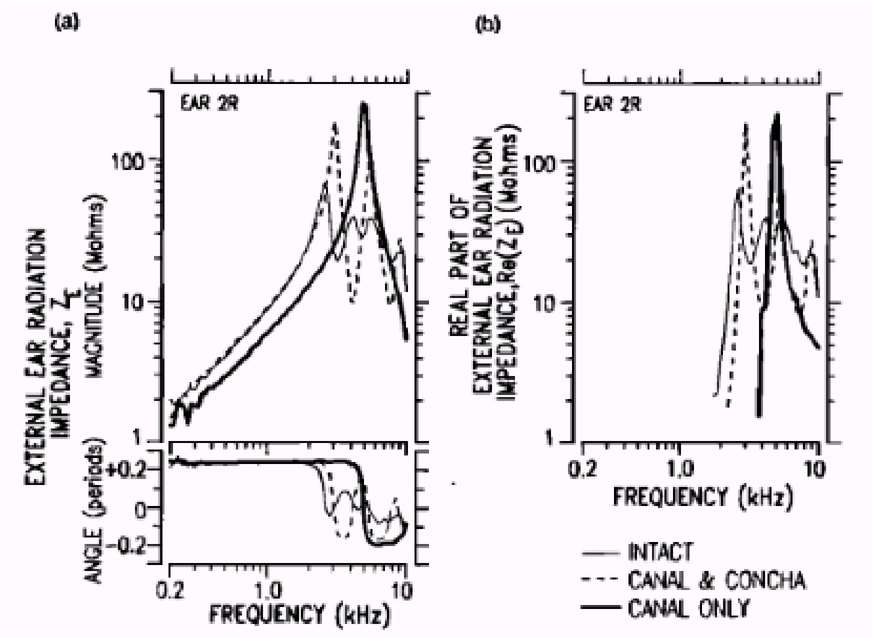

FIG. 10. Radiation impedance measured in one ear (2R), before surgical modification (intact), after removal of the pinna flange (canal and concha), and after removal of the concha (canal only). (a) Magnitude and angle of the impedance. (b) The real parts Re$\{Z_E\}$'s of the impedances

Figure 5: $Z_{rad}(L,s)$ *for an actual cat ear, looking out from the eardrum, with and without the pinna. (?, Fig. 10).*

2. Derive $\Gamma(s)$ from first principles.

3. Find the formula for the reflection coefficient at $x = L$

$$\Gamma(L,s) \equiv \frac{U_-(L,s)}{U_+(L,s)}.$$

[In this case the load impedance is $Z_c = R_c + 1/(sC_{al})$, thus

$$R(L,s) = \frac{R_c + 1/(sC_{al}) - z_0}{R_c + 1/(sC_{al}) + z_0}.$$

]

4. Find the formula for the reflection coefficient at $x = 0$

$$\Gamma(0,s) \equiv \frac{U_+(0,s)}{U_-(0,s)}.$$

[In this case the load is Z_{rad}.]

5. Find the formula for the input impedance $Z(0,s)$ of the middle ear at the entrance of the ear canal, when the cochlea is "blocked" ($Z_c = \infty$ or $R(L,s) = 1$)? (Hint, this has a simple solution, that you can easily derive.Start at $x = L$ and work backwards, in the time domain.) [When the end of the acoustic line is blocked there is a "short" across the end, namely the velocity (current) is zero. The reflectance at $x = 0$ is a delayed version of the reflectance at the cochlea ($x = L$), thus $R(0,s) = -e^{-j\omega 2L/c}$, which has an

inverse Fourier transform of $r(0,t) = -\delta(t-2L/c)$. Make sure you understand why this is! Do you understand where the delay is coming from?

It follows that the impedance is

$$Z(0,s) = z_0 \frac{1+R(0,s)}{1-R(0,s)} = z_0 \frac{1+e^{-s2L/c}}{1-e^{-s2L/c}}.$$

A little algebra and we find

$$Z(0,s) = jz_0 \frac{\cos(sL/c)}{\sin(sL/c)} = j\frac{\rho_o c}{A}\cot(\omega L/c) = z_0 \coth(sL/c).$$

This may be written in the time domain by a Taylor series, and it is a train of impulses spaced $2L/c$ apart. In other words, $R(0,s) \leftrightarrow r(x=0,t) = \delta(t-2L/c)$ is the same as impedance $Z(\omega) = -j\frac{\rho_o c}{A}\cot(\omega L/c)$.]

6. Find the two reflection coefficients and <u>sketch</u> out their magnitude frequency responses. [In each case

$$R(s) = \frac{Z(s)-z_0}{Z(s)+z_0},$$

with $z_0 = \rho_o c/A$ and $Z(s)$ as the impedance load, at each end.

The reflectance at each end is expressed as a numerator over denominator polynomial, namely we need four polynomials, $N(0,s)$ and $D(0,s)$ at the stapes end, defined as

$$R(0,s) = \frac{N(0,s)}{D(0,s)} = \frac{sL_{rad}||R_{rad} - z_0}{sL_{rad}||R_{rad} + z_0} \approx \frac{L_{rad}s - z_0}{L_{rad}s + z_0}$$

which has a first order pole and a zero as roots of $N(L,s)$ and $D(L,s)$. $R(0,s) \approx 1$ due to the high mass.

At the cochlear end

$$R(L,s) = \frac{N(L,s)}{D(L,s)} = \frac{R_c + 1/sC_{al} - z_0}{R_c + 1/sCal + z_0}$$

which also has a first order pole and zero. This last expression is 1 at low frequencies ($f < 0.8$ [kHz]), where the impedance of the annular ligament dominates, and becomes $1/3$ at high frequencies ($f > 0.8$ [kHz]).

The two reflectance functions may be inverse Laplace transformed to give impulse responses $r(0,t)$ and $r(L,t)$, which then may be convolved with $u^+(t)$ to give $u^-(t)$ at each end of the line. These filter are designed next.]