ECE 403
Univ. of Illinois

ExamII - Version 1.00 April 18, 2017
Tue, April 18, 2017; 7-10PM

Spring 2017
Prof. Allen

Topics: Acoustics, Transforms, Two-port networks

## Instructions:

- If you need more space for calculation, use the back of any page.
- NO Cell phones.
- NO Calculators
- Note that each problem has points assigned. More points means "harder."

You may open the exam at 7:00 PM; You must close it by 9:00 PM.

NAME: $\qquad$ NETID: $\qquad$
I will not share information during this exam, and I will discretely report others if I observe cheating: $\qquad$ (sign here)

| $\#$ | value | score |
| :--- | :---: | :---: |
| 1 | 13 |  |
| 2 | 6 |  |
| 3 | 11 |  |
| 4 | 15 |  |
| 5 | 14 |  |
| 6 | 11 |  |
| $\sum$ | 70 |  |

## Terminology, constants \& transforms

Definitions of common acoustic variables, ${ }^{1}$ the mathematical symbols and the units, as used in the text (see p. 180-181).

| Variable | Symbol | $[\mathrm{Units}]$ |
| :--- | :--- | :--- |
| Total pressure | $P=P_{o}+p(t)$ | $\left[\mathrm{N} / \mathrm{m}^{2}=P a\right]$ |
| temperature | $T=T_{o}+\tau(t)$ | $\left[\mathrm{K}^{\circ}\right]$ Kelvin |
| density | $\rho=\rho_{o}+\delta(t)$ | $\left[\mathrm{kg} / \mathrm{m}^{3}\right]$ |
| adiabatic law | $p / \delta^{\gamma}=$ Const. |  |
| Boyle's Law | $P_{o} /\left(\rho_{o} T_{o}\right)=$ Const. |  |
| Constants (air) |  |  |
| Atm Pressure | $P_{o}=10^{5}$ | $[\mathrm{~Pa}]$ |
| Abs temperature | $T_{o}=273$ | $\left[\mathrm{~K}^{\circ}\right]$ |
| density | $\rho_{o}=1.18$ | $\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$ |
| sound speed | $c=345=\sqrt{\gamma P_{o} / \rho_{o}}$ | $[\mathrm{~m} / \mathrm{s}]$ |
| specific impedance | $\rho_{o} c=407$ | $[\mathrm{Rayls}]$ |
| viscosity | $\mu=1.86 \times 10^{-5}$ | $\left[\mathrm{Ns} / \mathrm{m}^{2}\right]$ (Poiseuille) |
| thermal conductivity | $\kappa=25.4 \times 10^{3}$ | $[\mathrm{~N} / \mathrm{sK}]$ |
| specific heat cap @V $\mathrm{V}_{o}$ | $c_{v}$ | $[\mathrm{~J} / \mathrm{kg}]$ |
| specific heat cap @P | $c_{p}$ | $[\mathrm{~J} / \mathrm{kg}]$ |
| Boltzman's const. | $k=1.38 \times 10^{-23}$ | $\mathrm{~J} / \mathrm{K}$ |
| ratio of specific heats | $\gamma=c_{p} / c_{v}=1.4$ | $(\mathrm{unitless})$ |


| Transform | Time $\rightarrow$ Frequency | Frequency $\rightarrow$ Time |
| :--- | :--- | :--- |
| Laplace Transform (LT) | $F(s)=\int_{0}^{\infty} f(t) e^{-s t} d t$ | $f(t)=\frac{1}{2 \pi j} \oint_{C_{L}} F(s) e^{s t} d s$ |
|  |  | $C_{L}$ is the Laplace contour |
| Fourier Transform (FT) | $F(\omega)=\int_{-\infty}^{\infty} f(t) e^{-j \omega t} d t$ | $f(t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} F(\omega) e^{j \omega t} d \omega$ |
| Fourier Series (FS) | $F[k]=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(t) e^{-j n t} d t$ | $f(t)=\sum_{k=-\infty}^{\infty} F[k] e^{j n t}$ |
| Z-Transform (ZT) | $F(z)=\sum_{n=0}^{\infty} f[n] z^{-n}$ | $f[n]=\frac{1}{2 \pi j} \oint_{C_{z}} F(z) z^{n-1} d z$ |
|  |  | $C_{z}$ is the Z-transform contour |
| Discrete-Time FT (DTFT) | $F(\omega)=\sum_{-\infty}^{\infty} f[n] e^{-j \omega n}$ | $f[n]=\frac{1}{2 \pi} \int_{-\pi}^{\pi} F(\omega) e^{j \omega n} d \omega$ |
| Discrete FT (DFT) | $F[k] \sum_{n=0}^{N-1} x[n] e^{-\frac{2 \pi j}{N} k n}$ | $f[n]=\frac{1}{N} \sum_{k=0}^{N-1} F[k] e^{\frac{2 \pi j}{N} k n}$ |

Discrete convolution: $f[n] \star g[n]=\sum_{k=-\infty}^{\infty} f[k] g[n-k]$
Continuous convolution: $f(t) \star g(t)=\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d \tau$

[^0]

Figure 1: The Hunt model is composed of an electrical impedance $Z_{e}(s)$, a Gyrator having parameter $T=l B_{o}$, and a mechanical impedance $Z_{m}(s)$. The electrical side is driven by a voltage $\phi(t)$ and a current $i(t)$, while the mechanical side produces a force $f(t)$ and a velocity $v(t)$.

## 1 Transducer Thevenin Parameters

1. (3pts) Find the ABCD (Transmission) $\mathbf{T}(s)$ matrix for the Hunt transducer model in Figure 1

$$
\left[\begin{array}{c}
\Phi(\omega) \\
I(\omega)
\end{array}\right]=\mathbf{T}(\mathbf{s})\left[\begin{array}{c}
F(\omega) \\
-U(\omega)
\end{array}\right] .
$$

[The definition of the Transmission matrix is

$$
\left[\begin{array}{c}
\Phi(\omega) \\
I(\omega)
\end{array}\right]=\left[\begin{array}{cc}
1 & Z_{e} \\
0 & 1
\end{array}\right]\left[\begin{array}{cc}
0 & l B_{o} \\
1 / l B_{o} & 0
\end{array}\right]\left[\begin{array}{cc}
1 & Z_{m} \\
0 & 1
\end{array}\right]\left[\begin{array}{c}
F(\omega) \\
U(\omega)
\end{array}\right]=\frac{1}{T}\left[\begin{array}{cc}
Z_{e} & \Delta_{Z} \\
1 & Z_{m}
\end{array}\right]\left[\begin{array}{c}
F(\omega) \\
-U(\omega)
\end{array}\right]
$$

where $\Delta_{Z}$ is the determinent of the impedance $\mathbf{Z}(s)$ matrix. ]
2. (2pts) Find the impedance matrix $\mathbf{Z}(s)$.

$$
\left[\begin{array}{l}
\Phi(\omega) \\
F(\omega)
\end{array}\right]=\mathbf{Z}(s)\left[\begin{array}{c}
I(\omega) \\
U(\omega)
\end{array}\right]
$$

[This may be found via the table of relations between $\mathbf{T}$ and $\mathbf{Z}$ (VanValkenberg62). The answer is

$$
\left[\begin{array}{l}
\Phi(\omega) \\
F(\omega)
\end{array}\right]=\left[\begin{array}{ll}
Z_{e} & -T_{o} \\
T_{o} & Z_{m}
\end{array}\right]\left[\begin{array}{c}
I(\omega) \\
U(\omega)
\end{array}\right]
$$

where $T_{o}=l B_{o}$, with $l[\mathrm{~m}]$ the length of the wire and $B_{0}[\mathrm{Web}]$ the strength of the magnet $\left[\mathrm{Wb} / \mathrm{m}^{2}\right]$. ]
3. (2pts) Compute $\Delta_{Z}$ and $\Delta_{T}$. What is the significance of each of these determinants? $\left[\Delta_{Z}=Z_{e}(s) Z_{m}(s)+T_{o}^{2}\right.$ and $\Delta_{T}=-1$. If $\Delta_{T}=-1$ the system is 'anti-reciprocal,' and if $\Delta_{T}=+1$ the system is 'reciprocal.' ]
4. (3pts) Define, and then find the Thevenin equivalent force $F_{\text {thev }}(s)[\mathrm{N}]$, given an input voltage $\Phi_{o}$. Note: If you are not sure of your answers to the previous parts, you can solve this problem in terms of an arbitrary $A B C D$ matrix. [The Thevenin equivalent force is 'open circuit' force (analogous to voltage). Therefore we may read it off the top row of the ABCD matrix equation:

$$
\left[\begin{array}{c}
\Phi_{o} \\
I(\omega)
\end{array}\right]=\left[\begin{array}{ll}
\mathcal{A} & \mathcal{B} \\
\mathcal{C} & \mathcal{D}
\end{array}\right]\left[\begin{array}{c}
F(\omega) \\
-U(\omega)
\end{array}\right]=\frac{1}{T}\left[\begin{array}{cc}
Z_{e} & \Delta_{Z} \\
1 & Z_{m}
\end{array}\right]\left[\begin{array}{c}
F(\omega) \\
-U(\omega)
\end{array}\right]
$$

giving the equation

$$
\Phi_{o}=\mathcal{A} F-\left.\mathcal{B} U\right|_{U=0}=\mathcal{A} F
$$

therefore

$$
F_{\text {thev }}(s)=\frac{\Phi_{o} T_{o}}{Z_{e}(s)}
$$

]
5. (3pts) Define, and then find the Thevenin equivalent mechanical impedance in mechanical Ohms, $Z_{\text {thev }}(s)[\Omega]$. [The Thevenin impedance is the equivalen impedance looking into the right side of the loudspeaker model, with the voltage source shorted $(\Phi(\omega)=0)$. First, we must reverse the terminals of the ABCD matrix, so we can perform our usual left-to-right analysis:

$$
\left[\begin{array}{c}
F(\omega) \\
U(\omega)
\end{array}\right]=\frac{1}{\Delta_{T}}\left[\begin{array}{ll}
\mathcal{D} & \mathcal{B} \\
\mathcal{C} & \mathcal{A}
\end{array}\right]\left[\begin{array}{c}
\Phi(\omega) \\
-I(\omega)
\end{array}\right]
$$

Setting $\Phi=0$ and taking the ratio $Z=F / U$, we find the Thevenin impedance:

$$
Z_{\text {thev }}(s)=\left.\frac{F}{U}\right|_{\Phi=0}=\frac{\mathcal{B}}{\mathcal{A}}=\frac{Z_{e}(s) Z_{m}(s)+T_{o}^{2}}{Z_{e}(s)}
$$

## 2 (6pts) Nyquist Theorem on Thermal noise

A stub of transmission line having characteristic impedance $z_{0}$ is terminated in each end with this Thevenin model, with $R=z_{0}$. Then at $t=0$, the resistance is short or open circuited. This setup is shown in the figure.


Figure 2: On the left an ideal lossless transmission line, driven by two resistors. Each resistor has a noise source in series with it, that generates a voltage equal to the thermal noise in the resistor. At time $t=0$ the resistors are switched out, leaving the voltage in the transmission line frozen in place.

The transmission line stores the voltage as a function of $x$ at $t=0$ once the switch is opened removing the resistors (and also the Thevenin source). At that point voltage $V_{m}(L, t)$ becomes frozen.

1. Discuss what happens after $t=0$. [The noise traveling in the TL is frozen as it is reflected from the two ends, with no loss. As a result this frozen noise is periodic with a period of the round-trip time of the line, which is $T=2 L / c$. Every periodic signal may be expanded in a Fourier series, which has energy at the frequencies $\omega_{k}=k / T=k c / 2 L$. The Fourier coefficients will not be equal due to the random nature of the noise. On average, each spectral line will have a power of $k T$, but since the noise is frozen, the long term spectral values will not apply. Each time the experiment is repeated, there will be a randomness in the amplitude of each line. ]
2. What is the fundamental period of the noise? [The period is the round trip delay $T=2 L / c \approx 58[\mathrm{~ms}]$.
3. Every periodic signal has a Fourier series. If the period is $T[\mathrm{~s}]$, what can you say about the Fourier series frequencies? [The fourier frequencies are $\omega_{k}=2 \pi k / T$.]
4. The noise spectrum will differ depending on the time at which the switch is opened. Why? [While the long-term average power spectrum of each line is $k T$, a frozen piece of noise will not be equal to the long term average. Rather it will be a snap-shot of the power at that time. ]

## 3 Hilbert transform

Analyze the real impulse response

$$
h(t)=e^{-t / \tau_{0}} u(t),
$$

with $\tau_{0}=10[\mathrm{~ms}]$, in terms of its Hilbert transform (integral) relations. Note: In all parts of this problem $h(t) \leftrightarrow H(s)$ and $H(\omega)=\left.H(s)\right|_{s=j \omega}$.

1. (2pts) Find $H(s)$, the Laplace transform of $h(t)$. [Let $a=1 / \tau_{0}$. Then $e^{-a t} u(t) \leftrightarrow \frac{1}{s+a}$.
2. (1pt) Where are the poles of $H(s) \leftrightarrow e^{-t / \tau_{0}} u(t)$ ? [Let $a=1 / \tau_{0}$. Since $e^{-a t} u(t) \leftrightarrow \frac{1}{s+a}$ $H(s)$ has a simple pole at $s=-1 / \tau_{0}$. ]
3. (2pts) Find the real and imaginary parts of $\left.H(\omega) \equiv H(s)\right|_{s=j \omega}$. $\quad$ [First rationalize the denominator:

$$
H(s)=\frac{s^{*}+a}{(s+a)\left(s^{*}+a\right)}=\frac{\sigma-j \omega+a}{\sigma^{2}+\omega^{2}+a^{2}} .
$$

Next take the real $\mathfrak{R}$ and imaginary $\mathfrak{I}$ parts, and evaluate $s$ on the $\omega$ axis (set $\sigma=0$ ):

$$
\Re H(\omega)=a /\left(a^{2}+\omega^{2}\right), \quad \Im H(\omega)=-\omega /\left(a^{2}+\omega^{2}\right) .
$$

The real part is constant below the cutoff (resonant) frequency $\omega=a$, and goes at -12 $\mathrm{dB} /$ oct above the cutoff. The imaginary part is bandpass with $\pm 6 \mathrm{~dB} /$ Oct above and below the resonance frequency. ]
4. (1pt) Write out the symmetric $h_{e}(t)=h(t)+h(-t)$ and antisymmetric $h_{o}(t)=h(t)-$ $h(-t)$ functions. Describe why these equations from symmetric and antisymmetric functions. $\quad\left[2 h_{e}(t)=h(t)+h(-t) \equiv e^{-a t} u(t)+e^{a t} u(-t)\right.$, while $2 h_{o}(t)=h(t)-h(-t) \equiv$ $e^{-a t} u(t)-e^{a t} u(-t)$. It trivially follows that $h(t)=h_{e}(t)+h_{o}(t)$. ]
5. (3pts) Find the Fourier transforms of $h_{e}(t) \leftrightarrow H_{e}(\omega)$ and $h_{o}(t) \leftrightarrow H_{o}(\omega)$. How do they relate to the real and imaginary parts of $H(\omega)$ ?
[Since $h(-t) \leftrightarrow H^{*}(\omega)$, a symmetric time function is real in the frequency domain,

$$
2 h_{e}(t)=h(t)+h(-t) \leftrightarrow H(\omega)+H^{*}(\omega)=2 \Re H(\omega),
$$

thus $h_{e}(t) \leftrightarrow \Re H(\omega)$. In a similar fashion, an antisymmetric time function is pure imaginary

$$
h_{o}(t) \leftrightarrow j \Im H(\omega) .
$$

Again with $a \equiv 1 / \tau_{0}$ :

$$
\begin{aligned}
& H_{e}(\omega)=\Re H(\omega)=\frac{a}{\omega^{2}+a^{2}}, \\
& H_{o}(\omega)=j \Im H(\omega)=\frac{-j \omega}{\omega^{2}+a^{2}},
\end{aligned}
$$

thus

$$
H(\omega)=H_{e}(\omega)+H_{o}(\omega) \leftrightarrow h(t)=h_{e}(t)+h_{o}(t) .
$$

The inverse Fourier transform of $H_{o}(\omega)$ is zero at $t=0$, which makes it very different from the inverse Laplace transform, which is not defined at $t=0$. What is the inverse FT of $H_{e}(\omega)$ ? Be sure to discuss what happens at $t=0$. ]
6. (2pts) Find the Hilbert (integral) relations between $H_{r} \equiv \mathfrak{R} H(\omega)$ (real part) and $H_{i} \equiv \Im H(\omega)$ (imag part) of $H(\omega)$. Hint: These integrals come from a frequency-domain convolution.
[It follows from the above results that

$$
\begin{equation*}
j H_{i}(\omega)=\frac{1}{j \pi} \int \frac{H_{r}\left(\omega^{\prime}\right)}{\omega-\omega^{\prime}} d \omega^{\prime} \tag{1}
\end{equation*}
$$

which for the case at hand is

$$
\begin{equation*}
\frac{\omega}{\omega^{2}+a^{2}}=\frac{1}{\pi} \int \frac{a d \omega}{\left(\omega^{\prime}-\omega\right)\left(\omega^{\prime 2}+a^{2}\right)} . \tag{2}
\end{equation*}
$$

A second derivation of the requested integrals may be found from

$$
\begin{equation*}
h(t)=h(t) u(t), \tag{3}
\end{equation*}
$$

(note this is not exactly true at $t=0$ ) which after a FT, results in

$$
\begin{equation*}
H(\omega)=\frac{1}{2 \pi} H(\omega) \star\left(\pi \delta(\omega)+\frac{1}{j \omega}\right) \tag{4}
\end{equation*}
$$

which may be rewritten as

$$
\begin{equation*}
H_{r}(\omega)=\frac{1}{2} H_{r}(\omega)+H_{i}(\omega) \star \frac{1}{2 \pi \omega} . \tag{5}
\end{equation*}
$$

The final relations are [Papoulis (1977), Signal Analysis, McGraw Hill, page 251]

$$
\begin{equation*}
H_{r}(\omega)=\frac{1}{\pi} \int \frac{H_{i}\left(\omega^{\prime}\right)}{\omega-\omega^{\prime}} d \omega^{\prime} \quad \text { and } \quad H_{i}(\omega)=-\frac{1}{\pi} \int \frac{H_{r}\left(\omega^{\prime}\right)}{\omega-\omega^{\prime}} d \omega^{\prime} \tag{6}
\end{equation*}
$$

## 4 Wave equation

## 4.1 (3pts) History of the wave equation

1. What year did d'Alembert derive his solution to the wave equation? [d'Alembert first proved this in 1747. ]
2. What is the form of D'Alembert's solution? $[f(t-x / c)+g(t+x / c)]$
3. Who was the first person to calculate the speed of sound, and what was the result? [Newton did this in 1648. His formula was in error due to the dynamic stiffness of air, which is $\gamma P_{o}$. His result was too small by the factor of $\sqrt{1.4}$.]

### 4.2 The Webster horn equation:

In the time domain, in 2 x 2 matrix form, the Webster horn equation is given by

$$
\frac{\partial}{\partial x}\left[\begin{array}{l}
p(x, t)  \tag{7}\\
\nu(x, t)
\end{array}\right]=-\left[\begin{array}{cc}
0 & \frac{\rho_{o}}{A(x)} \\
\frac{A(x)}{\gamma P_{o}} & 0
\end{array}\right] \frac{\partial}{\partial t}\left[\begin{array}{l}
p(x, t) \\
\nu(x, t)
\end{array}\right] .
$$

1. (2pts) Transform the Horn equation to the frequency domain. [

$$
\frac{d}{d x}\left[\begin{array}{c}
P(x, \omega)  \tag{8}\\
V(x, \omega)
\end{array}\right]=-\left[\begin{array}{cc}
0 & Z_{s}(x, s) \\
Y_{s}(x, s) & 0
\end{array}\right]\left[\begin{array}{c}
P(x, \omega) \\
V(x, \omega) .
\end{array}\right]
$$

Here we use the complex Laplace frequencys when referring to the per-unit impedance

$$
\begin{equation*}
Z_{s}(s, x) \equiv s \frac{\rho_{o}}{A(x)}=s M(x) \tag{9}
\end{equation*}
$$

and per-unit admittance

$$
\begin{equation*}
Y_{s}(s, x) \equiv s \frac{A(x)}{\gamma P_{o}}=s C(x), \tag{10}
\end{equation*}
$$

where $M(x)=\rho_{o} / A(x)$ is the horn's per-unit-length mass, and $C(x)=A(x) / \gamma P_{o}$ per-unit-length compliance, to remind ourselves that these functions must be causal, and except at their poles, analytic in s. ]
2. (3pts) Assuming a conical horn, having area $A(x)=A_{o}\left(x / x_{o}\right)^{2}$ with $A_{o} \leq 4 \pi$, rewrite these equations as a second order equation solely in terms of the pressure $P$ (remove $U$ ), and thereby find the frequency domain solutions $P_{ \pm}(x, s)$ for the conical horn equation.
[If we let $P_{x} \equiv \partial P / \partial x$ (i.e., the partial with respect to space) then

$$
\begin{equation*}
P_{x}+\mathcal{Z} V=0 \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{x}+\mathcal{Y} P=0 \tag{12}
\end{equation*}
$$

Taking the partial wrt $x$ of the first equation, and then using the second, gives

$$
\begin{equation*}
P_{x x}+\mathcal{Z}_{x} U+\mathcal{Z} U_{x}=P_{x x}-\frac{\mathcal{Z}_{x}}{\mathcal{Z}} P_{x}-\mathcal{Z} \mathcal{Y} P=0 \tag{13}
\end{equation*}
$$

Using the relation

$$
\begin{equation*}
\mathcal{Z}_{x} / \mathcal{Z}=\frac{d}{d x} \ln \mathcal{Z}=-\frac{d}{d x} \ln A(x) \tag{14}
\end{equation*}
$$

with $A=A_{o}\left(x / x_{o}\right)^{2}$, we find

$$
\begin{equation*}
P_{x x}+\frac{2}{x} P=\frac{s^{2}}{c^{2}} P . \tag{15}
\end{equation*}
$$

Just as it was important to replace real frequency $\omega$ with the Laplace frequency $s$, since the roots are typically complex, using the same reasoning, we replace the "real wave number" $k=\omega / c$ with a complex wave number as $\kappa(s)$. Only in the case of non-dispersive waves (e.g., plane waves) is $\kappa(s)=s / c$.
Since the wave equation is

$$
\begin{equation*}
\frac{\partial^{2} P}{\partial x^{2}}=\frac{1}{c^{2}} \frac{\partial^{2} P}{\partial x^{2}} \tag{16}
\end{equation*}
$$

by inspection we see that $\omega^{2} / c^{2}=\mathcal{Z Y}$, which results in the final formula for the speed of sound. ]
3. (3pts) Assuming an exponential area function

$$
A(x)=A_{0} e^{2 m x}
$$

( $m$ is a positive constant, called the horn flair parameter), derive the exponential horn equation for the pressure.

$$
\begin{equation*}
\frac{\partial^{2} p(x, t)}{\partial x^{2}}+2 m \frac{\partial p(x, t)}{\partial x}=\frac{1}{c^{2}} \frac{\partial^{2} p(x, t)}{\partial^{2} t} \tag{17}
\end{equation*}
$$

[Starting from the basic definitions with $A(x)=A_{0} e^{2 m x}$ along with the basic equation for a horn, explicitly write out the two equations

$$
\begin{align*}
& \frac{d P}{d x}+s \frac{\rho_{o}}{A_{0}} e^{-2 m x} V=0  \tag{18}\\
& \frac{d V}{d x}+s \frac{A_{0} e^{2 m x}}{\gamma P_{o}} P=0 \tag{19}
\end{align*}
$$

Next solve for the pressure (remove $V$ ):

$$
\begin{equation*}
\frac{d^{2} P}{d x^{2}}+s \frac{\rho_{o}}{A_{0}}\left(e^{-2 m x} \frac{d V}{d x}+V \frac{d}{d x} e^{-2 m x}\right)=0 \tag{20}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{d^{2} P}{d x^{2}}+s \frac{\rho_{o}}{A(x)}\left(\frac{d V}{d x}-2 m V\right)=0 \tag{21}
\end{equation*}
$$

Going back to the basic equations, again removing $V$

$$
\begin{equation*}
\frac{d^{2} P}{d x^{2}}+s \frac{\rho_{o}}{A(x)}\left(-s \frac{A(x)}{\gamma P_{o}} P+2 m \frac{A(x)}{s \rho_{o}} \frac{d P}{d x}\right)=0 \tag{22}
\end{equation*}
$$

which simplifies to the requested result

$$
\begin{equation*}
\frac{d^{2} P}{d x^{2}}+2 m \frac{d P}{d x}-s^{2} \frac{\rho_{o}}{\gamma P_{o}} P=0 \tag{23}
\end{equation*}
$$

since $-s^{2}=\omega^{2}$ and $\gamma P_{o}=\rho_{o} c^{2}$.
4. (4pts) In general the soluton to a wave equation is of the form

$$
p(x, t)=P^{+}(\kappa) e^{\kappa(s) x} e^{s t}+P^{-}(\kappa) e^{-\kappa(s) x} e^{s t}
$$

where $s=\sigma+\jmath \omega$ is the Laplace frequency and $\kappa(s)$ is the complex "wave number."
(a) What is $\kappa(s)$ for the conical horn? $\quad[\kappa(s)= \pm s / c$.]
(b) What is the significance of $\kappa(s)$ ? [It is called the dispersion relation of the differential equation, that relates the wavelength to the frequency. ]
(c) Why is it a function of $s$ ? [In general the wavelength is a function of frequency. This is a generalization of the plane wave relation $\lambda f=c$, or $\kappa(s)=s / c$.]
(d) What is the role of $P^{ \pm}(\kappa, s)$ ? [These are aplitudes of the forward and reverse traveling waves, that are to be determined by applying boundary conditions to the solution. ]

## 5 Reflectance

1. (6pts) A tube transmission line with characteristic impedance $z_{0}$ and length $L$ is terminated in a load impedance $Z_{L}(s)$.
(a) Starting with the definition of the impedance

$$
Z(s, x) \equiv \frac{P(s, x)}{U(s, x)}
$$

show that the reflectance

$$
R(s, x)=\frac{Z(s, x)-z_{0}}{Z(s, x)+z_{0}} .
$$

You should express the pressure and velocity are sums of the forward and backward traveling waves $\left(P=P^{+}+P^{-}, U=U^{+}-U^{-}\right)$. What is $z_{0}$ in terms of $P^{ \pm}$and $U^{ \pm}$? $\quad[$ Starting from the definition of the impedance (at any point $x$ ),

$$
Z(x, s)=\frac{P(x)}{U(x)}=\frac{P^{+}+P^{-}}{U^{+}-U^{-}}=\frac{P^{+}}{U^{+}}\left(\frac{1+\Gamma}{1-\Gamma}\right)=z_{o} \frac{1+\Gamma}{1-\Gamma}
$$

]
(b) Find the formula for the reflectance at the load, $\left.R(s, x)\right|_{x=L}=R(s, L)$ for
i. $Z_{L}(s)=r\left[\mathrm{Nt-s} / \mathrm{m}^{5}\right]$
ii. $Z_{L}(s)=1 / s C\left[\mathrm{Nt}-\mathrm{s} / \mathrm{m}^{5}\right]$
iii. $Z_{L}(s)=r \| s M\left[\mathrm{Nt}-\mathrm{s} / \mathrm{m}^{5}\right] \quad\left[\right.$ Let $Z_{L}=\frac{r s M}{r+s M}$ then

$$
\begin{equation*}
R=\frac{r s M-z_{0}(r+s M)}{r s M+z_{0}(r+s M)}=\frac{\left(r-z_{0}\right) M s-z_{0} r}{\left(r+z_{0}\right) M s+z_{0} r} . \tag{24}
\end{equation*}
$$

2. (2pts) For the transmission line described in the previous problem, let $L=1, Z_{L}(s)=1$ and $z_{0}=2$.
(a) Find the frequency domain reflectance $R(s, 0)$ at $x=0$.
(b) Find the time-domain reflectance $r(t, 0)$ at $x=0 . \quad[r(t)=(1-2) /(1+2) \delta(t-$ $2 L / c)=-\delta(t-2) / 3]$
3. (3pts) Two transmission lines are in cascade, the first one having an area of $1\left[\mathrm{~cm}^{2}\right]$ and a second having an area of $2\left[\mathrm{~cm}^{2}\right]$, with lengths $L_{1}$ and $L_{2}$ respectively, terminated with a resistor $r=\rho c / A$, where $A=2\left[\mathrm{~cm}^{2}\right]$. Find $R(x=0, s)$. [Since the second line is terminated in its own impedance, it is just a resistor at its input, which makes the problem very simple. As a result

$$
\begin{equation*}
R(s)=\frac{1 / 1-1 / 2}{1 / 1+1 / 2} e^{-s 2 L_{1} / c}=1 / 3 e^{-s 2 L_{1} / c} \tag{25}
\end{equation*}
$$

where $L_{1}$ is the length of the first TL. Note that if the line were not matched at the end, the story would be very different. ]
4. (3pts) What is the inverse Laplace transform of
(a) $H(s)=1 /(s+1)$ ? Find $h(t) . \quad\left[h(t)=e^{-t} U(t)\right]$
(b) $H(s)=s /(s+1) ? \quad\left[\mathrm{H}(\mathrm{s})=1-1 /(\mathrm{s}+1) \leftrightarrow h(t)=\frac{d}{d t} e^{-t} U(t)=\delta(t)-e^{-t} U(t)\right]$

## 6 Model of the middle ear

As shown in the figure, the free field sound pressure, defined as $P_{0}(\omega)$ acts as a source in series with the radiation resistance $R_{\text {rad }}$. The total radiation impedance $Z_{\text {rad }}\left(s, A_{\text {rad }}\right)$ is a combination of the resistance and a reactive component $L_{\text {rad }}$, which represents the local stored field. The two impedances are in parallel

$$
Z_{r a d}(s)=s L_{r a d} R_{r a d} /\left(s L_{r a d}+R_{r a d}\right)=1 / Y_{r a d}(s) .
$$

where $s=\sigma+j \omega$ is the Laplace complex frequency variable. The radiation admittance for a sphere is

$$
Y_{r a d}=1 / Z_{r a d}=\frac{A_{r a d}}{s r_{c} \rho_{o}}+\frac{A_{r a d}}{\rho_{o} c}=\frac{1}{s L_{r a d}}+\frac{1}{R_{r a d}}
$$

where $r_{c}$ is the radius of the sphere and $A_{\text {rad }}$ is the effective area of the radiation.


Figure 3: Model of the ear canal, terminated by the radiation impedance $Z_{\text {rad }}(s)$ at the tragus $(x=0)$, and by the eardrum and cochlea at $x=L$.

On the left we terminate the ear in a radiation impedance

$$
Z_{r a d}(s)=\frac{s L_{r a d} R_{r a d}}{s L_{r a d}+R_{r a d}} .
$$

At the cochlear end we terminate the line with an impedance

$$
Z_{c}=R_{c}+1 /\left(s C_{a l}\right),
$$

where $R_{c}$ is the cochlear impedance and $C_{a l}$ is the stiffness of the annular ligament, which is the ligament that holds the stapes in the oval window. The cochlear resistance $\left(R_{c}\right)$ is assumed to be twice the characteristic impedance of the ear canal.

## To Do:

1. (3pts) Find the formula for the reflection coefficient at $x=L$. Sketch the magnitude $|R(s, L)|$ as a function of frequency. Hint: You should label any constants, or set them equal to 1 for plotting.

$$
\Gamma(L, s) \equiv \frac{U_{-}(L, s)}{U_{+}(L, s)}
$$

[In this case the load impedance is $Z_{c}=R_{c}+1 /\left(s C_{a l}\right)$, thus

$$
R(L, s)=\frac{R_{c}+1 /\left(s C_{a l}\right)-z_{0}}{R_{c}+1 /\left(s C_{a l}\right)+z_{0}} .
$$

2. (3pts) Find the formula for the reflection coefficient at $x=0$ (looking out towards the radiation impedance). Sketch its magnitude as a function of frequency.

$$
\Gamma(0, s) \equiv \frac{U_{+}(0, s)}{U_{-}(0, s)} .
$$

[In this case the load is $Z_{\text {rad }}$. ]
3. (2pts) Consider the radiation impedance looking out the ear canal, $Y_{\text {rad }}$.
(a) What is the frequency for which its real and imaginary parts are equal (in terms of $L_{r a d}$ and $\left.R_{r a d}\right)$ ?

$$
\frac{1}{2 \pi f_{0} L_{r a d}}=\frac{1}{R_{r a d}} .
$$

]
(b) Describe the dependence of $Y_{\text {rad }}$ on $L_{r a d}$ and $R_{r a d}$ below and above this frequency. [ $R_{\text {rad }}$ dominates at high frequencies above $f_{0}, L_{\text {rad }}$ dominates at low frequencies below $f_{0}$. ]
4. (3pts) Find the formula for the input impedance $Z(0, s)$ of the middle ear at the entrance of the ear canal, when the cochlea is "blocked" $\left(Z_{c}=\infty\right.$ or $\left.R(L, s)=1\right)$ ?
[When the end of the acoustic line is blocked there is a "short" across the end, namely the velocity (current) is zero. The reflectance at $x=0$ is a delayed version of the
reflectance at the cochlea $(x=L)$, thus $R(0, s)=-e^{-j \omega 2 L / c}$, which has an inverse Fourier transform of $r(0, t)=-\delta(t-2 L / c)$. Make sure you understand why this is! Do you understand where the delay is coming from?
It follows that the impedance is

$$
Z(0, s)=z_{0} \frac{1+R(0, s)}{1-R(0, s)}=z_{0} \frac{1+e^{-s 2 L / c}}{1-e^{-s 2 L / c}}
$$

A little algebra and we find

$$
Z(0, s)=j z_{0} \frac{\cos (s L / c)}{\sin (s L / c)}=j \frac{\rho_{o} c}{A} \cot (\omega L / c)=z_{0} \operatorname{coth}(s L / c) .
$$

This may be written in the time domain by a Taylor series, and it is a train of impulses spaced $2 L / c$ apart. In other words, $R(0, s) \leftrightarrow r(x=0, t)=\delta(t-2 L / c)$ is the same as impedance $Z(\omega)=-j \frac{\rho_{o} c}{A} \cot (\omega L / c)$. ]


[^0]:    ${ }^{1} c_{p}, c_{v}$ : https://en.wikipedia.org/wiki/Heat_capacity\#Specific_heat_capacity

