ECE 403	ExamII - Version 1.00 April 18, 2017	Spring 2017

Univ. of Illinois Tue, April 18, 2017; 7-10PM Prof. Allen

Topics: Acoustics, Transforms, Two-port networks

Instructions:

- If you need more space for calculation, use the back of any page.
- NO Cell phones.
- NO Calculators
- Note that each problem has points assigned. More points means "harder."

You may open the exam at 7:00 PM; You must close it by 9:00 PM.

NAME: ______ NETID: _____

#	value	score
1	13	
2	6	
3	11	
4	15	
5	14	
6	11	
\sum	70	

Terminology, constants & transforms

Definitions of common acoustic variables,¹ the mathematical symbols and the units, as used in the text (see p. 180-181).

Variable	Symbol	[Units]
Total pressure	$P = P_o + p(t)$	$[N/m^2 = Pa]$
temperature	$T = T_o + \tau(t)$	[K°] Kelvin
density	$\rho = \rho_o + \delta(t)$	$[kg/m^3]$
adiabatic law	$p/\delta^{\gamma} = \text{Const.}$	
Boyle's Law	$P_o/(\rho_o T_o) = \text{Const.}$	
Constants (air)		
Atm Pressure	$P_o = 10^5$	[Pa]
Abs temperature	$T_o = 273$	[K°]
density	$ \rho_o = 1.18 $	$[kg/m^3]$
sound speed	$c = 345 = \sqrt{\gamma P_o/\rho_o}$	[m/s]
specific impedance	$\rho_o c = 407$	[Rayls]
viscosity	$\mu = 1.86 \times 10^{-5}$	$[Ns/m^2]$ (Poiseuille)
thermal conductivity	$\kappa = 25.4 \times 10^3$	[N/sK]
specific heat cap $@V_o$	c_v	[J/kg]
specific heat cap $@P_o$	c_p	[J/kg]
Boltzman's const.	$k = 1.38 \times 10^{-23}$	J/K
ratio of specific heats	$\gamma = c_p/c_v = 1.4$	(unitless)

	i	1
Transform	Time \rightarrow Frequency	Frequency \rightarrow Time
Laplace Transform (LT)	$F(s) = \int_0^\infty f(t) e^{-st} dt$	$f(t) = \frac{1}{2\pi j} \oint_{C_L} F(s) e^{st} ds$
		C_L is the Laplace contour
Fourier Transform (FT)	$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$	$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$
Fourier Series (FS)	$F[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-jnt} dt$	$f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{jnt}$
Z-Transform (ZT)	$F(z) = \sum_{n=0}^{\infty} f[n] z^{-n}$	$f[n] = \frac{1}{2\pi j} \oint_{C_z} F(z) z^{n-1} dz$
		C_z is the Z-transform contour
Discrete-Time FT (DTFT)	$F(\omega) = \sum_{-\infty}^{\infty} f[n] e^{-j\omega n}$	$f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega$
Discrete FT (DFT)	$F[k] \sum_{n=0}^{N-1} x[n] e^{-\frac{2\pi j}{N}kn}$	$\int f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] e^{\frac{2\pi j}{N} kn}$

Discrete convolution: $f[n] \star g[n] = \sum_{k=-\infty}^{\infty} f[k]g[n-k]$ Continuous convolution: $f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$

 $^{^{1}}c_{p}, c_{v}$: https://en.wikipedia.org/wiki/Heat_capacity#Specific_heat_capacity



Figure 1: The Hunt model is composed of an electrical impedance $Z_e(s)$, a Gyrator having parameter $T = lB_o$, and a mechanical impedance $Z_m(s)$. The electrical side is driven by a voltage $\phi(t)$ and a current i(t), while the mechanical side produces a force f(t) and a velocity v(t).

1 Transducer Thevenin Parameters

1. (3pts) Find the ABCD (Transmission) $\mathbf{T}(s)$ matrix for the Hunt transducer model in Figure 1

$$\begin{bmatrix} \Phi(\omega) \\ I(\omega) \end{bmatrix} = \mathbf{T}(\mathbf{s}) \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix}.$$

[The definition of the Transmission matrix is

$$\begin{bmatrix} \Phi(\omega) \\ I(\omega) \end{bmatrix} = \begin{bmatrix} 1 & Z_e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & lB_o \\ 1/lB_o & 0 \end{bmatrix} \begin{bmatrix} 1 & Z_m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F(\omega) \\ U(\omega) \end{bmatrix} = \frac{1}{T} \begin{bmatrix} Z_e & \Delta_Z \\ 1 & Z_m \end{bmatrix} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix}$$

where Δ_Z is the determinent of the impedance $\mathbf{Z}(s)$ matrix.

2. (2pts) Find the impedance matrix $\mathbf{Z}(s)$.

$$\begin{bmatrix} \Phi(\omega) \\ F(\omega) \end{bmatrix} = \mathbf{Z}(s) \begin{bmatrix} I(\omega) \\ U(\omega) \end{bmatrix}.$$

[This may be found via the table of relations between \mathbf{T} and \mathbf{Z} (VanValkenberg62). The answer is

$$\begin{bmatrix} \Phi(\omega) \\ F(\omega) \end{bmatrix} = \begin{bmatrix} Z_e & -T_o \\ T_o & Z_m \end{bmatrix} \begin{bmatrix} I(\omega) \\ U(\omega) \end{bmatrix}$$

where $T_o = lB_o$, with l [m] the length of the wire and B_0 [Web] the strength of the magnet [Wb/m²].

- 3. (2pts) Compute Δ_Z and Δ_T . What is the significance of each of these determinants? $[\Delta_Z = Z_e(s)Z_m(s) + T_o^2 \text{ and } \Delta_T = -1. \text{ If } \Delta_T = -1 \text{ the system is 'anti-reciprocal,' and if } \Delta_T = +1 \text{ the system is 'reciprocal.' }]$
- 4. (3pts) Define, and then find the Thevenin equivalent force $F_{\text{thev}}(s)$ [N], given an input voltage Φ_o . Note: If you are not sure of your answers to the previous parts, you can solve this problem in terms of an arbitrary ABCD matrix. [The Thevenin equivalent force is 'open circuit' force (analogous to voltage). Therefore we may read it off the top row of the ABCD matrix equation:

$$\begin{bmatrix} \Phi_o \\ I(\omega) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix} = \frac{1}{T} \begin{bmatrix} Z_e & \Delta_Z \\ 1 & Z_m \end{bmatrix} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix}$$

giving the equation

$$\Phi_o = \mathcal{A}F - \mathcal{B}U|_{U=0} = \mathcal{A}F$$

therefore

$$F_{thev}(s) = \frac{\Phi_o T_o}{Z_e(s)}$$

1

5. (3pts) Define, and then find the Thevenin equivalent mechanical impedance in mechanical Ohms, $Z_{\text{thev}}(s)$ [Ω]. [The Thevenin impedance is the equivalen impedance looking into the right side of the loudspeaker model, with the voltage source shorted ($\Phi(\omega) = 0$). First, we must reverse the terminals of the ABCD matrix, so we can perform our usual left-to-right analysis:

$$\begin{bmatrix} F(\omega) \\ U(\omega) \end{bmatrix} = \frac{1}{\Delta_T} \begin{bmatrix} \mathcal{D} & \mathcal{B} \\ \mathcal{C} & \mathcal{A} \end{bmatrix} \begin{bmatrix} \Phi(\omega) \\ -I(\omega) \end{bmatrix}$$

Setting $\Phi = 0$ and taking the ratio Z = F/U, we find the Thevenin impedance:

$$Z_{thev}(s) = \frac{F}{U}\Big|_{\Phi=0} = \frac{\mathcal{B}}{\mathcal{A}} = \frac{Z_e(s)Z_m(s) + T_o^2}{Z_e(s)}$$

]

2 (6pts) Nyquist Theorem on Thermal noise

A stub of transmission line having characteristic impedance z_0 is terminated in each end with this Thevenin model, with $R = z_0$. Then at t = 0, the resistance is short or open circuited. This setup is shown in the figure.



Figure 2: On the left an ideal lossless transmission line, driven by two resistors. Each resistor has a noise source in series with it, that generates a voltage equal to the thermal noise in the resistor. At time t = 0 the resistors are switched out, leaving the voltage in the transmission line frozen in place.

The transmission line stores the voltage as a function of x at t = 0 once the switch is opened removing the resistors (and also the Thevenin source). At that point voltage $V_m(L,t)$ becomes frozen.

1. Discuss what happens after t = 0. [The noise traveling in the TL is frozen as it is reflected from the two ends, with no loss. As a result this frozen noise is periodic with a period of the round-trip time of the line, which is T = 2L/c. Every periodic signal may be expanded in a Fourier series, which has energy at the frequencies $\omega_k = k/T = kc/2L$. The Fourier coefficients will not be equal due to the random nature of the noise. On average, each spectral line will have a power of kT, but since the noise is frozen, the long term spectral values will not apply. Each time the experiment is repeated, there will be a randomness in the amplitude of each line.]

- 2. What is the fundamental period of the noise? [The period is the round trip delay $T = 2L/c \approx 58$ [ms].]
- 3. Every periodic signal has a *Fourier series*. If the period is T [s], what can you say about the Fourier series frequencies? [The fourier frequencies are $\omega_k = 2\pi k/T$.]
- 4. The noise spectrum will differ depending on the time at which the switch is opened. Why? [While the long-term average power spectrum of each line is kT, a frozen piece of noise will not be equal to the long term average. Rather it will be a snap-shot of the power at that time.]

3 Hilbert transform

Analyze the real impulse response

$$h(t) = e^{-t/\tau_0} u(t),$$

with $\tau_0 = 10 \text{ [ms]}$, in terms of its Hilbert transform (integral) relations. Note: In all parts of this problem $h(t) \leftrightarrow H(s)$ and $H(\omega) = H(s)|_{s=i\omega}$.

- 1. (2pts) Find H(s), the Laplace transform of h(t). [Let $a = 1/\tau_0$. Then $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$.]
- 2. (1pt) Where are the poles of $H(s) \leftrightarrow e^{-t/\tau_0} u(t)$? [Let $a = 1/\tau_0$. Since $e^{-at} u(t) \leftrightarrow \frac{1}{s+a}$ H(s) has a simple pole at $s = -1/\tau_0$.]
- 3. (2pts) Find the real and imaginary parts of $H(\omega) \equiv H(s)|_{s=j\omega}$. [First rationalize the denominator:

$$H(s) = \frac{s^* + a}{(s+a)(s^* + a)} = \frac{\sigma - j\omega + a}{\sigma^2 + \omega^2 + a^2}.$$

Next take the real \Re and imaginary \Im parts, and evaluate s on the ω axis (set $\sigma = 0$):

$$\Re H(\omega) = a/(a^2 + \omega^2), \qquad \Im H(\omega) = -\omega/(a^2 + \omega^2).$$

The real part is constant below the cutoff (resonant) frequency $\omega = a$, and goes at -12 dB/oct above the cutoff. The imaginary part is bandpass with ± 6 dB/Oct above and below the resonance frequency.

- 4. (1pt) Write out the symmetric $h_e(t) = h(t) + h(-t)$ and antisymmetric $h_o(t) = h(t) h(-t)$ functions. Describe why these equations from symmetric and antisymmetric functions. $[2h_e(t) = h(t) + h(-t) \equiv e^{-at}u(t) + e^{at}u(-t)$, while $2h_o(t) = h(t) h(-t) \equiv e^{-at}u(t) e^{at}u(-t)$. It trivially follows that $h(t) = h_e(t) + h_o(t)$.
- 5. (3pts) Find the Fourier transforms of $h_e(t) \leftrightarrow H_e(\omega)$ and $h_o(t) \leftrightarrow H_o(\omega)$. How do they relate to the real and imaginary parts of $H(\omega)$?

Since $h(-t) \leftrightarrow H^*(\omega)$, a symmetric time function is real in the frequency domain,

$$2h_e(t) = h(t) + h(-t) \leftrightarrow H(\omega) + H^*(\omega) = 2\Re H(\omega),$$

thus $h_e(t) \leftrightarrow \Re H(\omega)$. In a similar fashion, an antisymmetric time function is pure imaginary

$$h_o(t) \leftrightarrow j \Im H(\omega)$$

Again with $a \equiv 1/\tau_0$:

$$H_e(\omega) = \Re H(\omega) = \frac{a}{\omega^2 + a^2},$$
$$H_o(\omega) = j\Im H(\omega) = \frac{-j\omega}{\omega^2 + a^2},$$

thus

$$H(\omega) = H_e(\omega) + H_o(\omega) \leftrightarrow h(t) = h_e(t) + h_o(t).$$

The inverse Fourier transform of $H_o(\omega)$ is zero at t = 0, which makes it very different from the inverse Laplace transform, which is not defined at t = 0. What is the inverse FT of $H_e(\omega)$? Be sure to discuss what happens at t = 0.

6. (2pts) Find the Hilbert (integral) relations between $H_r \equiv \Re H(\omega)$ (real part) and $H_i \equiv \Im H(\omega)$ (imag part) of $H(\omega)$. Hint: These integrals come from a frequency-domain convolution.

It follows from the above results that

$$jH_i(\omega) = \frac{1}{j\pi} \int \frac{H_r(\omega')}{\omega - \omega'} d\omega'$$
(1)

which for the case at hand is

$$\frac{\omega}{\omega^2 + a^2} = \frac{1}{\pi} \int \frac{a \, d\omega}{(\omega' - \omega)(\omega'^2 + a^2)}.$$
(2)

A second derivation of the requested integrals may be found from

$$h(t) = h(t)u(t), \tag{3}$$

(note this is not exactly true at t = 0) which after a FT, results in

$$H(\omega) = \frac{1}{2\pi} H(\omega) \star \left(\pi \delta(\omega) + \frac{1}{j\omega} \right), \tag{4}$$

which may be rewritten as

]

$$H_r(\omega) = \frac{1}{2}H_r(\omega) + H_i(\omega) \star \frac{1}{2\pi\omega}.$$
(5)

The final relations are [Papoulis (1977), Signal Analysis, McGraw Hill, page 251]

$$H_r(\omega) = \frac{1}{\pi} \int \frac{H_i(\omega')}{\omega - \omega'} d\omega' \quad \text{and} \quad H_i(\omega) = -\frac{1}{\pi} \int \frac{H_r(\omega')}{\omega - \omega'} d\omega' \quad (6)$$

4 Wave equation

4.1 (3pts) History of the wave equation

- 1. What year did d'Alembert derive his solution to the wave equation? [d'Alembert first proved this in 1747.]
- 2. What is the form of D'Alembert's solution? [f(t x/c) + g(t + x/c)]
- 3. Who was the first person to calculate the speed of sound, and what was the result? [Newton did this in 1648. His formula was in error due to the dynamic stiffness of air, which is γP_o . His result was too small by the factor of $\sqrt{1.4}$.]

4.2 The Webster horn equation:

In the time domain, in 2x2 matrix form, the Webster horn equation is given by

$$\frac{\partial}{\partial x} \begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix} = -\begin{bmatrix} 0 & \frac{\rho_o}{A(x)} \\ \frac{A(x)}{\gamma P_o} & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix}.$$
(7)

1. (2pts) Transform the Horn equation to the frequency domain.

$$\frac{d}{dx} \begin{bmatrix} P(x,\omega) \\ V(x,\omega) \end{bmatrix} = - \begin{bmatrix} 0 & Z_s(x,s) \\ Y_s(x,s) & 0 \end{bmatrix} \begin{bmatrix} P(x,\omega) \\ V(x,\omega) \end{bmatrix}$$
(8)

Here we use the complex Laplace frequency s when referring to the per-unit impedance

$$Z_s(s,x) \equiv s \frac{\rho_o}{A(x)} = s M(x) \tag{9}$$

and per-unit admittance

$$Y_s(s,x) \equiv s \frac{A(x)}{\gamma P_o} = sC(x), \tag{10}$$

where $M(x) = \rho_o/A(x)$ is the horn's *per-unit-length mass*, and $C(x) = A(x)/\gamma P_o$ perunit-length compliance, to remind ourselves that these functions must be *causal*, and except at their poles, *analytic* in *s*.

2. (3pts) Assuming a conical horn, having area $A(x) = A_o(x/x_o)^2$ with $A_o \leq 4\pi$, rewrite these equations as a second order equation solely in terms of the pressure P (remove U), and thereby find the frequency domain solutions $P_{\pm}(x,s)$ for the conical horn equation.

[If we let $P_x \equiv \partial P / \partial x$ (i.e., the partial with respect to space) then

$$P_x + \mathcal{Z}V = 0 \tag{11}$$

and

$$V_x + \mathcal{Y}P = 0. \tag{12}$$

Taking the partial wrt x of the first equation, and then using the second, gives

$$P_{xx} + \mathcal{Z}_x U + \mathcal{Z} U_x = P_{xx} - \frac{\mathcal{Z}_x}{\mathcal{Z}} P_x - \mathcal{Z} \mathcal{Y} P = 0.$$
(13)

Using the relation

$$\mathcal{Z}_x/\mathcal{Z} = \frac{d}{dx}\ln\mathcal{Z} = -\frac{d}{dx}\ln A(x), \qquad (14)$$

with $A = A_o(x/x_o)^2$, we find

$$P_{xx} + \frac{2}{x}P = \frac{s^2}{c^2}P.$$
 (15)

Just as it was important to replace real frequency ω with the Laplace frequency s, since the roots are typically complex, using the same reasoning, we replace the "real wave number" $k = \omega/c$ with a *complex wave number* as $\kappa(s)$. Only in the case of non-dispersive waves (e.g., plane waves) is $\kappa(s) = s/c$.

Since the wave equation is

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 P}{\partial x^2} \tag{16}$$

by inspection we see that $\omega^2/c^2 = \mathcal{ZY}$, which results in the final formula for the speed of sound.

3. (3pts) Assuming an exponential area function

$$A(x) = A_0 e^{2mx}$$

(m is a positive constant, called the horn flair parameter), derive the exponential horn equation for the pressure.

$$\frac{\partial^2 p(x,t)}{\partial x^2} + 2m \frac{\partial p(x,t)}{\partial x} = \frac{1}{c^2} \frac{\partial^2 p(x,t)}{\partial^2 t}$$
(17)

[Starting from the basic definitions with $A(x) = A_0 e^{2mx}$ along with the basic equation for a horn, explicitly write out the two equations

$$\frac{dP}{dx} + s\frac{\rho_o}{A_0}e^{-2mx}V = 0 \tag{18}$$

$$\frac{dV}{dx} + s \frac{A_0 e^{2mx}}{\gamma P_o} P = 0 \tag{19}$$

Next solve for the pressure (remove V):

$$\frac{d^2P}{dx^2} + s\frac{\rho_o}{A_0} \left(e^{-2mx} \frac{dV}{dx} + V \frac{d}{dx} e^{-2mx} \right) = 0$$
(20)

or

$$\frac{d^2P}{dx^2} + s\frac{\rho_o}{A(x)}\left(\frac{dV}{dx} - 2mV\right) = 0.$$
(21)

Going back to the basic equations, again removing V

$$\frac{d^2P}{dx^2} + s\frac{\rho_o}{A(x)} \left(-s\frac{A(x)}{\gamma P_o}P + 2m\frac{A(x)}{s\rho_o}\frac{dP}{dx} \right) = 0$$
(22)

which simplifies to the requested result

$$\frac{d^2P}{dx^2} + 2m\frac{dP}{dx} - s^2\frac{\rho_o}{\gamma P_o}P = 0.$$
(23)

since $-s^2 = \omega^2$ and $\gamma P_o = \rho_o c^2$.

]

4. (4pts) In general the soluton to a wave equation is of the form

$$p(x,t) = P^+(\kappa)e^{\kappa(s)x}e^{st} + P^-(\kappa)e^{-\kappa(s)x}e^{st}$$

where $s = \sigma + j\omega$ is the Laplace frequency and $\kappa(s)$ is the complex "wave number."

- (a) What is $\kappa(s)$ for the conical horn? $[\kappa(s) = \pm s/c.]$
- (b) What is the significance of $\kappa(s)$? [It is called the dispersion relation of the differential equation, that relates the wavelength to the frequency.]
- (c) Why is it a function of s? [In general the wavelength is a function of frequency. This is a generalization of the plane wave relation $\lambda f = c$, or $\kappa(s) = s/c$.]
- (d) What is the role of $P^{\pm}(\kappa, s)$? [These are aplitudes of the forward and reverse traveling waves, that are to be determined by applying boundary conditions to the solution.]

5 Reflectance

- 1. (6pts) A tube transmission line with characteristic impedance z_0 and length L is terminated in a load impedance $Z_L(s)$.
 - (a) Starting with the definition of the impedance

$$Z(s,x) \equiv \frac{P(s,x)}{U(s,x)},$$

show that the reflectance

$$R(s,x) = \frac{Z(s,x) - z_0}{Z(s,x) + z_0}$$

You should express the pressure and velocity are sums of the forward and backward traveling waves $(P = P^+ + P^-, U = U^+ - U^-)$. What is z_0 in terms of P^\pm and U^\pm ? [Starting from the definition of the impedance (at any point x),

$$Z(x,s) = \frac{P(x)}{U(x)} = \frac{P^{+} + P^{-}}{U^{+} - U^{-}} = \frac{P^{+}}{U^{+}} \left(\frac{1+\Gamma}{1-\Gamma}\right) = z_{o} \frac{1+\Gamma}{1-\Gamma}$$

- (b) Find the formula for the reflectance at the load, $R(s,x)|_{x=L} = R(s,L)$ for
 - i. $Z_L(s) = r [\text{Nt-s/m}^5]$ ii. $Z_L(s) = 1/sC [\text{Nt-s/m}^5]$ iii. $Z_L(s) = r ||sM [\text{Nt-s/m}^5] [\text{Let } Z_L = \frac{rsM}{r+sM} \text{ then}$ $R = \frac{rsM - z_0(r+sM)}{rsM + z_0(r+sM)} = \frac{(r-z_0)Ms - z_0r}{(r+z_0)Ms + z_0r}.$ (24)

- 2. (2pts) For the transmission line described in the previous problem, let L = 1, $Z_L(s) = 1$ and $z_0 = 2$.
 - (a) Find the frequency domain reflectance R(s, 0) at x = 0.
 - (b) Find the time-domain reflectance r(t,0) at x = 0. $[r(t) = (1-2)/(1+2)\delta(t-2L/c) = -\delta(t-2)/3]$
- 3. (3pts) Two transmission lines are in cascade, the first one having an area of 1 [cm²] and a second having an area of 2 [cm²], with lengths L_1 and L_2 respectively, terminated with a resistor $r = \rho c/A$, where A = 2 [cm²]. Find R(x = 0, s). [Since the second line is terminated in its own impedance, it is just a resistor at its input, which makes the problem very simple. As a result

$$R(s) = \frac{1/1 - 1/2}{1/1 + 1/2} e^{-s2L_1/c} = 1/3e^{-s2L_1/c},$$
(25)

where L_1 is the length of the first TL. Note that if the line were not matched at the end, the story would be very different.

- 4. (3pts) What is the inverse Laplace transform of
 - (a) H(s) = 1/(s+1)? Find h(t). $[h(t) = e^{-t}U(t)]$ (b) H(s) = s/(s+1)? $[H(s) = 1-1/(s+1) \leftrightarrow h(t) = \frac{d}{dt}e^{-t}U(t) = \delta(t) - e^{-t}U(t)]$

6 Model of the middle ear

As shown in the figure, the free field sound pressure, defined as $P_0(\omega)$ acts as a source in series with the radiation resistance R_{rad} . The total radiation impedance $Z_{rad}(s, A_{rad})$ is a combination of the resistance and a reactive component L_{rad} , which represents the local stored field. The two impedances are in parallel

$$Z_{rad}(s) = sL_{rad}R_{rad}/(sL_{rad} + R_{rad}) = 1/Y_{rad}(s).$$

where $s = \sigma + j\omega$ is the Laplace complex frequency variable. The radiation admittance for a sphere is

$$Y_{rad} = 1/Z_{rad} = \frac{A_{rad}}{sr_c\rho_o} + \frac{A_{rad}}{\rho_o c} = \frac{1}{sL_{rad}} + \frac{1}{R_{rad}}$$

where r_c is the radius of the sphere and A_{rad} is the *effective area* of the radiation.



Figure 3: Model of the ear canal, terminated by the radiation impedance $Z_{rad}(s)$ at the tragus (x = 0), and by the eardrum and cochlea at x = L.

On the left we terminate the ear in a radiation impedance

$$Z_{rad}(s) = \frac{sL_{rad}R_{rad}}{sL_{rad} + R_{rad}}.$$

At the cochlear end we terminate the line with an impedance

$$Z_c = R_c + 1/(sC_{al}),$$

where R_c is the cochlear impedance and C_{al} is the stiffness of the annular ligament, which is the ligament that holds the stapes in the oval window. The cochlear resistance (R_c) is assumed to be twice the characteristic impedance of the ear canal.

To Do:

1. (3pts) Find the formula for the reflection coefficient at x = L. Sketch the magnitude |R(s,L)| as a function of frequency. *Hint: You should label any constants, or set them equal to 1 for plotting.*

$$\Gamma(L,s) \equiv \frac{U_{-}(L,s)}{U_{+}(L,s)}.$$

In this case the load impedance is $Z_c = R_c + 1/(sC_{al})$, thus

$$R(L,s) = \frac{R_c + 1/(sC_{al}) - z_0}{R_c + 1/(sC_{al}) + z_0}.$$

1

2. (3pts) Find the formula for the reflection coefficient at x = 0 (looking out towards the radiation impedance). Sketch its magnitude as a function of frequency.

$$\Gamma(0,s) \equiv \frac{U_+(0,s)}{U_-(0,s)}.$$

In this case the load is Z_{rad} .

- 3. (2pts) Consider the radiation impedance looking out the ear canal, Y_{rad} .
 - (a) What is the frequency for which its real and imaginary parts are equal (in terms of L_{rad} and R_{rad})? [

$$\frac{1}{2\pi f_0 L_{rad}} = \frac{1}{R_{rad}}.$$

- (b) Describe the dependence of Y_{rad} on L_{rad} and R_{rad} below and above this frequency. $[R_{rad} \text{ dominates at high frequencies above } f_0, L_{rad} \text{ dominates at low frequencies below } f_0.$
- 4. (3pts) Find the formula for the input impedance Z(0,s) of the middle ear at the entrance of the ear canal, when the cochlea is "blocked" ($Z_c = \infty$ or R(L,s) = 1)?

[When the end of the acoustic line is blocked there is a "short" across the end, namely the velocity (current) is zero. The reflectance at x = 0 is a delayed version of the reflectance at the cochlea (x = L), thus $R(0,s) = -e^{-j\omega^2 L/c}$, which has an inverse Fourier transform of $r(0,t) = -\delta(t-2L/c)$. Make sure you understand why this is! Do you understand where the delay is coming from?

It follows that the impedance is

$$Z(0,s) = z_0 \frac{1 + R(0,s)}{1 - R(0,s)} = z_0 \frac{1 + e^{-s2L/c}}{1 - e^{-s2L/c}}.$$

A little algebra and we find

$$Z(0,s) = jz_0 \frac{\cos(sL/c)}{\sin(sL/c)} = j\frac{\rho_o c}{A}\cot(\omega L/c) = z_0 \coth(sL/c).$$

This may be written in the time domain by a Taylor series, and it is a train of impulses spaced 2L/c apart. In other words, $R(0,s) \leftrightarrow r(x=0,t) = \delta(t-2L/c)$ is the same as impedance $Z(\omega) = -j\frac{\rho_o c}{A}\cot(\omega L/c)$.