ECE 403 **ExamII – Version 1.00 April 18, 2017** **Spring 2017**

Univ. of Illinois **Tue, April 18, 2017; 7-10PM** **Prof. Allen**

Topics: Acoustics, Transforms, Two-port networks

Instructions:

- If you need more space for calculation, use the back of any page.
- NO Cell phones.
- NO Calculators
- Note that each problem has points assigned. More points means "harder."

You may open the exam at 7:00 PM; You must close it by 9:00 PM.

NAME: ________________________ NETID: ________________________

I will not share information during this exam,
and I will discretely report others if I observe
cheating: ________________________ (sign here)

#	value	score
1	15	
2	7	
3	16	
4	17	
5	15	
6	15	
$\sum$	85	

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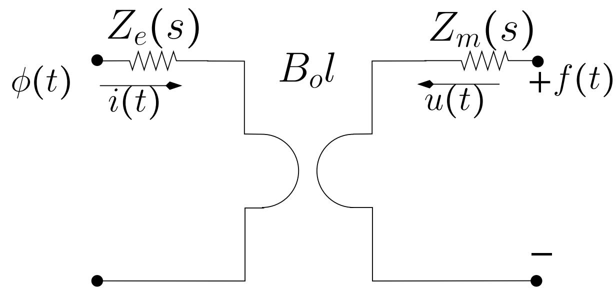

Figure 1: *The Hunt model is composed of an electrical impedance $Z_e(s)$, a Gyrator having parameter $T = lB_o$, and a mechanical impedance $Z_m(s)$. The electrical side is driven by a voltage $\phi(t)$ and a current $i(t)$, while the mechanical side produces a force $f(t)$ and a particle velocity $u(t)$.*

1 (15pts) Transducer Thevenin Parameters

1. **(3pts)** Find the ABCD (Transmission) $\mathbf{T}(s)$ matrix for the Hunt transducer model in Figure 1

$$\begin{bmatrix} \Phi(\omega) \\ I(\omega) \end{bmatrix} = \mathbf{T(s)} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix}.$$

[The definition of the Transmission matrix is

$$\begin{bmatrix} \Phi(\omega) \\ I(\omega) \end{bmatrix} = \begin{bmatrix} 1 & Z_e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & lB_o \\ 1/lB_o & 0 \end{bmatrix} \begin{bmatrix} 1 & Z_m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix} = \frac{1}{T} \begin{bmatrix} Z_e & \Delta_Z \\ 1 & Z_m \end{bmatrix} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix}$$

where Δ_Z is the determinent of the impedance $\mathbf{Z}(s)$ matrix.]

2. **(2pts)** Find the impedance matrix $\mathbf{Z}(s)$.

$$\begin{bmatrix} \Phi(\omega) \\ F(\omega) \end{bmatrix} = \mathbf{Z}(s) \begin{bmatrix} I(\omega) \\ U(\omega) \end{bmatrix}.$$

[This may be found via the table of relations between $\mathbf{T}$ and $\mathbf{Z}$ (VanValkenberg62). The answer is

$$\begin{bmatrix} \Phi(\omega) \\ F(\omega) \end{bmatrix} = \begin{bmatrix} Z_e & -T_o \\ T_o & Z_m \end{bmatrix} \begin{bmatrix} I(\omega) \\ U(\omega) \end{bmatrix}$$

where $T_o = lB_o$, with l [m] the length of the wire and B_0 [Web] the strength of the magnet [Wb/m^2]. $T_0 = z_{21} = -z_{12}$ due to the antireciprocal property.]

3. **(2pts)** Compute Δ_Z and Δ_T. What is the significance of each of these determinants? [$\Delta_Z = Z_e(s)Z_m(s) + T_o^2$ and $\Delta_T = -1$. If $\Delta_T = -1$ the system is 'anti-reciprocal,' and if $\Delta_T = +1$ the system is 'reciprocal.']

4. **(4pts)** Define, and then find the Thevenin equivalent force $F_{\text{thev}}(s)$ [N], given an input voltage Φ_o. *Note: If you are not sure of your answers to the previous parts, you can solve this problem in terms of an arbitrary ABCD matrix.* [The Thevenin equivalent

force is 'open circuit' force (analogous to voltage). Therefore we may read it off the top row of the ABCD matrix equation:

$$\begin{bmatrix} \Phi_o \\ I(\omega) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix} = \frac{1}{T} \begin{bmatrix} Z_e & \Delta_Z \\ 1 & Z_m \end{bmatrix} \begin{bmatrix} F(\omega) \\ -U(\omega) \end{bmatrix}$$

giving the equation

$$\Phi_o = \mathcal{A}F_{thev} - \mathcal{B}U|_{U=0} = \mathcal{A}F_{thev}$$

therefore

$$\frac{F_{thev}(s)}{\Phi_o} = \frac{T_o}{Z_e(s)}$$

]

5. **(4pts)** Define, and then find the Thevenin equivalent mechanical impedance in mechanical Ohms, $Z_{\text{thev}}(s)$ [Ω]. [The Thevenin impedance is the equivalent impedance looking into the right side of the loudspeaker model, with the voltage source shorted ($\Phi(\omega) = 0$). Setting $\Phi = 0$ and taking the ratio $Z = F/U$, we find the Thevenin impedance:

$$Z_{thev}(s) = \frac{F}{U}\bigg|_{\Phi=0} = \frac{\mathcal{B}}{\mathcal{A}} = \frac{Z_e(s)Z_m(s) + T_o^2}{Z_e(s)}$$

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2 (7pts) Nyquist Theorem on Thermal noise

A stub of transmission line having characteristic impedance z_0 is terminated in each end with this Thevenin model, with $R = z_0$. Then at $t = 0$, the resistance is short or open circuited. This setup is shown in the figure.

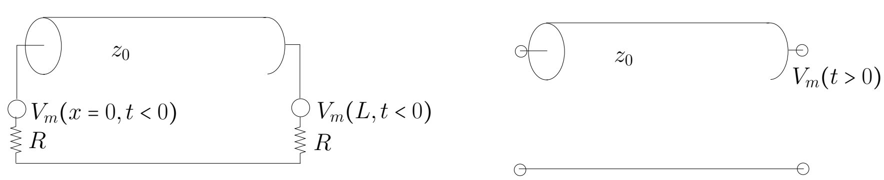

Figure 2: *On the left an ideal lossless transmission line, driven by two resistors. Each resistor has a noise source in series with it, that generates a voltage equal to the thermal noise in the resistor. At time $t = 0$ the resistors are switched out, leaving the voltage in the transmission line frozen in place.*

The transmission line stores the voltage as a function of x at $t = 0$ once the switch is opened removing the resistors (and also the Thevenin source). At that point voltage $V_m(L, t)$ becomes frozen.

1. **(1pt)** Discuss what happens after $t = 0$. [The noise traveling in the TL is frozen as it is reflected from the two ends, with no loss. As a result this frozen noise is periodic with a period of the round-trip time of the line, which is $T = 2L/c$. Every periodic signal may be expanded in a Fourier series, which has energy at the frequencies $f_k = k/T = kc/2L$ ($k = 0, 1, 2 \ldots$). The Fourier coefficients will not be equal due to the random nature of the noise. On average, each spectral line will have a power of kT, but since the noise

is frozen, the long term spectral values will not apply. Each time the experiment is repeated, there is a randomness in the amplitudes.]

2. **(2pt)** What is the fundamental period of the noise voltage, $v_m(L, t > 0)$? [The period is the round trip delay $T = 2L/c \approx 58$ [ms].]

3. **(2pts)** Every periodic signal has a *Fourier series*. If the period is T [s], what can you say about the Fourier series frequencies? [The Fourier frequencies are $\omega_k = 2\pi k/T$ [rad].]

4. **(2pts)** The noise spectrum will differ depending on the time at which the switch is opened. Why? [While the long-term average power spectrum of each line is kT, a frozen piece of noise will not be equal to the long term average. Rather it will be a snap-shot of the power at that time.]

3 (16pts) Hilbert transform

Analyze the real, causal impulse response

$$h(t) = e^{-t/\tau_0} u(t),$$

with $\tau_0 = 10$ [ms], in terms of its Hilbert transform (integral) relations. *Note: Use the notation $h(t) \leftrightarrow H(s)$ and $H(\omega) = H(s)|_{s=j\omega}$.*

1. **(2pts)** Find $H(s)$, the Laplace transform of $h(t)$. [Let $a = 1/\tau_0$. Then $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$.]

2. **(1pt)** Where are the poles of $H(s) \leftrightarrow e^{-t/\tau_0}u(t)$? [Let $a = 1/\tau_0$. Since $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$ $H(s)$ has a simple real pole at $s = -1/\tau_0$.]

3. **(4pts)** Find and *sketch* the real and imaginary parts of $H(\omega) \equiv H(s)|_{s=j\omega}$. [First rationalize the denominator:

$$H(s)\Big|_{s=j\omega} = \frac{s^* + a}{(s+a)(s^*+a)}\Bigg|_{s=j\omega} = \frac{-j\omega + a}{\omega^2 + a^2}.$$

Next take the real $\Re$ and imaginary $\Im$ parts:

$$\Re H(\omega) = a/(a^2 + \omega^2), \qquad \Im H(\omega) = -\omega/(a^2 + \omega^2).$$

The real part is constant below the cutoff (resonant) frequency $\omega = a$, and goes at -12 dB/oct (i.e. proportional to $1/\omega^2$) above the cutoff. The imaginary part is bandpass with ±6 dB/Oct (proportional to $1/\omega$) above and below the resonance frequency.]

4. **(1pt)** Write out the even $h_e(t) = h(t) + h(-t)$ and odd $h_o(t) = h(t) - h(-t)$ functions. Describe why these equations are symmetric and antisymmetric functions. [$2h_e(t) = h(t) + h(-t) \equiv e^{-at}u(t) + e^{at}u(-t)$, while $2h_o(t) = h(t) - h(-t) \equiv e^{-at}u(t) - e^{at}u(-t)$. It trivially follows that $h(t) = h_e(t) + h_o(t)$.]

5. **(4pts)** Find the Fourier transforms of $h_e(t) \leftrightarrow H_e(\omega)$ and $h_o(t) \leftrightarrow H_o(\omega)$. How do they relate to the real and imaginary parts of $H(\omega)$?

[Since $h(-t) \leftrightarrow H^*(\omega)$, a symmetric time function is real in the frequency domain,

$$2h_e(t) = h(t) + h(-t) \leftrightarrow H(\omega) + H^*(\omega) = 2\Re H(\omega),$$

thus $h_e(t) \leftrightarrow \Re H(\omega)$. In a similar fashion, an antisymmetric time function is pure imaginary

$$h_o(t) \leftrightarrow j\Im H(\omega).$$

Again with $a \equiv 1/\tau_0$:

$$H_e(\omega) = \Re H(\omega) = \frac{a}{\omega^2 + a^2},$$

$$H_o(\omega) = j\Im H(\omega) = \frac{-j\omega}{\omega^2 + a^2},$$

thus

$$H(\omega) = H_e(\omega) + H_o(\omega) \leftrightarrow h(t) = h_e(t) + h_o(t).$$

The inverse Fourier transform of $H_o(\omega)$ is zero at $t = 0$, which makes it very different from the inverse Laplace transform, which is not defined at $t = 0$.]

6. **(4pts)** Find the Hilbert (integral) relations between $H_r \equiv \Re H(\omega)$ (real part) and $H_i \equiv \Im H(\omega)$ (imag part) of $H(\omega)$. *Hint: These integrals come from a frequency-domain convolution.*

[

The derivation of the requested integrals may be found from

$$h(t) = h(t)u(t) \tag{1}$$

(note this is not always true at $t = 0$) which after a FT, results in

$$H(\omega) = \frac{1}{2\pi} H(\omega) \star \left(\pi\delta(\omega) + \frac{1}{j\omega}\right), \tag{2}$$

which may be rewritten as

$$H(\omega) = \frac{1}{\pi j\omega} \star H(\omega). \tag{3}$$

Taking the $\Re$ and $\Im$ parts gives

$$H_\Re(\omega) = \frac{1}{\pi\omega} \star H_\Im(\omega) \tag{4}$$

$$H_\Im(\omega) = \frac{-1}{\pi\omega} \star H_\Im(\omega). \tag{5}$$

The final relations are [Papoulis (1977), *Signal Analysis*, McGraw Hill, page 251]

$$H_r(\omega) = \frac{1}{\pi} \int \frac{H_i(\omega')}{\omega - \omega'} d\omega' \qquad \text{and} \qquad H_i(\omega) = -\frac{1}{\pi} \int \frac{H_r(\omega')}{\omega - \omega'} d\omega' \tag{6}$$

]

4 (17pts) Wave equation

4.1 History of the wave equation

1. **(1pt)** What year did d'Alembert derive his solution to the wave equation? [d'Alembert first proved this in 1747.]

2. **(1pt)** What is the form of D'Alembert's solution? [$f(t-x/c)+g(t+x/c)$ or in 3D spherical coordinates, $f(t-r/c)/r+g(t+r/c)/r$]

3. **(1pt)** Who was the first person to calculate the speed of sound, and what was the result? [Newton did this in 1648. His formula was in error due to the dynamic stiffness of air, which is γP_o. His result was too small by the factor of $\sqrt{1.4}$.]

4.2 The Webster horn equation:

In the time domain, in 2x2 matrix form, the Webster horn equation is given by

$$\frac{\partial}{\partial x}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix} = -\begin{bmatrix} 0 & \frac{\rho_o}{A(x)} \\ \frac{A(x)}{\gamma P_o} & 0 \end{bmatrix} \frac{\partial}{\partial t}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix}. \tag{7}$$

where $p(x,t) \leftrightarrow \mathcal{P}(x,\omega)$ is the pressure and $\nu(x,t) \leftrightarrow \mathcal{V}(x,\omega)$ is the *volume velocity*.

1. **(2pts)** Transform the Horn equation to the frequency domain. [

$$\frac{d}{dx}\begin{bmatrix} P(x,\omega) \\ V(x,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & Z_s(x,s) \\ Y_s(x,s) & 0 \end{bmatrix}\begin{bmatrix} P(x,\omega) \\ V(x,\omega). \end{bmatrix} \tag{8}$$

Here we use the complex *Laplace frequency* s when referring to the *per-unit impedance*

$$Z_s(x,s) \equiv s\frac{\rho_o}{A(x)} = sM(x) \tag{9}$$

and *per-unit admittance*

$$Y_s(x,s) \equiv s\frac{A(x)}{\gamma P_o} = sC(x), \tag{10}$$

where $M(x) = \rho_o/A(x)$ is the horn's *per-unit-length mass*, and $C(x) = A(x)/\gamma P_o$ *per-unit-length compliance*, to remind ourselves that these functions must be *causal*, and except at their poles, *analytic* in s.]

2. **(4pts)** Assuming a *conical horn*, having area $A(x) = A_o(x/x_o)^2$ with $A_o \le 4\pi$, rewrite Equation 1 as a second order equation solely in terms of the pressure $\mathcal{P}(x,\omega)$ (remove $\mathcal{V}$), and thereby find the frequency domain solutions $P_\pm(x,s)$ for the conical horn equation.

[If we let $P_x \equiv \partial P/\partial x$ (i.e., the partial with respect to space) then

$$P_x + \mathcal{Z}V = 0 \tag{11}$$

and

$$V_x + \mathcal{Y}P = 0. \tag{12}$$

Taking the partial wrt x of the first equation, and then using the second, gives

$$P_{xx} + \mathcal{Z}_x U + \mathcal{Z} U_x = P_{xx} - \frac{\mathcal{Z}_x}{\mathcal{Z}} P_x - \mathcal{Z}\mathcal{Y} P = 0. \tag{13}$$

Using the relation

$$\mathcal{Z}_x / \mathcal{Z} = \frac{d}{dx} \ln \mathcal{Z} = -\frac{d}{dx} \ln A(x), \tag{14}$$

with $A = A_o (x/x_o)^2$, we find

$$P_{xx} + \frac{2}{x} P = \frac{s^2}{c^2} P. \tag{15}$$

Just as it was important to replace real frequency ω with the Laplace frequency s, since the roots are typically complex, using the same reasoning, we replace the "real wave number" $k = \omega/c$ with a *complex wave number* as $\kappa(s)$. Only in the case of non-dispersive waves (e.g., plane waves) is $\kappa(s) = s/c$.

Since the wave equation is

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 P}{\partial x^2} \tag{16}$$

by inspection we see that $\omega^2/c^2 = \mathcal{Z}\mathcal{Y}$, which results in the final formula for the speed of sound.]

3. **(4pts)** Assuming an exponential area function

$$A(x) = A_0 e^{2mx}$$

(m is a positive constant, called the *horn flair parameter*), derive the exponential horn equation for the pressure.

$$\frac{\partial^2 p(x,t)}{\partial x^2} + 2m \frac{\partial p(x,t)}{\partial x} = \frac{1}{c^2} \frac{\partial^2 p(x,t)}{\partial^2 t} \tag{17}$$

[Starting from the basic definitions with $A(x) = A_0 e^{2mx}$ along with the basic equation for a horn, explicitly write out the two equations

$$\frac{dP}{dx} + s \frac{\rho_o}{A_0} e^{-2mx} V = 0 \tag{18}$$

$$\frac{dV}{dx} + s \frac{A_0 e^{2mx}}{\gamma P_o} P = 0 \tag{19}$$

Next solve for the pressure (remove V):

$$\frac{d^2 P}{dx^2} + s \frac{\rho_o}{A_0} \left(e^{-2mx} \frac{dV}{dx} + V \frac{d}{dx} e^{-2mx} \right) = 0 \tag{20}$$

or

$$\frac{d^2 P}{dx^2} + s \frac{\rho_o}{A(x)} \left(\frac{dV}{dx} - 2mV \right) = 0. \tag{21}$$

Going back to the basic equations, again removing V

$$\frac{d^2 P}{dx^2} + s \frac{\rho_o}{A(x)} \left(-s \frac{A(x)}{\gamma P_o} P + 2m \frac{A(x)}{s\rho_o} \frac{dP}{dx} \right) = 0 \tag{22}$$

which simplifies to the requested result

$$\frac{d^2P}{dx^2} + 2m\frac{dP}{dx} - s^2\frac{\rho_o}{\gamma P_o}P = 0. \tag{23}$$

since $-s^2 = \omega^2$ and $\gamma P_o = \rho_o c^2$.

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4. **(4pts)** In general the solution to a wave equation is of the form

$$p(x,t) = P^+(\kappa)e^{\kappa(s)x}e^{st} + P^-(\kappa)e^{-\kappa(s)x}e^{st}$$

where $s = \sigma + \jmath\omega$ is the Laplace frequency and $\kappa(s)$ is the complex "wave number."

(a) What is $\kappa(s)$ for the conical horn? [$\kappa(s) = \pm s/c$.]

(b) What is the significance of $\kappa(s)$? [It is called the dispersion relation of the differential equation, that relates the wavelength to the frequency.]

(c) Why is it a function of s? [In general the wavelength is a function of frequency. This is a generalization of the plane wave relation $\lambda f = c$, or $\kappa(s) = s/c$.]

(d) What is the role of $P^\pm(\kappa, s)$? [These are aplitudes of the forward and reverse traveling waves, that are to be determined by applying boundary conditions to the solution.]

5 (15pts) Reflectance

1. **(6pts)** A tube transmission line with characteristic impedance z_0 and length L is terminated in a load impedance $Z_L(s)$.

(a) The reflectance at any location x along the tube transmission line is

$$\Gamma(x,s) = \frac{Z(x,s) - z_0}{Z(x,s) + z_0}.$$

Starting with this formula, show that the impedance

$$Z(x,s) \equiv \frac{\mathcal{P}(x,s)}{\mathcal{V}(x,s)}.$$

Hint: What is z_0 in terms of the forward and reverse traveling waves $\mathcal{P}^\pm$ and $\mathcal{V}^\pm$?
[Starting from the definition of the impedance (at any point x),

$$Z(x,s) = \frac{\mathcal{P}(x)}{\mathcal{V}(x)} = \frac{\mathcal{P}^+ + \mathcal{P}^-}{\mathcal{P}^+ - \mathcal{V}^-} = \frac{\mathcal{P}^+}{\mathcal{V}^+}\left(\frac{1+\Gamma}{1-\Gamma}\right) = z_o\frac{1+\Gamma}{1-\Gamma}$$

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(b) Find the formula for the reflectance at the load (simplify your answer, if applicable), $\Gamma(x,s)|_{x=L} = \Gamma(L,s) = \Gamma_L(s)$ for

i. $Z_L(s) = r$ [Nt-s/m^5] [Let $Z_L = \frac{rsM}{r+sM}$ then

$$\Gamma_L(s) = \frac{r - z_0}{r + z_0} \tag{24}$$

]

ii. $Z_L(s) = 1/sC$ [Nt-s/m^5] [Let $Z_L = \frac{rsM}{r+sM}$ then

$$\Gamma_L(s) = \frac{1/sC - z_0}{1/sC + z_0} = \frac{(1 - z_0 sC}{1 + z_0 sC}. \tag{25}$$

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iii. $Z_L(s) = r \| sM$ [Nt-s/m^5] [Let $Z_L = \frac{rsM}{r+sM}$ then

$$\Gamma_L(s) = \frac{rsM - z_0(r + sM)}{rsM + z_0(r + sM)} = \frac{(r - z_0)Ms - z_0 r}{(r + z_0)Ms + z_0 r}. \tag{26}$$

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2. **(3pts)** For the transmission line described in the previous problem, let $L = 1$, $Z_L(s) = 1$ and $z_0 = 2$.

   (a) Find the frequency domain reflectance $\Gamma(0, s)$ at $x = 0$. [$r(t) = ((1-2)/(1+2))e^{-j]\omega 2L/c}$]

   (b) Find the time-domain reflectance $\gamma(0, t) \leftrightarrow \Gamma(0, s)$ at $x = 0$. [$r(t) = (1-2)/(1+2)\delta(t - 2L/c) = -\delta(t-2)/3$]

3. **(3pts)** Two transmission lines are in cascade, the first one having an area of 1 [cm^2] and a second having an area of 2 [cm^2], with lengths L_1 and L_2 respectively, terminated with a resistor $r = \rho c/A$, where $A = 2$ [cm^2]. Find $R(x = 0, s)$. [Since the second line is terminated in its own impedance, it is just a resistor at its input, which makes the problem very simple. As a result

$$R(s) = \frac{1/1 - 1/2}{1/1 + 1/2} e^{-s2L_1/c} = 1/3 e^{-s2L_1/c}, \tag{27}$$

where L_1 is the length of the first TL. Note that if the line were not matched at the end, the story would be very different.]

4. **(3pts)** What is the inverse Laplace transform of

   (a) $H(s) = 1/(s+1)$? Find $h(t)$. [$h(t) = e^{-t}U(t)$]

   (b) $H(s) = s/(s+1)$? [H(s) = 1-1/(s+1) $\leftrightarrow h(t) = \frac{d}{dt}e^{-t}U(t) = \delta(t) - e^{-t}U(t)$]

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6 (15pts) Model of the middle ear

As shown in the figure, the free field sound pressure, defined as $P_0(\omega)$ acts as a source in series with the radiation resistance R_{rad}. The total radiation impedance $Z_{rad}(s)$ is a combination of the resistance and a reactive component L_{rad}, which represents the local stored field. The two impedances are in parallel

$$Z_{rad}(s) = sL_{rad}R_{rad}/(sL_{rad} + R_{rad}) = 1/Y_{rad}(s).$$

where $s = \sigma + j\omega$ is the Laplace complex frequency variable. The *radiation admittance for a sphere* is

$$Y_{rad} = 1/Z_{rad} = \frac{A_{rad}}{sr_c\rho_o} + \frac{A_{rad}}{\rho_o c} = \frac{1}{sL_{rad}} + \frac{1}{R_{rad}},$$

where r_c is the radius of the sphere and A_{rad} is the *effective area* of the radiation.

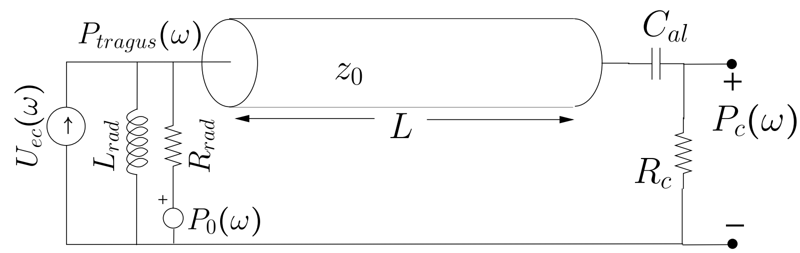

Figure 3: *Model of the ear canal, terminated by the radiation impedance $Z_{rad}(s)$ at the tragus ($x = 0$), and by the eardrum and cochlea at $x = L$.*

On the left we terminate the ear in a radiation impedance

$$Z_{rad}(s) = \frac{sL_{rad}R_{rad}}{sL_{rad} + R_{rad}}.$$

At the cochlear end we terminate the line with an impedance

$$Z_c = R_c + 1/(sC_{al}),$$

where R_c is the cochlear impedance and C_{al} is the stiffness of the annular ligament, which is the ligament that holds the stapes in the oval window. The cochlear resistance (R_c) is assumed to be twice the characteristic impedance of the ear canal.

To Do:

1. **(4pts)** Find the formula for the reflection coefficient at $x = L$. Sketch the magnitude $|R(L, s)|$ as a function of frequency. *Hint: You should label any constants, or set them equal to 1 for plotting.* [In this case the load impedance is $Z_c = R_c + 1/(sC_{al})$, thus

$$\Gamma_L(s) = \frac{R_c + 1/(sC_{al}) - z_0}{R_c + 1/(sC_{al}) + z_0}.$$

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2. **(4pts)** Find the formula for the reflection coefficient at $x = 0$ (looking out towards the radiation impedance). Sketch its magnitude as a function of frequency. [In this case the load is Z_{rad}.]

3. **(3pts)** Consider the radiation impedance looking out the ear canal, Y_{rad}.

   (a) What is the frequency for which its real and imaginary parts are equal (in terms of L_{rad} and R_{rad})? [

$$\frac{1}{2\pi f_0 L_{rad}} = \frac{1}{R_{rad}}.$$

   ]

   (b) Describe the dependence of Y_{rad} on L_{rad} and R_{rad} below and above this frequency. [R_{rad} dominates at high frequencies above f_0, L_{rad} dominates at low frequencies below f_0.]

4. **(4pts)** Find the formula for the input impedance $Z(0,s)$ of the middle ear at the entrance of the ear canal, when the cochlea is "blocked" ($Z_c = \infty$ or $\Gamma(L,s) = 1$)?

   [When the end of the acoustic line is blocked there is a "short" across the end, namely the velocity (current) is zero. The reflectance at $x = 0$ is a delayed version of the reflectance at the cochlea ($x = L$), thus $\Gamma(0,s) = -e^{-j\omega 2L/c}$, which has an inverse Fourier transform of $\gamma(0,t) = -\delta(t - 2L/c)$. Make sure you understand why this is! Do you understand where the delay is coming from?

   It follows that the impedance is

$$Z(0,s) = z_0 \frac{1 + \Gamma(0,s)}{1 - \Gamma(0,s)} = z_0 \frac{1 + e^{-s2L/c}}{1 - e^{-s2L/c}}.$$

   A little algebra and we find

$$Z(0,s) = jz_0 \frac{\cos(sL/c)}{\sin(sL/c)} = j\frac{\rho_o c}{A} \cot(\omega L/c) = z_0 \coth(sL/c).$$

   This may be written in the time domain by a Taylor series, and it is a train of impulses spaced $2L/c$ apart. In other words, $\Gamma(0,s) \leftrightarrow \gamma(x = 0, t) = \delta(t - 2L/c)$ is the same as impedance $Z(\omega) = -j\frac{\rho_o c}{A} \cot(\omega L/c)$.]