ECE 403 **HW c – Ver. 1.26 April 10, 2017** **Spring 2017**

Univ. of Illinois **2/21, Disc: 2/28, Due: 3/2** **Prof. Allen**

Topic of this homework: Loudspeaker Impedance; Analytic power series; Acoustic Signal processing Acoustics; Fourier Transform; Signal processing;

Deliverable: Show your work.

If you hand it in late, you will get zero credit (I will be handing out my solution at that time). You will only get credit for what you hand in. I want a paper copy, with your name on it. Please **no** files.doc.

No matter how limited your results, on the due date submit what ever you have. Some credit is better than NO credit.

Note: This homework will be discussed by the entire class on Disc: 2/28. You need to be there. Each person is to do there own final writeup, but obviously you can discuss it as much as you like between yourselves.

1 Model of a loudspeaker

1.1 ABCD model

The attached figure shows the equivalent circuit for an electro-dynamic earphone.

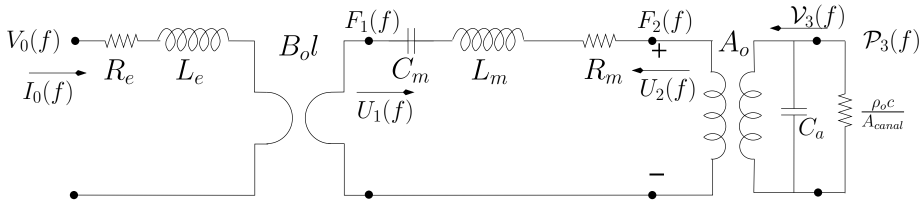


Figure 1: *Equivalent circuit for an earphone (Beranek and Mellow, 2012, p. 109). This model is simplified from Kim and Allen (2013). There are three sections, the electrical input (left), the mechanical response (center), and the acoustic output (right). The electrical input in in terms of the voltage $V_0(f)$ and current $I_0(f)$. There are two elements, the coil resistance R_e and its inductance L_e. The center section corresponds to the mechanical components, with a compliance C_m (compliance/stiffness), mass L_m and mechanical damping R_m. The mechanical force $F_2(f)$ and a velocity $U_2(f)$ are the input to the transformer which converts the force into a pressure. The diaphragm has an area A, which results in a pressure $\mathcal{P}_3(f) = F_2/A$, and a volume velocity $\mathcal{V}_3(f) = AU_2(f)$ at the right.*

This figure generates the two-port relationship (most of the following quantities are functions of frequency - we will show it explicitly in the first equation)

$$\begin{bmatrix} V_0(f) \\ I_0(f) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(f) & \mathcal{B}(f) \\ \mathcal{C}(f) & \mathcal{D}(f) \end{bmatrix} \begin{bmatrix} \mathcal{P}_3(f) \\ -\mathcal{V}_3(f) \end{bmatrix}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & (R_e + sL_e) \\ 0 & 1 \end{bmatrix}}_{electrical} \underbrace{\begin{bmatrix} 0 & G_o \\ \frac{1}{G_o} & 0 \end{bmatrix}}_{gyrator} \underbrace{\begin{bmatrix} 1 & \frac{1}{sC_m} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & sL_m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & R_m \\ 0 & 1 \end{bmatrix}}_{mechanical} \underbrace{\begin{bmatrix} A_o & 0 \\ 0 & \frac{1}{A_o} \end{bmatrix}}_{transformer} \underbrace{\begin{bmatrix} 1 & 0 \\ sC_a & 1 \end{bmatrix}}_{acoustical}$$

where the gyrator ($G_o = B_o l$) and the transformer[1] (A_o) are used to cross between modalities. The gyrator converts variables from electrical (voltage (force) = V_0, current (flow) = I_0) to mechanical

[1] This matrix is slightly different from Kim et al. (2013), where A is $1/area$.

(force = F_1, velocity (flow) = U_1). The transformer converts variables from mechanical (force = F_2, velocity (flow) = U_2) to acoustical (pressure (force) = $\mathcal{P}_3$, volume velocity (flow) = $\mathcal{V}_3$).

When analyzing this model, it may help to think of the ABCD matrix as

$$\begin{bmatrix} force_1 \\ flow_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} force_2 \\ -flow_2 \end{bmatrix}$$

which has a corresponding impedance matrix

$$\begin{bmatrix} force_1 \\ force_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} flow_1 \\ flow_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_T \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} flow_1 \\ flow_2 \end{bmatrix}.$$

In the figure, we show the load at the output of the speaker to be $Z_l = \rho c / A_{canal}$. The input impedance of a 2-port network with load Z_l is

$$Z_{in} = z_{11} - \frac{z_{12} z_{21}}{Z_l + z_{22}}$$

where the elements z_{ij} are the elements of the impedance matrix of the network.

In this homework assignment, you are asked to compute forces, flows, and transfer functions at different places in the model (denoted by subscripts 0, 1, 2, and 3). Therefore, you will consider T and Z matrices that represent subsections of the model.

Calculating transfer functions: In general,

$$force_1 = \mathcal{A} \times force_2 - \mathcal{B} \times flow_2.$$

Therefore, if either $force_2$ or $flow_2$ is set to zero (open or short circuit), you can calculate the ratio $force_2/force_1$ or $flow_2/force_1$.

Calculating forces and flows: To calculate forces and flows, you need to know how the loudspeaker is driven (driving voltage $V_0 = 1$ for this homework), and what the load is at the speaker output (in Figure 1, the load is $Z_l = \rho c / A_{canal}$). In this case

$$force_1 = V_0 = 1 \text{ [Volt]} = \mathcal{A} \times force_2 - \mathcal{B} \times flow_2.$$

To calculate $force_2$ or $flow_2$, you need an additional item of information

$$Z = \frac{force_2}{-flow_2},$$

where Z is the input impedance of the remainder of the model, looking to the right.

1.2 `MATLAB` code

```
% The following code generates a simplified model of the balanced armature
% receiver (small hearing aid speaker) from Kim et al. 2013. It does not
% include the semi-inductor or transmission line

NFT=1024; NF=1+NFT/2;%number of non-negative frequencies
Fmax=1e4; Fs=2*Fmax; Fmin=Fs/NFT;
f=0:Fmin:Fmax; %sweep frequency from 0 to Fmax [Hz] in steps of Fmin [Hz]
f = f(10:end); % we don't need the lowest frequencies

% define load condition
A_canal = pi*0.0075^2; % ear canal area m^2
```

```
% useful physical constants
rho = 1.2; % density of air kg/m^3
c = 342; % speed of sound m/s
P0 = 1E5; % atmospheric pressure
eta = 1.4; % ratio of specific heats

%Define basic parameters based on Kim and Allen
Re=195;
Le=9e-3; %electrical section
Lem = 52e-3;
LeTot = Le+Lem;
G = 7.5; %Gryator Bo l: Electrical to Mechanical transformation
Cm=1.25e-3; Lm=4.3e-6; Rm=3e-3; %Mechanical section
A = 2.4e-6; %Mechanical to Acoustical transformation (earphone port radius ~ 1 mm)
Ca = 4.3E-15; % acoustic volume
z0tx = 1e9; ltx = 1e-4; % Kim et al 2013 transmission line parameters
Vtx = ltx*(rho*c/z0tx); % volume of transmission line tube
Ctx = Vtx/(eta*P0); % capacitance of transmission line volume
CaTot = Ca+Ctx;

for k=1:length(f);
    s=2*pi*f(k)*1j; %Define complex frequency

    Te = [1 Re+s*LeTot;0 1]; %electrical
    Tem = [0 G; 1/G 0]; % gyrator electrical to mechanics
    Tm = [1 1./(s*Cm); 0 1]*[1 s*Lm; 0 1]*[1 Rm; 0 1]; % mechanical
    Tma = [A 0; 0 1/A]; % coupling of mechanics to acoustics
    Ta = [1 0; s*CaTot 1]; % acoustical

    % indices: 0= front end; 1 = left side of mechanical part (right of
    % gyrator); 2 = right side of mechanical part (left of transformer); 3
    % = left side of acoustical part (right side of transformer)
    T03(k,:,:)=Te*Tem*Tm*Tma*Ta; %avoids a nightmare of algebra
    T01(k,:,:)=Te*Tem;
    T02(k,:,:)=Te*Tem*Tm;
    T13(k,:,:)=Tm*Tma*Ta;
    T23(k,:,:)=Tma*Ta;

    % calculate Z matrices here to make calculations easier later
    Z03(k,1,1) = T03(k,1,1)/T03(k,2,1); % A/C
    Z03(k,1,2) = -1/T03(k,2,1); % det(T)/C where det(T) = -1 for anti-reciprocal system (gyrator)
    Z03(k,2,1) = 1/T03(k,2,1); % 1/C
    Z03(k,2,2) = T03(k,2,2)/T03(k,2,1); % D/C

    Z02(k,1,1) = T02(k,1,1)/T02(k,2,1); % A/C
    Z02(k,1,2) = -1/T02(k,2,1); % det(T)/C where det(T) = -1 for anti-reciprocal system (gyrator)
    Z02(k,2,1) = 1/T02(k,2,1); % 1/C
    Z02(k,2,2) = T02(k,2,2)/T02(k,2,1); % D/C

    Z13(k,1,1) = T13(k,1,1)/T13(k,2,1); % A/C
    Z13(k,1,2) = 1/T13(k,2,1); % det(T)/C where det(T) = 1 for recriprocal system
    Z13(k,2,1) = 1/T13(k,2,1); % 1/C
    Z13(k,2,2) = T13(k,2,2)/T13(k,2,1); % D/C

    Z23(k,1,1) = T23(k,1,1)/T23(k,2,1); % A/C
    Z23(k,1,2) = 1/T23(k,2,1); % det(T)/C where det(T) = 1 for reciprocal system
    Z23(k,2,1) = 1/T23(k,2,1); % 1/C
    Z23(k,2,2) = T23(k,2,2)/T23(k,2,1); % D/C

end %end for freq loop

% Solve for transfer functions:
```

```
I0divV0_U10 = T01(:,2,1)./T01(:,1,1);
% TO DO: Solve for other transfer functions

% Solve for input impedance looking to the right of solution point. Use
% ABCD matrix to model transmission between input (V=1) and solution point.
ZL = rho*c/A_canal; % ear canal
Zin2 = Z23(:,1,1) - Z23(:,1,2).*Z23(:,2,1)./(ZL + Z23(:,2,2));
Zin1 = Z13(:,1,1) - Z13(:,1,2).*Z13(:,2,1)./(ZL + Z13(:,2,2));
Zin0 = Z03(:,1,1) - Z03(:,1,2).*Z03(:,2,1)./(ZL + Z03(:,2,2));

% forces
V0 = ones(1,length(f)); % input 1 V
% TO DO: Calculate F1, F2, P3

% flows
I0 = 1./Zin0;
% TO DO: Calculate U1, U2, V3

% PLOT INPUT IMPEDANCE
figure('units','normalized','outerposition',[.05 .15 .7,.7])
subplot(121)
plot(f/1000, abs(Zin0)); hold on
set(gca,'xscale','log','yscale','log','xlim',[.2 10]);
title('|Z_{in}|')
xlabel('Frequency [kHz]')
subplot(122)
plot(f/1000, unwrap(angle(Zin0))/pi); hold on
set(gca,'xscale','log','xlim',[.2 10],'ylim',[-.6 .6]);
title('\angle Z_{in} [rad/pi]')
xlabel('Frequency [kHz]')
% TO DO: Calculate Zin with other load conditions, compare

% PLOT TRANSFER FUNCTIONS
figure('units','normalized','outerposition',[.05 .15 .7,.7])
subplot(231) %(a)
plot(f/1000, abs(I0divV0_U10));
set(gca,'xscale','log','xlim',[.2 10]); title('(a) |I_0/V_0|_{U_1=0}')
xlabel('Frequency [kHz]')
% TO DO: add subplots for (b)-(f)

% PLOT FORCES AND FLOWS
[val,i1k] = min(abs(f-1000)); % find 1kHz index for normalizing curves
figure('units','normalized','outerposition',[.05 .15 .7,.7])
subplot(121)
plot(f/1000, abs(V0)/abs(V0(i1k)),'k','linewidth',2); hold on
legend('|V_0|')
xlabel('Frequency [kHz]'); set(gca,'xscale','log','xlim',[.2 10]); title('Generalized Forces')
% TO DO: Add F1, F2, P3
subplot(122)
plot(f/1000, abs(I0)/abs(I0(i1k)),'k','linewidth',2); hold on
legend('|I_0|')
xlabel('Frequency [kHz]'); set(gca,'xscale','log','xlim',[.2 10]); title('Generalized Flows')
% TO DO: Add U1, U2, V3
```

To do:

1. Explain how this code works. For full credit you must be clear. Identify the key lines of code, and explain how they work.
2. Plot the magnitude of each of the following transfer functions:
   (a) The current into the gyrator $I_0(f)/V_0(f)$ with the speaker motor blocked ($U_1 = 0$)

(b) $F_1(f)/V_0(f)$ with $U_1 = 0$ (blocked)

(c) $U_1(f)/V_0(f)$ with $F_1(f) = 0$ (open circuit)

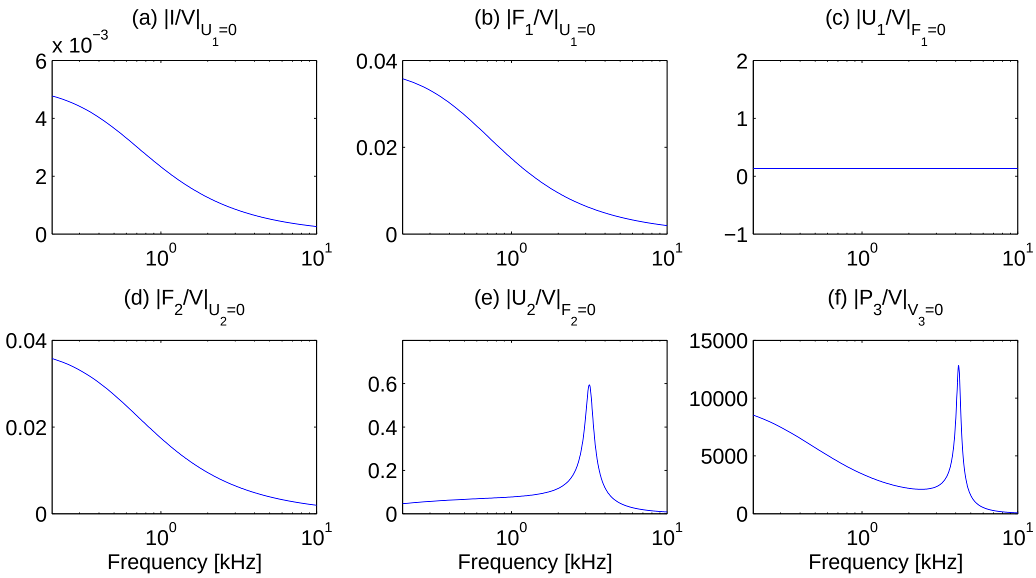
(d) $F_2(f)/V_0(f)$ with $U_2 = 0$

(e) $U_2(f)/V_0(f)$ with $F_2(f) = 0$

(f) $\mathcal{P}_3(f)/V_0(f)$ with $\mathcal{V}_3(f) = 0$, expressed in Pascals [Pa].

In each of the above response, discuss the bandwidth and properties of each transfer function in terms of the circuit elements. Solution: You may find the transfer functions using the following equations:

```
I0divV0_U10 = T01(:,2,1)./T01(:,1,1);
F1divV0_U10 = 1./T01(:,1,1);
U1divV0_F10 = -1./T01(:,1,2);
F2divV0_U20 = 1./T02(:,1,1);
U2divV0_F20 = -1./T02(:,1,2);
P3divV0_V30 = 1./T03(:,1,1);
```

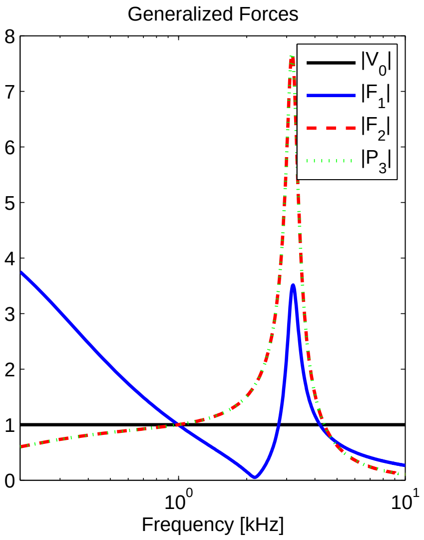


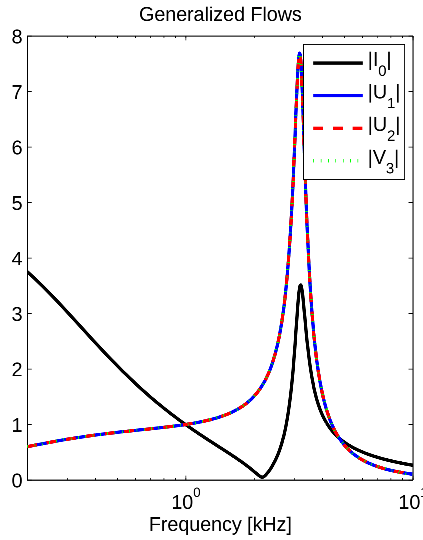
3. Assume the earphone is terminated in an infinitely long tube of area A_{canal}, having an impedance of $\rho_o c/A_{canal}$. Also assume the earphone is driven by a voltage $V_0 = 1$ [Volt]. Plot the *magnitudes* of the following variables as a function of frequency: generalized forces $V_0, F_1, F_2, \mathcal{P}_3$, and (in a second plot) the generalized flows $I_0, U_1, U_2, \mathcal{V}_3$. Normalize all the plots to a single value at 1 [kHz]. Explain what each curve is telling you (identify resonances, and explain their source). Be sure your curves are labeled properly in a legend, so that the different variables are easily distinguished. Solution: You can find the forces and flows from the following equations:

```
% forces V0 = ones(1,length(f)); % input 1 V
F1 = 1./(T01(:,1,1) + T01(:,1,2)./Zin1);
F2 = 1./(T02(:,1,1) + T02(:,1,2)./Zin2);
P3 = 1./(T03(:,1,1) + T03(:,1,2)./ZL);

% flows
I0 = 1./Zin0;
U1 = -1./(Zin1.*T01(:,1,1) + T01(:,1,2));
U2 = -1./(Zin2.*T02(:,1,1) + T02(:,1,2));
```

```
V3 = -1./(ZL.*T03(:,1,1) + T03(:,1,2));
```





4. The given code generates a plot of the input impedance of the speaker, given $Z_{in} = \rho c/A_{canal}$. On the same plot, show the input impedance when

   (a) The speaker is in the ear canal ($Z_l = \rho c/A_{canal}$).

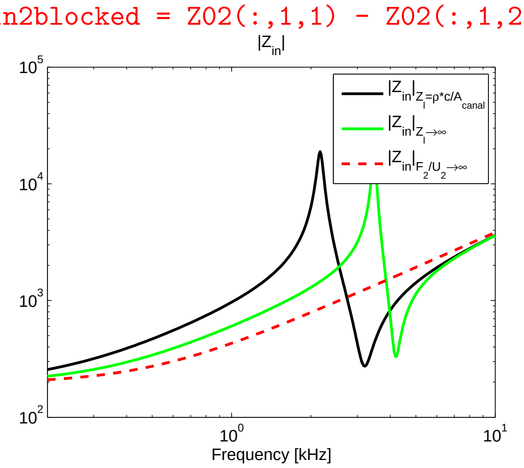
   (b) The speaker port is blocked (instead of $Z_l = \rho c/A_{canal}$, $Z_l \to \infty$; in Matlab, set `ZL=Inf`).

   (c) The diaphragm is blocked (this requires the impedance $F_2/U_2 \to \infty$).

   Describe the differences in the speaker's electrical input impedance under these three conditions. Can you relate these differences to the motion of the diaphragm? Solution: Difference in loads causes diaphragm resonance to move. When the diaphragm itself is blocked, there is no resonance!

   You can use the following code to calculate these input impedances:

```
ZL = rho*c/A_canal; % ear canal
Zin0 = Z03(:,1,1) - Z03(:,1,2).*Z03(:,2,1)./(ZL + Z03(:,2,2)); % (a)
Zin3blocked = Z03(:,1,1) - Z03(:,1,2).*Z03(:,2,1)./(Inf + Z03(:,2,2)); % (b)
Zin2blocked = Z02(:,1,1) - Z02(:,1,2).*Z02(:,2,1)./(Inf + Z02(:,2,2)); % (c)
```

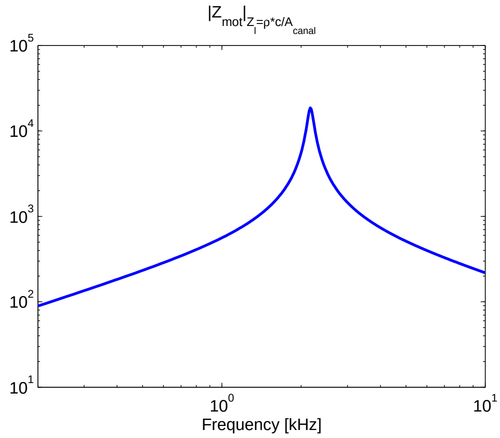


5. Define the *motional impedance*, and explain why it is an important concept. *Hint: See the lab manual (Lab 3) for help!* Using your input impedances from the previous problem, calculate and plot the motional impedance for the case where $Z_l = \rho c/A_{canal}$. Solution: The motional impedance is described in the Lab Manual for Lab 3. There are references to pages in the

text that discuss this more.
The measured electrical impedance $Z_{in} = Z_e + Z_{mot}$, where Z_{mot} is the impedance due to the motion of the diaphragm. Therefore, $Z_{in} = Z_e$ when the diaphragm is blocked. Once we know Z_e, we can calculate Z_{mot} with any mechanical or acoustical load, given the electrical input impedance. You can use the code:

```
Zmot = Zin0-Zin2blocked
```

to calculate Z_{mot} for the case where the acoustic load is $\rho c/A_{canal}$.



2 Filter classes

In the following let $s = \sigma + i\omega$ be the Laplace (complex) frequency. *Filters* are causal functions (one sided in time) that modify a *signal* (any function of time) into another signal. For example if $h(t)$ is a filter, and $x(t)$ a signal then

$$y(t) = h(t) \star x(t) \equiv \int_{\infty}^{t} h(t-\tau)x(\tau)d\tau$$

where $\star$ defines convolution.

Both *transfer functions* and *impedances* are in the class of filters.

An important property of a filter is that it is causal, and has a *Laplace transform*. However, a signal has a Fourier transform and may not always be causal. A physical real-world filter is **always** causal. Mathematically one may easily define an anti-causal filter (e.g., $(u(-t))$, but it is hard (i.e., impossible) to understand exactly what that would mean in practice.

Background:

1. A *causal* filter $h(t) \leftrightarrow H(s)$ is one that is zero for negative time. It necessarily has a Laplace transform.

2. A *finite impulse response* (FIR) filter has finite duration, namely if $f(t)$ is FIR, then it is zero for $t < 0$ (it is causal) and for $t > T$ where T is a positive constant (time). FIR filters only have *zeros* (they do not have poles).

3. An *Infinite impulse response* (IIR) filter is one that is non-zero in magnitude as $t \to \infty$, bu it is still causal ($h(t < 0) = 0$). All IIR filters have poles (as well as zeros), namely if $h(t) \leftrightarrow H(s)$ then $H(s)$ has poles in the region $\sigma \leq 0$. IIR filters have the characteristic of 'feedback' (output depends on previous inputs), which is why the impulse response is infinitely long.

4. A *Minimum Phase* filter $m(t) \leftrightarrow M(s)$ is a filter (it must be causal) having the smallest phase (i.e., $\angle M(j\omega)$) of any filter with magnitude $|M(\omega)|$, on the $j\omega$ axis. A minimum phase

filter also satisfies the very special condition

$$\aleph(t) \leftrightarrow N(s) \equiv \frac{1}{M(s)},$$

where the inverse of $m(t)$ is causal. Thus $m(t) \star \aleph(t) = \delta(t)$ where $\star$ represents convolution. All impedances are minimum phase (every impedance $Z(s)$ has a corresponding admittance $Y(s) \equiv 1/Z(s)$).

5. An *all-pass* filter modifies the *phase* but not the *magnitude* of a signal; namely if $a(t)$ is a causal impulse response of a causal all-pass filter, having Fourier Transform $A(\omega) \equiv |A(\omega)|e^{i\phi(\omega)}$, then $|A(\omega)| = 1$. Thus the phase $\phi(\omega)$ completely specifies the all-pass filter. The group delay is defined as

$$\tau_g(\omega) \equiv -\frac{\partial\phi(\omega)}{\partial\omega},$$

Since

$$\phi(\omega) = \int_0^\omega \tau_g(\omega)d\omega,$$

the group delay also may be used as the definition of an all-pass (i.e., the filter can be derived the group delay).

A causal all-pass filter having a real time response ($a(t)$ real and causal), must have its poles and zeros symmetrically located across both the σ and $j\omega$ axes. For example a pole at $s_p = -1 + j$ and a zero at $s_z = 1 + j$

$$\tilde{A}(s) = \frac{s-1-j}{s+1-j}$$

would produce an all-pass response. This is because $|\tilde{A}|_{s=j\omega} = 1$ (verify this for yourself). The inverse Laplace transform $a(t) \leftrightarrow A(s)$ is be complex because the conjugate poles and zeros have been ignored in this example. To repair this (to force $a(t)$ to be real), the full filter must be

$$A(\omega) \equiv \tilde{A}(\omega) \cdot \tilde{A}^*(\omega) = \frac{s-1-j}{s+1-j} \cdot \frac{s-1+j}{s+1+j}.$$

6. A *positive real* (PR) filter $z(t) \leftrightarrow Z(s) = R(s) + iX(s)$ is both minimum phase, and has a positive real part in the right half s plane, namely

$$R(\sigma > 0) > 0$$

that is, for $\sigma > 0$ $\Re Z > 0$.

Every impedance $z(t) \leftrightarrow Z(s)$ is PR. Since impedance is used in the definition of power, it represents a *positive definite* operator (a fancy name for a filter). For example, if one convolves a current $i(t)$ with an impedance, a voltage results. Namely $v(t) = z(t) \star i(t)$. Since power is voltage times current, the complex power is $\mathcal{P}(t) \equiv v(t)i(t)$. The *time average power* is the time average of $\mathcal{P}(t)$, $\overline{\mathcal{P}(t)} \equiv \int_T \mathcal{P}(t)dt$ (if $P(t)$ is periodic, then we average over a period T).

To do: Prove (or discuss in detail) each of the following:

1. The relation $h(t) \star g(t) \leftrightarrow H(\omega)G(s)$.

   (a) Start by writing out the formula for the convolution of $h(t)$ and $g(t)$, denoted $h(t) \star g(t)$.
   Solution:

   $$h(t) \star g(t) \equiv \int_{\tau=0}^{-\infty} h(t-\tau)g(\tau)d\tau$$

   The limits have been chosen to be consistent with the fact that $g(t) = 0$ for $\tau < 0$.

(b) Then show that the inverse transform of a product of filters is a convolution. Solution: Start from the definition of the product of the FT of $h(t) \leftrightarrow H(\omega)$ and the LT of $g(t) \leftrightarrow G(s)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)G(i\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \underbrace{\left[\int_{-\infty}^{\infty} h(\xi)e^{-i\omega\xi}d\xi\right]}_{H(\omega)} \underbrace{\left[\int_{0}^{\infty} g(\tau)e^{-(\sigma_0+i\omega)\tau}d\tau\right]}_{G(\sigma_0+i\omega)} e^{i\omega t}$$

Here we let $\sigma_0 = 0$ placing the ROC in the RHP.
Changing the order of the integration gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} d\tau \; h(\xi)\; g(\tau) \underbrace{\int_{\omega=-\infty}^{\infty} d\omega e^{i\omega(t-\xi-\tau)}}_{2\pi\delta(t-\xi-\tau)} = \frac{1}{2\pi}\int_{\xi=0}^{\infty} h(\xi)g(t-\xi)d\xi$$

This may also be written as the convolution given as the first equation if we manage the delta function differently.

(c) The point of this example is that one function is of ω while the other a function of s. How does this impact the convolution? Solution: The function $h(t)$ may exist over all time. Because $G(s)$ is causal however, one limit of the convolution is affected. This may be written in two different ways, but the simpler way is the first method, since there it is clear where the limit is for $g(\tau)$ (i.e., at $\tau = 0$).

2. Can an all-pass be filter minimum phase? Explain. Solution: In some sense all-pass is the opposite of minimum phase. Any causal filter may be factored into the product of an all-pass ($A(\omega)$) and a minimum phase $M(\omega)$ filter. Namely given any causal transfer function $H(\omega) = A(\omega)M(\omega)$, where $|A| = 1$ is only a frequency dependent delay, and $M(\omega)$ is a filter with the smallest phase possible given $|M(\omega)|$. The real and imaginary parts of a minimum phase filter are Hilbert transforms of each other.

3. Prove that $\delta(t-5)$ is all-pass. Solution: $|e^{-5s}| = 1$

4. Is $\delta(t+5)$ all-pass? Solution: This response is not causal, but still is all-pass, since $|e^{5s}| = 1$. One can take the LT of an anti-causal signal, since it is still one-sided. However, this situation seems unlikely to arise in a real-world problem.

5. Is e^{-t} all-pass or minimum phase? Justify your answer. Solution: if you assume its causal, then yes it has a single pole at $s = -1$. If the function exists for all time, then it doesn't have either a L or F transform.

6. For what conditions on a and b is $F(s) = \frac{s-a}{s+b}$ all-pass? Solution: In the trivial case, if b = -a, then F(s) =(s-a)/(s-a)=1 is all-pass. Otherwise, we require that b is equal to the complex conjugate of a. If b=a*, then the real part of the pole will be opposite the real part of the zero, so it falls on the other side of the jw axis. We need to take the complex conjugate so that the imaginary parts of the pole and zero will be the same.

7. For what conditions on a and b is $F(s) = \frac{s-a}{s+b}$ minimum phase? Solution: Only if the pole and zero are in the LHP. i.e., $a, -b$ both have negative real parts.

8. Does $F(s) = \frac{s+j}{s-j}$ have a real impulse response? Solution: No: $f(t) = \frac{d}{dt}e^{jt}u(t) + je^{jt}u(t)$

9. In the continuous time domain, a pure delay by T [s] may be written as $\delta(t-T) \leftrightarrow e^{-i\omega T}$. Find the expression for z^{-N}, a pure delay of N samples in the discrete time domain. *Hint:*

Discrete time n relates to continuous time via $t = nT_s$, where T_s is the sampling period. If a delay $T = NT_s$, relate this to the continuous time delay. What must z be? Solution: In the discrete time domain, a delay of N samples is $\delta(nT - NT)$, where T is the sample period, is n is the time index and $t_n = nT$ is the delay we wish to represent. Taking the z transform gives $z^{-N} = e^{-i\omega NT}$. Thus $z^{-1} = e^{-i\omega T}$.

10. Where are the poles and zeros for

    (a) a Stable filter Solution: poles in LHP, zeros anywhere

    (b) an all-pass filter Solution: poles LHP, zeros symmetrically in the RHP

    (c) a Minimum phase filter Solution: All poles and zeros in LHP

    (d) an Impedance (PR function) Solution: Poles and zeros must be carefully place in the LHP only, such that the phase of the impedance is always between $\pm\pi$. This places a *very* tight constraint on the pole–zero locations, much more than their simplifying being in the LHP.

11. Describe the mathematical relationship between $i(t)$ and $v(t)$ if they are related by the Laplace transform via Ohm's law
$$V(\omega) = Z(s)I(\omega).$$
Solution: From Ohm's law $Z = V/I$, convolution relates the voltage and current. That is
$$v(t) \equiv z(t) \star i(t) = \int_{\tau=-\infty}^{t} z(t-\tau)i(\tau)d\tau = \int_{\tau=0}^{\infty} z(\tau)i(t-\tau)d\tau \leftrightarrow V(\omega) = Z(s)I(\omega),$$
where $z(t) \leftrightarrow Z(s) \equiv \frac{V(\omega)}{I(\omega)}$. Here $v(t) \leftrightarrow V(\omega)$ and $i(t) \leftrightarrow I(\omega)$ are FTs of the voltage and current. In general $v(t)$ and $i(t)$ do not have LTs, since they are signals, not systems, thus the system properties (*linear*, *causality*, *active*, ...) have no physical significance. Impedances are the most common functions having a LT, and therefore, poles and zeros. Most books do not discuss this most obvious case, of mixing signals with impedances. It only considers signal that have a LT. For example, when a noise is filtered by a low-pass RC filter having a linear causal time-invariant *transfer function* $H(s) \equiv V_o/V_i = R/(R + 1/sC)$, the input an output noise signals will not have a LT. Such problems abound (are common).

3 Name that transform

1. You are given a specification of the time and frequency properties of some signals and you are asked to name the type of transform that would be used to analyze these signals in the frequency domain (e.g. Fourier transform, Laplace transform, z-transform, discrete FT (DFT), discrete time FT (DTFT), Fourier series, etc.).

   (a) The time response is zero for $t < 0$ and the frequency response is a function of the radian frequency $\omega = 2\pi f$ Solution: Laplace since it is strictly causal.

   (b) The time response is zero for $t > 0$ and the frequency response is a function of the radian frequency $\omega = 2\pi f$ Solution: This is anticausal, it must be analytic in the left half plane. This never happens, thus we never talk about it.

   (c) The time response is given at points $t_n = nT$, where $T = 1/F_s$ with $F_s = 44100$ kHz, and the frequency response is specified outside the unit circle. Solution: Since the frequency resp is specified outside the unit circle, then it must be the z transform. It is a causal sampled systems.

(d) The time response is given at times $t[n] = nT$ for integer n and constant T, and the frequencies are given at $f[k] = k/T$ Solution: This sounds like the DTFT. Since no time range is specified, it must be for all time, and thus a DTFT.

2. Find the Fourier series expansion of the periodic function

$$f((t))_T \equiv \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Give the formula for $F[k]$. Show your work. Explain how you get the solution. Solution: This is exactly the example I gave in class, of a train of delta functions. The FS was found to be $\frac{2\pi}{T} \sum_k \delta(\omega - k\frac{2\pi}{T})$

3. Find the Laplace transform of $h(t) = 3e^{-t/\tau}u(t)$. Give an example of an electrical circuit that has this impulse response. Solution: This is a bit of a classic result with $H(s) = 3/(s+(1/\tau))$.

4. If the impulse response of some system is $h(t) = 3e^{t/\tau}u(t+1)$, describe the interesting things about the system. Solution: This is unstable (active) and "advanced-causal," meaning it is causal but advanced in time by 1 second. This is a special case of one-sided, but it is *not* causal.

5. What is the basic idea behind an *analytic function?* Give an example of a function that is analytic, and one that is not. Solution: All impedances are analytic in the right half plane, since they are causal. A causal function has a power series in frequency that converges everywhere in the *region of convergence* (ROC).

6. Laplace vs. Fourier

   (a) When do you use a Laplace transform and when do you use the Fourier transform? Solution: Use the Laplace on causal impulse responses, such as impedances, and the Fourier transform on "signals" that go on forever, or at least for a long time

   (b) Give an example where you can use both. Solution: A causal function typically has a FT, but not always. For example $e^{-at}u(t)$ has both, but $e^t u(t)$ has no FT.

   (c) Give an example where you cannot use the Laplace transform. Solution: The Laplace transform is always analytic in some region. Functions that only have transforms on the $j\omega$ axis do not have FTs. A speech signal does not have a LT, for example.

7. Given the transform pair $f(t) \leftrightarrow F(\omega)$ one may prove that $F^*(t) \leftrightarrow 2\pi f^*(\omega)$. Solution: This isn't that easy. I need to show this in class. Its a matter of carefully applying the definitions of the FT.

   Apply this relationship to the following transform pairs, to derive new transform pairs (I worked out the first question, as an example):

   (a) $\delta(t) \leftrightarrow 1$
   Solution: $f(t) = \delta(t) \leftrightarrow F(\omega) = 1$. Thus applying the above relationship we find that if the time function is 1 then the transform is $2\pi\delta(\omega)$.

   (b) $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$ Solution: $\delta(t - t_0) \leftrightarrow e^{-j\omega t_0}$. This should be known to you.

   (c) $u(t) \leftrightarrow \pi\delta(\omega) + 1/j\omega$ Solution: $\delta(t - 1/jt) \leftrightarrow 2u(\omega)$

4 History

1. Describe some interesting things about Pythagoras. Be sure to include when, where, and why. What might this have to do with Audio Engineering? Solution: Look this up on Wiki.

2. Give a few reasons that Newton might be relevant to Audio Engineering. Solution: He was the first to calculate the speed of sound. He may have been off by $\sqrt{1.4}$, but this is a pretty impressive feat.

3. What year did Fourier work out his analysis of heat transfer? How did he do it? Solution: Around 1822 (Maxwell was c1865) he did this in the frequency domain. The heat transfer equation is not the wave equation, it is the diffusion equation. Yet he solved it in the frequency domain.

References

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