ECE 403 **HW #1 – Version 1.00 February 20, 2016** **Spring 2016**

Univ. of Illinois **Due W, Feb 4, 2016** **Prof. Allen**

Topic of this homework: Analytic functions of a complex variable; Fourier Transforms

Deliverable: Show your work.

If you hand it in late, you will get zero credit! I need a **stapled paper copy**, with your name on it. (Some ceedit is better than NO credit.) **No** files.doc

Note: Each person is to do their own final writeup, but (obviously) you may, and should, discuss it as much as you wish between yourselves. Your crossing the line if you share computer files. The general rule is, "look but don't touch." In otherwords, you need to process all the words you write through your eyes and your fingers. When ever you use material from someone else, you must give them credit. I got at least some of my ideas for homework from Wikipedia (I must give Wikipedia credit when I do this).

1. Convolution: Given two "causal" sequences $a_n = [\cdots, \therefore, 0, 1, 0, -1, 0, \cdots]$ and $b_n = [\cdots, \therefore, 1, -1, 0, 0, \cdots]$. Here the rising dots $\therefore$ define $t = 0$, before and at which time the signal is zero.
   (a) Find causal sequence $c \equiv a \star b$ by direction convolution
   (b) Form polynomial $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$, and find $C(z) = A(z) \cdot B(z)$
   (c) thus demonstrate that sequence c_n and the coefficients of $C(z)$ are identical.

2. Laplace transforms: given that $f(t) \leftrightarrow F(s)$
   (a) Find the Laplace transform of $\delta(t)$, $df(t)/dt$, $\int_{-\infty}^{t} \delta(t)dt$, and $\int_{-\infty}^{t} u(t)dt$.
   (b) If $f(t) = 1/\sqrt{\pi t}$ has a Laplace transform $F(s) = 1/\sqrt{s}$:
      i. What is the inverse Laplace transform of $\sqrt{s}$?
      ii. What is $f(-1)$?
      iii. Integrate $I = \int_C \frac{1}{s} ds$ around the unit circle centered on $s = 0$.
      iv. Integrate $\int_C \frac{1}{s} ds$ around the unit circle centered on $s = .5$ (i.e., $\sigma = .5, \omega = 0$), and $s = -2$.
   (c) Assume that $s = \sigma + j\omega$ and $F(s) = U(\sigma, \omega) + jV(\sigma, \omega)$ then explain the following formula, in words:

$$\oint_\gamma F(s)\,ds = \oint_\gamma (U + jV)(d\sigma + jd\omega) = \oint_\gamma (Ud\sigma - Vd\omega) + j\oint_\gamma (Vd\sigma + Ud\omega)$$

   (d) What are the conditions on γ and D that

$$\int_\gamma (Ud\sigma - Vd\omega) = -\iint_D \left(\frac{\partial U}{\partial \omega} + \frac{\partial V}{\partial \sigma}\right) d\sigma d\omega$$

   the above is true?
   (e) Name this condition:

$$\frac{\partial U}{\partial \omega} + \frac{\partial V}{\partial \sigma} = 0$$

3. **Fourier and Laplace transforms**
   (a) Derive the Fourier transform for the step function $u(t-1)$.

(b) Derive the Laplace transform for the step function $u(t-1)$.

(c) Find the Fourier transform $X(\omega)$ of

$$x(t) = u(t) - u(t - .001) \leftrightarrow X(\omega)$$

(d) Find the Laplace transform $\mathcal{X}(s)$ of $x(t) = u(t) - u(t - .001) \leftrightarrow \mathcal{X}(s)$

(e) If $\tilde{u}(t) \leftrightarrow \widetilde{U}(\omega)$ is the $\mathcal{F}$ step function, what is $\tilde{u}(t) \star \tilde{u}(t)$?

(f) Hand-plot (or describe the plot of) $|X(\omega)|$ and $|\mathcal{X}(s)|$.

(g) Where are the poles and zeros in each case above.

4. **Plotting of complex functions** $Z(s)$**: Domain:** $s \equiv \sigma + i\omega$, Range: $Z(s) \equiv R(s) + iX(s)$.

There are two ways you can plot complex functions of a complex variable: First one can plot the contour $R(s)$ vs $X(s)$ as a function of the specific domain of s. Alternatively one may plot $R(s)$ and $X(s)$, that is plot the *Range* $Z(s)$ in terms of the specified *Domain*. In the first case its like a polar plot, parametric in the range variable. In the second, its two plots, $R(s)$ and $X(s)$ as a function of the Domain variable in the s plane. These are very different ways of plotting the same information, and both are important.

(a) Domain: $s = \sigma$, Range: $Z(s) = 1 + s$.

(b) Domain: $s = j\omega$, Range: $Z(s) = 1 + s$.

(c) Domain: $s = j\omega$, Range: $H(s) = 1 + s^2$.

(d) Domain: real, but Reverse the range and domain. Thus the Domain is the real part of $H(s) = 1 + s^2$, while the Range $s(\Re H)$. Plot range for domain $\Re H$. Hint: use the notation $w = H(s)$, that is write $s(w)$ and restrict the Domain w to be real. (What I'm asking for here that you to find the inverse function $s(w)$, and restrict w to be real, and plot the result.)

5. Find the solutions (numerical values in the form $x = a + jb$) of the following:

(a) $x^2 + 1 = 0$

(b) $x^3 + 8 = 0$

(c) $x = i^i$ (Show your work, as always!)

(d) What is the frequency of a^t for any constant a? What if $a = -1$?

6. Harmonic functions

(a) Show that if $F(s) = e^s$ that the real and imaginary parts obey the Cauchy-Riemann conditions.

(b) If $F(s) = \log(s)$, where are the Cauchy-Riemann conditions valid, or not? Explain. Note the CR conditions in polar form are

$$U_r = \frac{1}{r} V_\theta \text{ (CR-1)} \qquad V_r = -\frac{1}{r} U_\theta \text{ (CR-2)}$$

(c) If $F(s) = \sqrt{1 + s^2}$, where are the Cauchy-Riemann conditions valid, or not? Explain. Hint: Take the log first.

(d) If $F(s) = s/(1 + s)$, where are the Cauchy-Riemann conditions valid, or not valid? Explain.

Hilbert Transforms

(a) Determine the Hilbert (integral) relations between $H_e(\omega)$ and $H_o(\omega)$ by use of the Fourier transform relations $\frac{1}{2}\text{sgn}(t) \leftrightarrow \frac{1}{i\omega}$ and/or $u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{i\omega}$. Note: $\text{sgn}(t) = t/|t|$.

(b) Find the Hilbert (integral) relations between $H_r \equiv \Re H(\omega)$ (real part) and $H_i \equiv \Im H(\omega)$ (imag part) of $H(\omega)$.

7. Filter classes

In the following let $s = \sigma + i\omega$ be the Laplace (complex) frequency. *Filters* are causal functions (one sided in time) that modify a *signals* (any function of time) into another signal. For example if $h(t)$ is a filter, and $s(t)$ a signal then

$$y(t) = h(t) \star s(t) \equiv \int_{\infty}^{t} h(t-\tau)x(\tau)d\tau$$

where $\star$ defines convolution.

Both *transfer functions* and *impedances* are in the class of filters.

The main signature of a filter is that it is causal, and has a *Laplace transform.* The signature of a signal is that it has a Fourier transform and is NOT causal. A signal may be causal, but that is not the norm. An physical real-world filter is always causal. Mathematically one may easily define a non-causal filter (e.g., $(u(-t))$, but it is hard (i.e., impossible) to understand exactly what that would mean in practice, other than in some quantum-black-hole-world.

Background:

(a) A *causal* filter $h(t) \leftrightarrow H(s)$ is one that is zero for negative time. It necessarily has a Laplace transform.

(b) An *finite impulse response* (FIR) filter has finite duration, namely if $f(t)$ is FIR, then it is zero for $t < 0$ (it is causal) and for $t > T$ where T is positive constant (time). FIR filters only have *zeros* (they do not have poles).

(c) An *Infinite impulse response* (IIR) filter is one that is non-zero in magnitude as $t \to \infty$. All IIR filters have poles, namely if $h(t) \leftrightarrow H(s)$ then $H(s)$ has poles in the region $\sigma \le 0$.

(d) A *Minimum Phase* filter $m(t) \leftrightarrow M(s)$ is a filter (it must be causal) having the smallest phase (i.e., $\angle M(j\omega)$) of any filter with magnitude $|M(\omega)|$, on the $j\omega$ axis. A minimum phase filter also satisfies the very special condition

$$\aleph(t) \leftrightarrow N(s) \equiv \frac{1}{M(s)},$$

namely the inverse of $m(t)$ is causal. Thus $m(t) \star \aleph(t) = \delta(t)$ where $\star$ represents convolution. All impedances are minimum phase (every impedance $Z(s)$ has a corresponding admittance $Y(s) \equiv 1/Z(s)$.

(e) An *all-pass* filter modifies the *phase* but not the *magnitude* of a signal; namely if $a(t)$ is a causal impulse response of a causal all-pass filter, having Fourier Transform $A(\omega) \equiv |A(\omega)|e^{i\phi(\omega)}$, then $|A(\omega)| = 1$. Thus the phase $\phi(\omega)$ completely specifies the all-pass filter. The group delay is defined as

$$\tau_g(\omega) \equiv -\frac{\partial\phi(\omega)}{\partial\omega},$$

Since

$$\phi(\omega) = \int_0^\omega \tau_g(\omega) d\omega,$$

the group delay also may be used as the definition of an all-pass.

A causal all-pass filter having a real time response ($a(t)$ real and causal), must have its poles and zeros symmetrically located across both theω and $j\omega$ axes. For example a pole at $s_p = -1 + j$ and a zero at $s_z = 1 + j$

$$\tilde{A}(s) = \frac{s - 1 - j}{s + 1 - j}$$

would produce an all-pass response. This is because $|\tilde{A}|_{s=j\omega} = 1$ (verify this for yourself). The inverse Laplace transform $a(t) \leftrightarrow A(s)$ is be complex because the conjugate poles and zeros have been ignored in this example. To repair this (to force $a(t)$ to be real), the full filter must be

$$A(\omega) \equiv \tilde{A}(\omega) + \tilde{A}^*(\omega) = \frac{s - 1 - j}{s + 1 - j} + \frac{s - 1 + j}{s + 1 - j}.$$

(f) A *positive real* (PR) filter $z(t) \leftrightarrow Z(s) = R(s) + iX(s)$ is both minimum phase, and has a positive real part in the right half s plane, namely

$$R(\sigma > 0) > 0$$

that is, for $\sigma > 0$ $\Re Z > 0$.

Every impedance $z(t) \leftrightarrow Z(s)$ is PR. Since impedance is used in the definition of power, it represents a *positive definite* operator (a fancy name for a filter). For example, if one convolves a current $i(t)$ with an impedance, a voltage results. Namely $v(t) = z(t) \star i(t)$. Since power is voltage times current, the complex power is $\mathcal{P}(t) \equiv v(t)i(t)$. The *time average power* is the time average of $\mathcal{P}(t)$, $\overline{\mathcal{P}(t)} \equiv \int_t \mathcal{P}(t)dt$.

To do: Prove (or discuss in detail) each of the following:

(a) The relation $h(t) \star g(t) \leftrightarrow H(\omega)G(s)$.

  i. Start by writing out the formula for the convolution of $h(t)$ and $g(t)$, denoted $h(t) \star g(t)$.

  ii. Then show that the inverse transform of a product of filters is a convolution.

  iii. The point of this example is that one function is of ω while the other a function of s. How does this impact the convolution?

(b) Is an all-pass filter minimum phase? Explain.

(c) Prove that $\delta(t - 5)$ is all-pass.

(d) Is $\delta(t + 5)$ all-pass?

(e) Is e^{-t} all-pass or minimum phase? Justify your answer.

(f) When is $F(s) = \frac{s-a}{s+b}$ all-pass?

(g) When is $F(s) = \frac{s-a}{s+b}$ minimum phase?

(h) Does $F(s) = \frac{s+j}{s-j}$ have a real impulse response?

(i) In the continuous time domain, a pure delay by T [s] may be written as $\delta(t-T) \leftrightarrow e^{-i\omega T}$. Find the expression for z^{-N}, a pure delay of N samples in the discrete time domain.

(j) Where are the poles and zeros for

   i. a Stable filter
   ii. an all-pass filter
   iii. a Minimum phase filter
   iv. an Impedance (PR function)

8. Describe the mathematical relationship between $i(t)$ and $v(t)$ if they are related by the Laplace transform via Ohm's law

$$V(\omega) = Z(s)I(\omega).$$