ECE 403 **HWb – Ver. 1.00 January 22, 2017** **Spring 2017**

Univ. of Illinois **Feb 7, Disc: Feb 14, Due: Feb 21** **Prof. Allen**

Topic of this homework: Experimental measurements of a 2-port network. The first objective is become familiar with the computer measurement system. This is a lab study where you will use the computer and Network analyzer (MU box) to measure the 2-port properties of two simple electrical networks.

Deliverable: Measure the circuits shown in the Fig. below. Using Matlab or Octave, show the four $\mathbf{T}(s)$ matrix elements, magnitude and phase. Explain how you measured and computed these functions. Also compute the four $\mathbf{Z}(s)$ matrix parameters. Explain how you measured them.

Make sure the plots look professional, have log-log scales, and restrict the results between 50 to 10^4 [Hz], (i.e., the audio, audible, frequency range). Write a report (i.e., a few pages long) that outlines what you found and how you did the analysis.

I request a paper copy, with your name on it, stapled. **No** files.doc, no emailed homework.

Due Date: If you hand it in late, you will get zero credit. Some credit is better than NO credit.

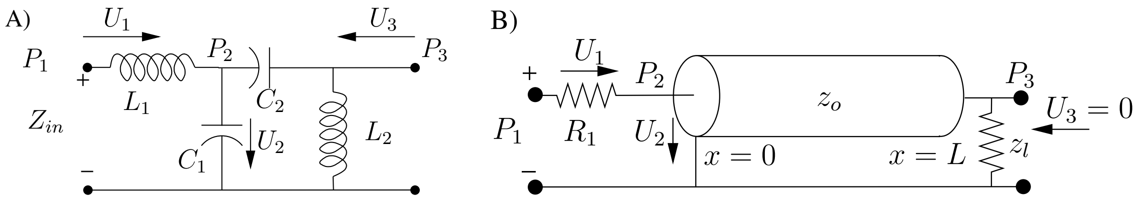
Notes:

- This homework will be discussed in class on Disc: Feb 14, so you may want to be there.
- You should work as a team to collect the raw data. But then each person is to do there own analysis and write-up. Obviously you can, and should, discuss how to do the analysis, as much as you wish, with your team (or others). However, you should not share computer files, including the final graphs. In other words, you need to process all the words you write through your eyes and your fingers. When ever you use material from someone else, give a citation.

1 ABCD Matrix method

We seek to demonstrate that the traditional methods of circuit analysis can require much more algebra than using the transmission matrix method, which is based on 2x2 matrix algebra.

Two circuits are shown In the figure below: Left: A) and Right: B).



1. Analyize circuit A:

   (a) Use a "traditional" analysis (define the impedances, and use formulas for series $z = z_1 + z_2$ and parallel $z = z_1 || z_2 = \frac{z_1 z_2}{z_1 + z_2}$ combinations), to obtain the input impedance $Z_{in}(s)$, where s is the complex frequency ($s = \sigma + j\omega$), as defined for the *Laplace Transform.* For example, the input impedance is

   $$Z_{in} = sL_1 + Z_p$$

   where

   $$Z_p = \frac{1}{sC_1} || \left(\frac{1}{sC_2} + sL_2 \right)$$

   Write out the formula for $Z_{in}(s)$ in terms of the component values L_1, C_1, etc.

   (b) Next find the formula for $H_{32} = P_3/P_2$ (the pressure transfer function)?

   (c) Note that the related formula for $H_{31} = P_3/P_1$ is more complicated. Comment on, or if you can, write out the formula.

   (d) Use the ABCD (Transmission) matrix approach to find the transmission (ABCD) matrix for circuit A.

2. Analyize circuit B: This circuit represents a transmission line (TL), terminated at the far end $x = L$ with impedance $z_l(s)$. Every TL has four parameters: a length L [m], wave speed $c = 345$ [m/s], cross-sectional area A_o [m^2] and *characteristic impedance* $z_o = \rho_o c / A_o$ [Rayls].

   When sound enters the input, at $x = 0$, the wave travels at speed c, thus is therefore delayed by $T = L/c$ [s]. At $x = L$ the wave is reflected with a reflection coefficient

   $$\Gamma = \frac{P^-}{P^+} = \frac{z_o - z_l}{z_o + z_l} \tag{1}$$

   The reflected wave is again delayed by T as it travels back to the input ($x = 0$), giving a total delay of $2L/c$.

   When the TL is terminated with its characteristic impedance, namely when $z_l = z_o$, their are no reflections since $\Gamma = 0$. If $L = \infty$ there are no reflections, as the load at $x = L$ is simply z_o

To Do: Justify Eq. 1. The idea is to express the pressure and velocity as sums of the forward and backward traveling waves. Starting with the definition of the impedance

$$Z(x, s) = \frac{P(x, s)}{U(x, s)},$$

rewrite it in terms of the forward and backward plane waves ($P = P^+ + P^-$, $U = U^+ - U^-$). Recall the definition of $z_o = P^+/U^+$. Hint: $\Gamma(s)$ is derived in the Reading assignment for Lec 8 of Feb 9 (i.e., "pp. 87-90: An Invitation").

3. (a) Assuming $z_l = z_o$, find the input impedance $Z_{in}(s)$:

   (b) If the load impedance $z_l = z_o$, determine the transfer function.

2 2-port Network Measurements and Analysis

This assignment will require making measurements (Lab 2) on the two electrical 2-port networks shown below, and then performing a simple analysis. First you need to learn how to measure the response using the network analyizer tool. Second you need to analyze the resulting files using simple Matlab/Octave tool.

Your results need to be clearly explained in a short report, with figures and equations, that shows your analysis and explains your conclusions.[1]

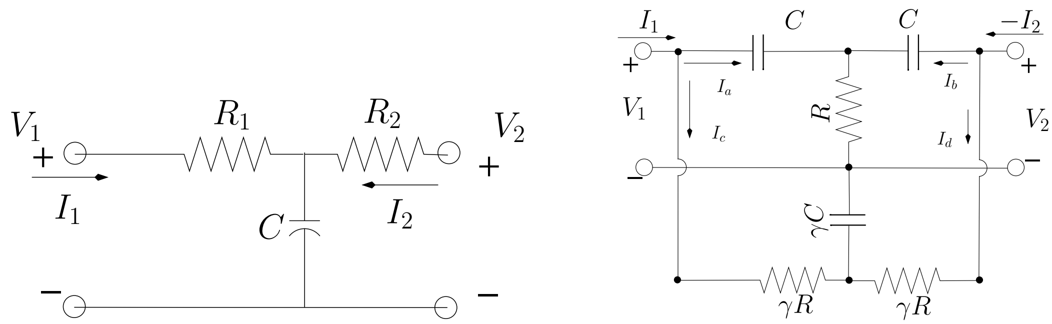


Figure 1: **Left:** *This RC-lowpass circuit is intended to be trivial to analyze.* **Right:** *This parallel pair of 3-element (R and C) circuit is called a "Twin-T notch filter." The values of C and R will be pick by the TA, and the circuit will be pre-configured for you. The constant γ is typically 2.*

1. For the RC lowpass circuit (Fig. 1, **Left**), find the ABCD matrix, the input impedance $Z_1 = V_1/I_2$, and the open circuit voltage transfer function

$$H_{21} = \left.\frac{V_1}{I_1}\right|_{I_2=0}.$$

Compare the calculated results to the data collected in Lab 2. The resistors are in $[k\Omega]$. Assume $Z_1 = 10\ k\Omega$ and $C = 0.1\ \mu$Fd. *Note:* An Matlab/Octave script that models this circuit is provide below, which is available on the website.

2. For the circuit on the right, the following method is used:

(a) Find the T matrix for the upper and lower circuits. Then convert them into an admittance matrix using the transformation given in the Vanvalkenberg handout

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{B} \begin{bmatrix} D & -\Delta_T \\ -1 & A \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{2}$$

Cal this matrix Y_1.

(b) Do the same for the lower circuit, and call this admittance matrix Y_2

$$\begin{bmatrix} I_a \\ I_c \end{bmatrix} = \frac{1}{B} \begin{bmatrix} D & -\Delta_T \\ -1 & A \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \tag{3}$$

(c) The addinging the two matrix equations together, adds the currents, giving the desired result.

[1] This report should be part of your *Final Report* (i.e., due on May 5, 2017).

To Do:

1. Evaluate the impedance matrix and the ABCD matrix for the two circuits of Fig. 1. A Matlab/Octave script is provide below (and on line)[2] to compute and display the solution, so you may verify your Lab 2 results.

2. For the two circuits: Using the network analyizer (MU-box) provided in the lab, measure the input impedance and transfer function, in both directions, and then from your measurements, compute the impedance matrix elements. This is the basis of Lab 2.

Matlab/Octave script to solve RC lowpass system (Left circuit of Fig. 1).

```
% Matlab code to evaluate the RC lowpass filter response of the first 2-port problem
% HWb, Feb 16, 2017, Section 2 Fig 1, left.

NFT=1024; NF=1+NFT/2;%number of non-negative frequencies
Fmax=1e4; Fs=2*Fmax; Fmin=Fs/NFT;
 f=0:Fmin:Fmax; %sweep frequency from 0 to Fmax [Hz] in steps of Fmin [Hz]

R1=1e4; R2=1e4; %in [Ohms]
C=0.1e-6; %[Farads]
T=zeros(NF,2,2); %Init output array

for k=1:NF;
s=2*pi*f(k)*1j; %Define complex frequency

%Define three matrices for the three elements in the network
T1=[1 R1;0 1];
T2=[1 0; s*C 1];
T3=[1 R2;0 1];
T(k,:,:)=T1*T2*T3; %avoid all the messy algebra

end %end for freq loop

%Save T matrix elements
kf=2:(NF-1); %freq limits; avoid ends
A=T(kf,1,1); B=T(kf,1,2);
C=T(kf,2,1); D=T(kf,2,2);

%compute Z matrix Z1, Z2, Z12
Z1=D./C; Z12=1./C;
Z2=A./C; Z21=12; % Assumes reciprocity
```

[2] https://web.engr.illinois.edu/~jontalle/uploads/403/EvalResp.m

3 Laplace transforms

Given a *Laplace transform* ($\mathcal{L}$) pair $f(t) \leftrightarrow F(s)$, the time domain signal, $f(t)$, is written in a lower case font, while the frequency domain representation $F(s)$ is written in an upper-case font. The definition of the *forward* $\mathcal{L}$, $f(t) \rightarrow F(s)$ is

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt.$$

Here $s = \sigma + j\omega$ is the complex Laplace radian frequency and t is time [s]. Note that the input time function must be causal $[f(t < 0) = 0]$.

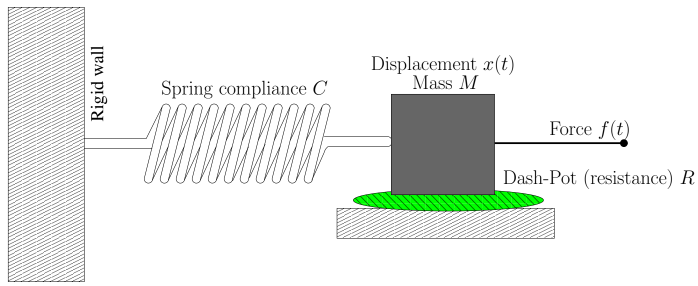


Figure 2: *This three element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).*

An example: Hooke's Law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dash-pot is $f(t) = Rv(t)$, and Newton's famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for constant M is $f(t) = Mdv/dt$ (Newton's second Law of motion).

The equation of motion for the mechanical oscillator in Fig. 2 is given by Newton's third law.[3] Namely, the sum of the *reaction forces* (those due to the mass, spring, dashpot load impedances) must exactly balance the driving force $f(t)$:

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t). \tag{4}$$

These three constants, the mass $M > 0$, resistance $R > 0$ and stiffness $K > 0$ are all real ($\in \mathbb{R}$), and positive.[4] The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

Newton's second law (1687) is the mechanical equivalent of Kirchhoff's (1845) voltage law (KCL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$).

Equation 4 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

3.1 Impedance

For each of the three mechanical items, the Mass, the dash-pot and the spring, define an impedance [Mechanical Ohms], defined as the ratio of the force $f(t) \leftrightarrow F(s)$ [N] over the velocity ($v(t) \leftrightarrow V(s) = sX(s)$ [m/s]).

[3]It is useful to note that Newton's laws of motion (c1687) are the same as Kirchhoff's (1845) current and voltage laws of electricity.

[4]It is not physical for these to be zero, with the interesting exception of a super-conductors, for which $R = 0$. Later we shall learn about low–viscosity Ferric–fluids, widely used in loudspeakers, to concentrate the magnetic flux.

To Do:

1. Find the $\mathcal{L}$ of the three force relations in terms of the force $F(s)$ and the velocity V(s), along with the electrical equivalent impedance:

   (a) Hooke's Law $f(t) = Kx(t)$.

   (b) Dash-pot resistance $f(t) = Rv(t)$.

   (c) Newton's Law for Mass $f(t) = Mdv(t)/dt$.

2. Take the $\mathcal{L}$ of Eq. 4, and find the total impedance $Z(s)$ of the mechanical circuit.

3. The impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in *residue form* as

$$Z(s) = Ms + R + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}. \tag{5}$$

   Note: The mass M and resistance R terms must be factored out of $Z(s)$ when computing the residues c_k, which have a special role in the definition of the impedance and its $\mathcal{L}^{-1}$. This is actually a very subtle point, which will be developed in a later homework.

4. What are $N(s)$ and $D(s)$ (Eq. 5) for the LT of Eq. 4

5. Find the roots of $D(s)$

6. Assuming that $M/2 = R = K = 1$, find the residue form of the admittance $Y(s) = 1/Z(s)$ (e.g. Eq. 5) in terms of the roots $s_{\pm}$. You may check your answer with the Matlab's `residue` command.

7. Use the residue form method (Eq. **??**) of the expression that you derived in the previous exercise.

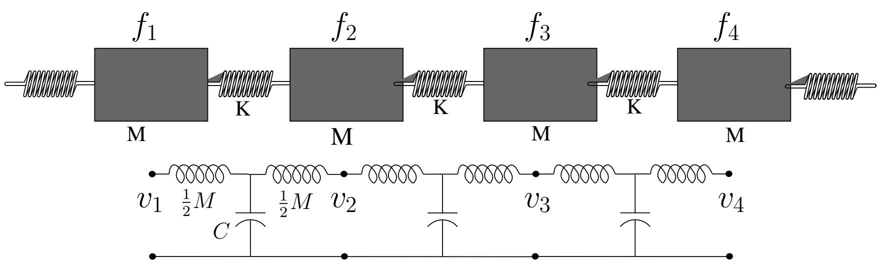


Figure 3: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $v_n(t)$.*

3.2 Transfer functions of a transmission line

In this problem, we will look at the transfer function of a two-port network, shown in Fig. 3. We wish to model the dynamics of a freight-train having N such cars. The model of the train consists of masses connected by springs.

The velocity *transfer function* for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity V_1 and each car responding with a velocity of V_n. Then

$$H(s) = \frac{V_N(s)}{V_1(s)}$$

is the frequency domain ratio of the last car having velocity V_N to V_1, the velocity of the engine, at the left most spring (i.e., coupler).

To do: Use the ABCD method to find the matrix representation of Fig. 3. Consistent with the figure, break the model into cells each consisting of three elements: a series inductor representing half the mass ($L = M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$). Each cell is symmetric, making the model a cascade of identical cells.

At each node define the force $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$ at junction n.

1. Write the ABCD matrix $\boldsymbol{T}$ for a single cell, composed of series mass $M/2$, shunt compliance C and series mass $M/2$, that relates the input node 1 to node 2 where

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} F_2(\omega) \\ -V_2(\omega) \end{bmatrix}$$

   Note that here the mechanical force F is analogous to electrical voltage, and the mechanical velocity V is analogous to electrical current.

2. Assuming that $N = 2$ and that $F_2 = 0$ (two mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\boldsymbol{T}$ matrix you determined above, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

3. Find $h_{21}(t)$, the inverse Laplace transform of $H_{21}(s)$.

4. What is the input impedance $Z_2 = F_2/V_2$ if $F_3 = -r_0 V_3$?

5. Simplify the expression for Z_2 with $N \to \infty$ by assuming that:
   1) $F_3 = -r_0 V_3$ (i.e., V_3 cancels), 2) $s^2 MC << 1$: 3) $r_0 = \sqrt{M/C}$

6. State the ABCD matrix relationship between the first and Nth node, in terms of of the cell matrix.

7. Given a $\boldsymbol{T}$ (ABCD) transmission matrix, the eigenvalues are and vectors are given in Appendix C of the Notes (p. 143), repeated here.

   Eigenvalues:

$$\begin{bmatrix} \lambda_+ \\ \lambda_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (A+D) - \sqrt{(A-D)^2 + 4BC} \\ (A+D) + \sqrt{(A-D)^2 + 4BC} \end{bmatrix}$$

   Due to symmetry, $A = D$, this simplifies to $\lambda_\pm = A \mp \sqrt{BC}$ so that the eigen matrix is

$$\Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix}$$

Eigenvectors: The eigenvectors simplifying even more

$$\left[\boldsymbol{E}_{\pm}\right] = \begin{bmatrix} \frac{1}{2C}\left[(A-D) \mp \sqrt{(A-D)^2+4BC}\right] \\ 1 \end{bmatrix} = \begin{bmatrix} \mp\sqrt{\frac{B}{C}} \\ 1 \end{bmatrix}$$

Eigen-matrix:

$$\boldsymbol{E} = \begin{bmatrix} -\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{E}^{-1} = \frac{1}{2}\begin{bmatrix} -\sqrt{\frac{C}{B}} & 1 \\ +\sqrt{\frac{C}{B}} & 1 \end{bmatrix}$$

There are two important invariants for the transmission lines, the wave number $k = \sqrt{BC} = s/c$ and the characteristic resistance $r_o = \sqrt{B/C}$.

Transfer functions of the transmission line: The velocity transfer function $H_{N,1} = \frac{V_N}{V_1}$ is given by