ECE 403 **HW #1 – Version 1.00 January 26, 2016** **Spring 2016**

Univ. of Illinois **Due W, Feb 4, 2016** **Prof. Allen**

Topic of this homework: Analytic functions of a complex variable; Fourier Transforms

Deliverable: Show your work.

If you hand it in late, you will get zero credit! I need a **stapled paper copy**, with your name on it. (Some ceedit is better than NO credit.) **No** files.doc

Note: Each person is to do their own final writeup, but (obviously) you may, and should, discuss it as much as you wish between yourselves. Your crossing the line if you share computer files. The general rule is, "look but don't touch." In otherwords, you need to process all the words you write through your eyes and your fingers. When ever you use material from someone else, you must give them credit. I got at least some of my ideas for homework from Wikipedia (I must give Wikipedia credit when I do this).

1. Convolution: Given two "causal" sequences $a_n = [\cdots, \therefore, 0, 1, 0, -1, 0, \cdots]$ and $b_n = [\cdots, \therefore, 1, -1, 0, 0, \cdots]$. Here the rising dots $\therefore$ define $t = 0$, before and at which time the signal is zero.

   (a) Find causal sequence $c \equiv a \star b$ by direction convolution Solution: Time reverse either a or b and slide it against the other, forming the output sequence $c_n = [0, 1, -1, -1, 1, 0, 0, \cdots]$

   (b) Form polynomial $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$, and find $C(z) = A(z) \cdot B(z)$ Solution: $C(z) = (z - z^3)(1, -z) = z - z^2 - z^3 + z^4$ which has a coef vector $[0, 1, -1, -1, 1, 0, \cdots]$

   (c) thus demonstrate that sequence c_n and the coefficients of $C(z)$ are identical. Solution: Indeed they are the same.

2. Laplace transforms: given that $f(t) \leftrightarrow F(s)$

   (a) Find the Laplace transform of $\delta(t)$, $df(t)/dt$, $\int_{-\infty}^{t} \delta(t)dt$, and $\int_{-\infty}^{t} u(t)dt$.
   Solution:

      i. $\delta(t) \leftrightarrow 1$ is too trivial to repeat here.

      ii. $df/dt \leftrightarrow sF(s)$. This is shown using integration-by-parts, as follows:

$$d[f(t)e^{-st}] = e^{-st}\frac{df}{dt}dt - se^{-st}f(t)dt$$

      Next integrate this from 0^- to ∞, giving

$$f(t)e^{-st}\Big|_{0^-}^{\infty} = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}dt - s\int_{0^-}^{\infty} e^{-st}f(t)dt$$

      Rearranging these and evaluating the limits gives the desired result

$$\int_{0^-}^{\infty} \frac{df}{dt}e^{-st}dt = f(0^-) + sF(s),$$

      where $f(0^-) = 0$.

      iii. The integral of a delta function is a step function $u(t) \leftrightarrow 1/s$, while

      iv. the integral of a step is $tu(t) \leftrightarrow 1/s^2$.

   In each case it is important to carry along the $u(t)$ but it can be implied if you know its a causal function (i.e., if you are told the transform is a function of s (e.g., $F(s)$).

(b) If $f(t) = 1/\sqrt{\pi t}$ has a Laplace transform $F(s) = 1/\sqrt{s}$:

i. What is the inverse Laplace transform of $\sqrt{s}$?
Solution: Since $d/dt \leftrightarrow s$ then $\sqrt{s} = s/\sqrt{s}$, thus $\frac{d}{dt}\frac{u(t)}{\sqrt{\pi t}} \leftrightarrow \sqrt{s}$. In fact there is a major difficulty here, since $\int \frac{\delta(t)}{\sqrt{t}} dt = \infty$. This problem may be resolved by a small delay in either the numerator of denominator terms so that the delta function does not resolve at $t = 0$ in the $\sqrt{t}$ term.

ii. What is $f(-1)$?
Solution: Since $f(t)$ is causal, at $t = -1$ it is zero.

iii. Integrate $I = \int_C \frac{1}{s} ds$ around the unit circle centered on $s = 0$.
Solution: Let C be the unit circle, then $s = e^{j\theta}$, so

$$I = \int_0^{2\pi} \frac{de^{j\theta}}{e^{j\theta}} = \int_{\theta=0}^{2\pi} \frac{je^{j\theta}}{e^{j\theta}} d\theta = j \int_{\theta=0}^{2\pi} d\theta = j\ \theta|_{\theta=0}^{2\pi} = 2\pi j$$

iv. Integrate $\int_C \frac{1}{s} ds$ around the unit circle centered on $s = .5$ (i.e., $\sigma = .5, \omega = 0$), and $s = -2$. Solution: The first is $2\pi j$ and the second is 0. Why?

(c) Assume that $s = \sigma + j\omega$ and $F(s) = U(\sigma,\omega) + jV(\sigma,\omega)$ then explain the following formula, in words:

$$\oint_\gamma F(s)\, ds = \oint_\gamma (U + jV)(d\sigma + jd\omega) = \oint_\gamma (Ud\sigma - Vd\omega) + j \oint_\gamma (Vd\sigma + Ud\omega)$$

(d) What are the conditions on γ and D that

$$\int_\gamma (Ud\sigma - Vd\omega) = -\iint_D \left(\frac{\partial U}{\partial \omega} + \frac{\partial V}{\partial \sigma}\right) d\sigma d\omega$$

the above is true?

(e) Name this condition:

$$\frac{\partial U}{\partial \omega} + \frac{\partial V}{\partial \sigma} = 0$$

Solution: This is one of the two equations that define the Cauchy Riemann condition. If the two conditions are satisfied, then $F(s)$ is analytic in the region enclosed by γ over the enclosed region D in the complex plane s.

3. **Fourier and Laplace transforms**

(a) Derive the Fourier transform for the step function $u(t-1)$.
Solution: Since the integral does not converge, one must fake it by using the time-symmetric relationship $2u(t) = 1 - \text{sgn}(t)$, delayed:

$$\widetilde{U}(\omega) \equiv \int_{-\infty}^{\infty} \tilde{u}(t-1)e^{-j\omega t} dt = \mathcal{F}\left\{\frac{1 - \text{sgn}(t-1)}{2}\right\} = \pi\tilde{\delta}(\omega) + \frac{e^{-j\omega}}{j\omega}$$

$$\neq \int_1^\infty e^{-j\omega t} dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_1^\infty = \frac{e^{-j\omega} - e^{-j\omega\infty}}{j\omega} = \frac{e^{-j\omega}}{j\omega} - \frac{e^{-j\omega\infty}}{j\omega}$$

(b) Derive the Laplace transform for the step function $u(t-1)$. Solution: e^{-s}/s

(c) Find the Fourier transform $X(\omega)$ of

$$x(t) = u(t) - u(t - .001) \leftrightarrow X(\omega)$$

Solution: Note the $\pi\delta(\omega)$ term cancels out in this case, making the $\text{rect}(t)$ function much nicer than the step allowing one to compute $\text{rect}(t) \star \text{rect}(t)$.

(d) Find the Laplace transform $\mathcal{X}(s)$ of $x(t) = u(t) - u(t - .001) \leftrightarrow \mathcal{X}(s)$
Solution: $(1 - e^{-s/1000})/s$.

(e) If $\tilde{u}(t) \leftrightarrow \widetilde{U}(\omega)$ is the $\mathcal{F}$ step function, what is $\tilde{u}(t) \star \tilde{u}(t)$? Solution: This is a really dirty question, because its not in any book I know of, and the solution is a surprise: The problem here is that $2\tilde{u}(t) \equiv 1 + \text{sgn}(t)$, and $1 \star 1$ badly blows up, and does not exist. Another way to say this is to work in the frequency domain and use the fact that $1 \leftrightarrow 2\pi\delta(\omega)$, and that convolution in time is the product in frequency. In this case $1 \star 1 \leftrightarrow (2\pi)^2 (\delta(\omega))^2$. Probably nobody told you, that you cannot make a function out of $\delta()$. (e.g., It is not legal to square a delta function. (The Taylor series of a delta functions cannot be defined.) Please show me if you find this written down anywhere.)

(f) Hand-plot (or describe the plot of) $|X(\omega)|$ and $|\mathcal{X}(s)|$. Solution: Since the numerator has $1 - e^{-j\omega T}$ the magnitude is sinusodial varying. The phase factor $e^{-j\omega T/2}$ is removed by taking the magnitude.

(g) Where are the poles and zeros in each case above. Solution: Fourier transforms dont have poles and zeros. The Laplace transforms for these examples have poles at $s = 0$, and zeros at $e^{-sT} = 1$ and $e^{s} = 0$, depending on the case.

4. **Plotting of complex functions $Z(s)$: Domain:** $s \equiv \sigma + i\omega$, Range: $Z(s) \equiv R(s) + iX(s)$.

There are two ways you can plot complex functions of a complex variable: First one can plot the contour $R(s)$ vs $X(s)$ as a function of the specific domain of s. Alternatively one may plot $R(s)$ and $X(s)$, that is plot the *Range* $Z(s)$ in terms of the specified *Domain*. In the first case its like a polar plot, parametric in the range variable. In the second, its two plots, $R(s)$ and $X(s)$ as a function of the Domain variable in the s plane. These are very different ways of plotting the same information, and both are important.

(a) Domain: $s = \sigma$, Range: $Z(s) = 1 + s$.

Solution: Make two axes, one for the s plane and one for the $Z(s) = R(s) + iX(s)$ plane. Label the two sets of axes: On the left (s plane), the horizontal axis (abscissa) is labeled σ, while the vertical axis (ordinate) is $j\omega$. For the $Z(s)$ axis (on the right), the abscissa is labeled $R(s)$ and the ordinate axis is $jX(s)$.

In engineering terms think of $Z(s) = R + jX$ as an *impedance* having a real part (the resistance) R, and an imaginary part (the *reactance*) $X(s)$. In this specific example, the impedance consists of a 1 ohm [Ω] resistor $R = 1$, in series with a $L = 1$ Henry [H] inductor of impedance $X = sL$. Note that $L = 1$ is *not* an impedance (it is an *inductance*) whereas sL is an impedance.

Next indicate the *Range* $s = \sigma$ on the s axis. This will be a line along the σ (x) axis. Label several points on this line, including $A = -1$, $B = 0$ and $C = 1$.

On the second axis plot $Z(\sigma) = 1 + \sigma$ This will also be a line along the x axis, but in this case, an axis that is labeled R. Note that $Z(\sigma) = R = \sigma + 1$. Thus in the Z plane, our three points are offset by 1. On the x axis of the Domain plot, the mapping dictates $A = 0$, $B = 1$ and $C = 2$.

(b) Domain: $s = j\omega$, Range: $Z(s) = 1 + s$. Solution: The range is a vertical line defined by $\sigma = 0$. Pick three points as $-j, 0, j$. The range here is $Z(j\omega) = 1 + j\omega$, which is a vertical line running to the right of the $j\omega$ axis by 1 unit. Our three points are at $1-j, 1, 1+j$. You could also plot the real and imaginary parts of Z

(c) Domain: $s = j\omega$, Range: $H(s) = 1 + s^2$. Solution: First, an impedance cannot have this form, because its real part can be negative. However a transfer function (Output voltage over an input voltage) can have this form. For this reason I have changed the Range to be $H(s)$.

Since the domain is $j\omega$ we wish to look at $H(j\omega) = 1 + (j\omega)^2 = 1 - \omega^2$. This is therefore simply a real function of radian frequency ω with $R(\omega) = 1 - \omega^2$ being a parabola (in standard (x, y) form: $y(x) = 1 - x^2$). In the s (Range) plane, s traverses the vertical $s = j\omega$ axis.

The Range is $R \le 1$, and $X = 0$.

(d) Domain: real, but Reverse the range and domain. Thus the Domain is the real part of $H(s) = 1 + s^2$, while the Range $s(\Re H)$. Plot range for domain $\Re H$. Hint: use the notation $w = H(s)$, that is write $s(w)$ and restrict the Domain w to be real. (What I'm asking for here that you to find the inverse function $s(w)$, and restrict w to be real, and plot the result.) Solution: As suggested, use the more transparent notation $w(s) \equiv H(s) = u(s) + iv(s)$. Invert the function $w(s)$, so that the range becomes the domain, and then restricted the new domain to be real ($v = 0$).

Specifically, solving for $s(w) = \pm\sqrt{w-1}$, restricting $v = 0$ (i.e., H real):

$$s(u) = \begin{cases} \pm\sqrt{u-1} & \text{if } u \ge 0 \\ \pm i\sqrt{1+|u|} & \text{if } u < 0 \end{cases}$$

This is a double-valued parabola, on its side, for $u > 0$ but switches to a purely imaginary function $\pm\sqrt{1+|u|}$ for $u < 0$. We shall later learn this happens in an acoustic horn, and I believe in semiconductors in the "valence band," but then that's not my area of expertise.

5. Find the solutions (numerical values in the form $x = a + jb$) of the following:

(a) $x^2 + 1 = 0$ Solution: This requires finding the roots of this quadratic equation, which are $x_\pm = \pm\sqrt{-1} = \pm i$

(b) $x^3 + 8 = 0$ Solution: $x_k = \sqrt[3]{-8}$ with $k = 1, 2, 3$. Thus $x_k = 2e^{i\pi(1+2k)/3}$ for integers k.

(c) $x = i^i$ (Show your work, as always!) Solution: According to Matlab the answer is $\sqrt{-1}^{\sqrt{-1}} = 0.2079$. To get this number, we know that $a^x = e^{x\ln(a)}$. Thus $\ln(j) = j\pi/2$, and therefore $j^j = e^{j \cdot j\pi/2} = e^{-\pi/2} = 0.2079$. Done correctly, one must add $jk2\pi$ to the exponent, revealing an ∞ number of real numbers as the solution! For more on this topic read "**An imaginary tale** THE STORY OF $\sqrt{-1}$ " (Princeton University Press, 1998) by Paul J. Nahin.

(d) What is the frequency of a^t for any constant a? What if $a = -1$? Solution: $a^t = |a|^t e^{j(\angle a)t} = e^{\sigma t} e^{j\omega t}$, where ω defines the frequency we wish to solve for $\omega_k = \angle a$, thus

$$f = \angle a)/2\pi.$$

Example: $a = -1$ then $\angle a = \pi$ and $f = 1/2$ [Hz].

6. Harmonic functions

(a) Show that if $F(s) = e^s$ that the real and imaginary parts obey the Cauchy-Riemann conditions. Solution: Let $x = \sigma$ and $y = \omega$: The CR conditions are $u_x = v_y$ and $u_y = -v_x$, which we wish to show hold for e^s. Expressing e^s in real and imaginary parts

$$u(x,y) + iv(x,y) = e^{x+iy} = e^x(\cos(y) + i\sin(y)).$$

Thus $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$. From the CR conditions we find

$$u_x = e^x\cos(y) = v_y = e^x\cos(y)$$

furthermore

$$u_y = -e^x\sin(y) = -v_x = e^x\sin(y),$$

and we see that the CR conditions are satisfied everywhere. We see that the function is therefore *entire* (i.e., analytic everywhere in the s plane).

(b) If $F(s) = \log(s)$, where are the Cauchy-Riemann conditions valid, or not? Explain. Note the CR conditions in polar form are

$$U_r = \frac{1}{r}V_\theta \text{ (CR-1)} \qquad V_r = -\frac{1}{r}U_\theta \text{ (CR-2)}$$

Solution: This is nicely explained in Greenberg, *Advanced Engineering Mathematics 2d Edition* (ECE-493 Text) on page 1144, Example 7: To understand this case we must write $s = re^{j\theta}$ in polar coordinates, where

$$F(r,\theta) = \log(r) + j(\theta + k2\pi))$$

with k any integer. Thus $U(r,\theta) = \log(r)$ and $V(r,\theta) = \theta + k2\pi$.
As in the previous example we must show CR conditions in polar coordinates. In this case it is fairly easy however

$$U_r = 1/r \text{ and } V_\theta/r = 1/r,$$

thus CR (1) is satisfied. From CR (2)

$$V_r = 0 \text{ and } -U_\theta/r = 0$$

so we find equality again. Thus the CR conditions are obeyed. The function in this case is multi-valued due to the $ik2\pi$ term in $v(\theta)$ (a *branch cut* is required, not discussed in class).

(c) If $F(s) = \sqrt{1+s^2}$, where are the Cauchy-Riemann conditions valid, or not? Explain. Hint: Take the log first. Solution: The hint is *really* important, else the math is really tedious. The function is analytic except at $s = \pm i$.
Going the hard way, it looks like working in polar coordinates is useful. We may write the real part as $u(r,\theta) \equiv (F(s) + \overline{F(s)})/2$. Thus

$$2u(r,\theta) = \sqrt{1 + r^2e^{i2\theta}} + \sqrt{1 + r^2e^{-i2\theta}}$$

and

$$2v(r,\theta) = \sqrt{1 + r^2e^{i2\theta}} - \sqrt{1 + r^2e^{-i2\theta}}$$

for which we need to show the CR conditions $u_r = v_\theta/r$ and $v_r = -u_\theta/r$ hold. As I say, don't do the problem this way.
Or the easy way

$$\frac{d\log\sqrt{1+s^2}}{ds} = s/(1+s^2)$$

which has poles at $s = \pm j$ so its not analytic there. Thus using $d\log F(s)/ds$ makes the problem trivial.

(d) If $F(s) = s/(1+s)$, where are the Cauchy-Riemann conditions valid, or not valid? Explain. Solution: As above we need $u(\sigma,\omega)$ and $v(\sigma,\omega)$:

$$F(s) = \frac{(\sigma+i\omega)(1+\sigma-i\omega)}{(1+\sigma+i\omega)(1+\sigma-i\omega)} = \frac{\sigma(1+\sigma)+\omega^2+i\omega}{(1+\sigma)^2+\omega^2}.$$

Thus

$$u(\sigma,\omega) = \frac{\sigma(1+\sigma)+\omega^2}{(1+\sigma)^2+\omega^2}$$

and

$$v(\sigma,\omega) = \frac{\omega}{(1+\sigma)^2+\omega^2}.$$

To compute u_σ and v_ω is starting to get ugly.

Lets start over again with a transformation $z = 1+s$, and then work in polar coordinates ($z = re^{i\theta}$). This gives

$$F(z) = 1 - 1/z = 1 - e^{-i\theta}/r = 1 - \cos\theta/r + i\sin\theta/r,$$

thus $u(r,\theta) = 1 - \cos\theta/r$ and $v(r,\theta) = \sin\theta/r$. Now use the CR conditions in polar coordinates, as given above. as $u_r = v_\theta/r$ and $v_r = -u_\theta/r$. Proceeding with $F(z)$

$$u_r = -\cos\theta\frac{\partial}{\partial r}r^{-1} = \cos\theta/r^2 \text{ and } v_\theta/r = \cos\theta/r^2$$

thus these two are equal. Next

$$v_r = -\sin\theta/r^2 \text{ and } -u_\theta/r = \frac{\partial}{\partial\theta}\cos\theta/r^2 = -\sin\theta/r^2,$$

which are also equal. Thus both CR conditions are satisfied — *except* at $r = 0$, where nothing works because the angle is not defined, and the partials do not exist.

Using the log on this function and you can do it in your head. By inspection we see that there are poles in the s plane at -1 and ∞.

Hilbert Transforms

(a) Determine the Hilbert (integral) relations between $H_e(\omega)$ and $H_o(\omega)$ by use of the Fourier transform relations $\frac{1}{2}\mathrm{sgn}(t) \leftrightarrow \frac{1}{i\omega}$ and/or $u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{i\omega}$. Note: $\mathrm{sgn}(t) = t/|t|$.

Solution: This follows simply from the following obvious relationship

$$h_o(t) = h_e(t)\mathrm{sgn}(t) \leftrightarrow \frac{1}{2\pi}H_e(\omega) \star \left(\frac{2}{i\omega}\right) \tag{1}$$

since $\mathrm{sgn}(t) \leftrightarrow \frac{2}{i\omega}$. Thus

$$H_o(\omega) = \frac{1}{\pi i\omega} \star H_e(\omega) \tag{2}$$

(b) Find the Hilbert (integral) relations between $H_r \equiv \Re H(\omega)$ (real part) and $H_i \equiv \Im H(\omega)$ (imag part) of $H(\omega)$.

Solution: It follows from the above results that

$$iH_i(\omega) = \frac{1}{i\pi}\int \frac{H_r(\omega')}{\omega-\omega'}d\omega' \tag{3}$$

which for the case at hand is

$$\frac{\omega}{\omega^2 + a^2} = \frac{1}{\pi} \int \frac{a d\omega}{(\omega' - \omega)(\omega'^2 + a^2)}. \tag{4}$$

A *second* derivation of the requested integrals may be found from

$$h(t) = h(t)u(t), \tag{5}$$

(note this is not exactly true at $t = 0$) which after a FT, results in

$$H(\omega) = \frac{1}{2\pi} H(\omega) \star \left(\pi\delta(\omega) + \frac{1}{i\omega}\right), \tag{6}$$

which may be rewritten as

$$H_r(\omega) = \frac{1}{2} H_r(\omega) + H_i(\omega) \star \frac{1}{2\pi\omega}. \tag{7}$$

The final relations are [Papoulis (1977), *Signal Analysis*, McGraw Hill, page 251]

$$H_r(\omega) = \frac{1}{\pi} \int \frac{H_i(\omega')}{\omega - \omega'} d\omega' \text{ and } H_i(\omega) = -\frac{1}{\pi} \int \frac{H_r(\omega')}{\omega - \omega'} d\omega'$$

7. Filter classes

In the following let $s = \sigma + i\omega$ be the Laplace (complex) frequency. *Filters* are causal functions (one sided in time) that modify a *signals* (any function of time) into another signal. For example if $h(t)$ is a filter, and $s(t)$ a signal then

$$y(t) = h(t) \star s(t) \equiv \int_{\infty}^{t} h(t - \tau)x(\tau)d\tau$$

where $\star$ defines convolution.

Both *transfer functions* and *impedances* are in the class of filters.

The main signature of a filter is that it is causal, and has a *Laplace transform*. The signature of a signal is that it has a Fourier transform and is NOT causal. A signal may be causal, but that is not the norm. An physical real-world filter is always causal. Mathematically one may easily define a non-causal filter (e.g., $(u(-t))$, but it is hard (i.e., impossible) to understand exactly what that would mean in practice, other than in some quantum-black-hole-world.

Background:

(a) A *causal* filter $h(t) \leftrightarrow H(s)$ is one that is zero for negative time. It necessarily has a Laplace transform.

(b) An *finite impulse response* (FIR) filter has finite duration, namely if $f(t)$ is FIR, then it is zero for $t < 0$ (it is causal) and for $t > T$ where T is positive constant (time). FIR filters only have *zeros* (they do not have poles).

(c) An *Infinite impulse response* (IIR) filter is one that is non-zero in magnitude as $t \to \infty$. All IIR filters have poles, namely if $h(t) \leftrightarrow H(s)$ then $H(s)$ has poles in the region $\sigma \leq 0$.

(d) A *Minimum Phase* filter $m(t) \leftrightarrow M(s)$ is a filter (it must be causal) having the smallest phase (i.e., $\angle M(j\omega)$) of any filter with magnitude $|M(\omega)|$, on the $j\omega$ axis. A minimum phase filter also satisfies the very special condition

$$\aleph(t) \leftrightarrow N(s) \equiv \frac{1}{M(s)},$$

namely the inverse of $m(t)$ is causal. Thus $m(t) \star \aleph(t) = \delta(t)$ where $\star$ represents convolution. All impedances are minimum phase (every impedance $Z(s)$ has a corresponding admittance $Y(s) \equiv 1/Z(s)$.

(e) An *all-pass* filter modifies the *phase* but not the *magnitude* of a signal; namely if $a(t)$ is a causal impulse response of a causal all-pass filter, having Fourier Transform $A(\omega) \equiv |A(\omega)|e^{i\phi(\omega)}$, then $|A(\omega)| = 1$. Thus the phase $\phi(\omega)$ completely specifies the all-pass filter. The group delay is defined as

$$\tau_g(\omega) \equiv -\frac{\partial \phi(\omega)}{\partial \omega},$$

Since

$$\phi(\omega) = \int_0^\omega \tau_g(\omega) d\omega,$$

the group delay also may be used as the definition of an all-pass.

A causal all-pass filter having a real time response ($a(t)$ real and causal), must have its poles and zeros symmetrically located across both theω and $j\omega$ axes. For example a pole at $s_p = -1 + j$ and a zero at $s_z = 1 + j$

$$\tilde{A}(s) = \frac{s - 1 - j}{s + 1 - j}$$

would produce an all-pass response. This is because $|\tilde{A}|_{s=j\omega} = 1$ (verify this for yourself). The inverse Laplace transform $a(t) \leftrightarrow A(s)$ is be complex because the conjugate poles and zeros have been ignored in this example. To repair this (to force $a(t)$ to be real), the full filter must be

$$A(\omega) \equiv \tilde{A}(\omega) + \tilde{A}^*(\omega) = \frac{s - 1 - j}{s + 1 - j} + \frac{s - 1 + j}{s + 1 - j}.$$

(f) A *positive real* (PR) filter $z(t) \leftrightarrow Z(s) = R(s) + iX(s)$ is both minimum phase, and has a positive real part in the right half s plane, namely

$$R(\sigma > 0) > 0$$

that is, for $\sigma > 0$ $\Re Z > 0$.

Every impedance $z(t) \leftrightarrow Z(s)$ is PR. Since impedance is used in the definition of power, it represents a *positive definite* operator (a fancy name for a filter). For example, if one convolves a current $i(t)$ with an impedance, a voltage results. Namely $v(t) = z(t) \star i(t)$. Since power is voltage times current, the complex power is $\mathcal{P}(t) \equiv v(t)i(t)$. The *time average power* is the time average of $\mathcal{P}(t)$, $\overline{\mathcal{P}(t)} \equiv \int_t \mathcal{P}(t)dt$.

To do: Prove (or discuss in detail) each of the following:

(a) The relation $h(t) \star g(t) \leftrightarrow H(\omega)G(s)$.

i. Start by writing out the formula for the convolution of $h(t)$ and $g(t)$, denoted $h(t) \star g(t)$.
Solution:

$$h(t) \star g(t) \equiv \int_{\tau=0}^{-\infty} h(t-\tau)g(\tau)d\tau$$

The limits have been chosen to be consistent with the fact that $g(t) = 0$ for $\tau < 0$.

ii. Then show that the inverse transform of a product of filters is a convolution. Solution: Start from the definition of the product of the FT of $h(t) \leftrightarrow H(\omega)$ and the LT of $g(t) \leftrightarrow G(s)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)G(i\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \underbrace{\left[\int_{-\infty}^{\infty} h(\xi)e^{-i\omega\xi}d\xi\right]}_{H(\omega)} \underbrace{\left[\int_{0}^{\infty} g(\tau)e^{-(\sigma_0+i\omega)\tau}d\tau\right]}_{G(\sigma_0+i\omega)} e^{i\omega t}$$

Here we let $\sigma_0 = 0$ placing the ROC in the RHP.
Changing the order of the integration gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} d\tau\ h(\xi)\ g(\tau)\ \underbrace{\int_{\omega=-\infty}^{\infty} d\omega e^{i\omega(t-\xi-\tau)}}_{2\pi\delta(t-\xi-\tau)} = \frac{1}{2\pi}\int_{\xi=0}^{\infty} h(\xi)g(t-\xi)d\xi$$

This may also be written as the convolution given as the first equation if we manage the delta function differently.

iii. The point of this example is that one function is of ω while the other a function of s. How does this impact the convolution? Solution: The function $h(t)$ may exist over all time. Because $G(s)$ is causal however, one limit of the convolution is affected. This may be written in two different ways, but the simpler way is the first method, since there it is clear where the limit is for $g(\tau)$ (i.e., at $\tau = 0$).

(b) Is an all-pass filter minimum phase? Explain. Solution: In some sense all-pass is the opposite of minimum phase. Any causal filter may be factored into the product of an all-pass ($A(\omega)$) and a minimum phase $M(\omega)$ filter. Namely given any causal transfer function $H(\omega) = A(\omega)M(\omega)$, where $|A| = 1$ is only a frequency dependent delay, and $M(\omega)$ is a filter with the smallest phase possible given $|M(\omega)|$. The real and imaginary parts of a minimum phase filter are Hilbert transforms of each other.

(c) Prove that $\delta(t-5)$ is all-pass. Solution: $|e^{-5s}| = 1$

(d) Is $\delta(t+5)$ all-pass? Solution: This response is not causal, but still is all-pass, since $|e^{5s}| = 1$. Now one could argue (and you should) that it cannot have a Laplace transform if its not causal. I agree, but actually one sided functions can be generalized to have a LT. So its an unfair question (or at least very confusing) in that sense.

(e) Is e^{-t} all-pass or minimum phase? Justify your answer. Solution: if you assume its causal, then yes it has a single pole at $s = -1$. If the function exists for all time, then it doesn't have either a L or F transform.

(f) When is $F(s) = \frac{s-a}{s+b}$ all-pass? Solution: When $a = \pm b$ it is all-pass.

(g) When is $F(s) = \frac{s-a}{s+b}$ minimum phase? Solution: Only if the pole and zero are in the LHP. i.e., $a, -b$ both have negative real parts.

(h) Does $F(s) = \frac{s+j}{s-j}$ have a real impulse response? Solution: No: $f(t) = \frac{d}{dt}e^{jt}u(t) + je^{jt}u(t)$

(i) In the continuous time domain, a pure delay by T [s] may be written as $\delta(t-T) \leftrightarrow e^{-i\omega T}$. Find the expression for z^{-N}, a pure delay of N samples in the discrete time domain. Solution: In the discrete time domain, a delay of N samples is $\delta(nT - NT)$, where T is the sample period, is n is the time index and $t_n = nT$ is the delay we wish to represent. Taking the z transform gives $z^{-N} = e^{-i\omega NT}$. Thus $z^{-1} = e^{-i\omega T}$.

(j) Where are the poles and zeros for

i. a Stable filter Solution: poles in LHP, zeros anywhere

ii. an all-pass filter Solution: poles LHP, zeros symmetrically in the RHP

iii. a Minimum phase filter Solution: All poles and zeros in LHP

iv. an Impedance (PR function) Solution: Poles and zeros must be carefully place in the LHP only, such that the phase of the impedance is always between $\pm\pi$. This places a *very* tight constraint on the pole–zero locations, much more than their simplifying being in the LHP.

8. Describe the mathematical relationship between $i(t)$ and $v(t)$ if they are related by the Laplace transform via Ohm's law

$$V(\omega) = Z(s)I(\omega).$$

Solution: From Ohm's law $Z = V/I$, convolution relates the voltage and current. That is

$$v(t) \equiv z(t) \star i(t) = \int_{\tau=-\infty}^{t} z(t-\tau)i(\tau)d\tau = \int_{\tau=0}^{\infty} z(\tau)i(t-\tau)d\tau \leftrightarrow V(\omega) = Z(s)I(\omega),$$

where $z(t) \leftrightarrow Z(s) \equiv \frac{V(\omega)}{I(\omega)}$. Here $v(t) \leftrightarrow V(\omega)$ and $i(t) \leftrightarrow I(\omega)$ are FTs of the voltage and current. In general $v(t)$ and $i(t)$ do not have LTs, since they are signals, not systems, thus the system properties (*linear*, *causality*, *active*, ...) have no physical significance. Impedances are the most common functions having a LT, and therefore, poles and zeros. Most books do not discuss this most obvious case, of mixing signals with impedances. It only considers signal that have a LT. For example, when a noise is filtered by a low-pass RC filter having a linear causal time-invariant *transfer function* $H(s) \equiv V_o/V_i = R/(R + 1/sC)$, the input an output noise signals will not have a LT. Such problems abound (are common).