Eigenvectors: The eigenvectors simplifying even more

$$[\boldsymbol{E}_\pm] = \begin{bmatrix} \frac{1}{2C}\left[(A-D) \mp \sqrt{(A-D)^2+4BC}\right] \\ 1 \end{bmatrix} = \begin{bmatrix} \mp\sqrt{\frac{B}{C}} \\ 1 \end{bmatrix}$$

Eigen-matrix:

$$\boldsymbol{E} = \begin{bmatrix} -\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{E}^{-1} = \frac{1}{2}\begin{bmatrix} -\sqrt{\frac{C}{B}} & 1 \\ +\sqrt{\frac{C}{B}} & 1 \end{bmatrix}$$

There are two important invariants for the transmission lines, the wave number $k = \sqrt{BC} = s/c$ and the characteristic resistance $r_o = \sqrt{B/C}$.

Transfer functions of the transmission line: The velocity transfer function $H_{N,1} = \frac{V_N}{V_1}$ is given by Solution:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T}^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}$$

along with the eigenvalue expansion

$$\boldsymbol{T}^N = E\Lambda^N E^{-1} = E \begin{bmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{bmatrix} E^{-1}.$$

Topic of this homework: Experimental measurements of a 2-port network. The first objective is become familiar with the computer measurement system. This is a lab study where you will use the computer and Network analyzer (MU box) to measure the 2-port properties of two simple electrical networks.

Deliverable: Measure the circuits shown in the Fig. below. Using Matlab or Octave, show the four $\mathbf{T}(s)$ matrix elements, magnitude and phase. Explain how you measured and computed these functions. Also compute the four $\mathbf{Z}(s)$ matrix parameters. Explain how you measured them.

Make sure the plots look professional, have log-log scales, and restrict the results between 50 to 10^4 [Hz], (i.e., the audio, audible, frequency range). Write a report (i.e., a few pages long) that outlines what you found and how you did the analysis.

I request a paper copy, with your name on it, stapled. **No** files.doc, no emailed homework.

Due Date: If you hand it in late, you will get zero credit. Some credit is better than NO credit.

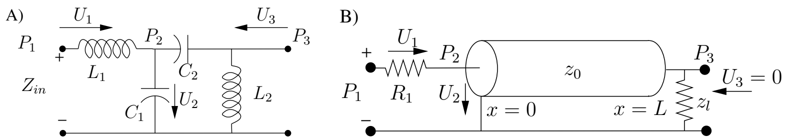
Notes:

- This homework will be discussed in class on Disc: Feb 14, so you may want to be there.
- You should work as a team to collect the raw data. But then each person is to do there own analysis and write-up. Obviously you can, and should, discuss how to do the analysis, as much as you wish, with your team (or others). However, you should not share computer files, including the final graphs. In other words, you need to process all the words you write through your eyes and your fingers. When ever you use material from someone else, give a citation.

1 ABCD Matrix method

We seek to demonstrate that the traditional methods of circuit analysis can require much more algebra than using the transmission matrix method, which is based on 2x2 matrix algebra.

Two circuits are shown In the figure below: Left: A) and Right: B).



1. Analyze circuit A, as follows:

   (a) Use a "traditional" analysis (define the impedances, and use formulas for series $z = z_1 + z_2$ and parallel $z = z_1 || z_2 = \frac{z_1 z_2}{z_1 + z_2}$ combinations), to obtain the input impedance $Z_{in}(s)$, where s is the complex frequency ($s = \sigma + j\omega$), as defined for the *Laplace Transform*. For example, the input impedance is

   $$Z_{in} = sL_1 + Z_p$$

   where

   $$Z_p = \frac{1}{sC_1} || \left(\frac{1}{sC_2} + sL_2 \right)$$

   Write out the formula for $Z_{in}(s)$ in terms of the component values L_1, C_1, etc. Solution:

   $$Z_{in} = sL_1 + \frac{1}{sC_1} || \left(\frac{1}{sC_2} + sL_2 \right) = \frac{sL_1 + (1 + s^2 L_2 C_2 / C_1)}{s(C_1 + C_2 + s^2 L_2 C_1 C_2)}$$

   I hope you agree that the algebra is error-prone.

   (b) Next find the formula for $H_{32} = P_3/P_2$ (the pressure transfer function)? Solution: Using the traditional approach requires the use of the "voltage divider rule," which in this case is is the "pressure rule"

   $$H_{32} = \frac{P_3}{P_2} = \frac{1/sC_2}{sL_2 + sC_2}.$$

   (c) Note that the related formula for $H_{31} = P_3/P_1$ is more complicated. Comment on, or if you can, write out the formula. Solution:

   $$H_{31} = \frac{P_3}{P_1} = \frac{sL_1}{sL_1 + Z_p}$$

   with

   $$Z_p = (1/sC_1) || (sL_2 + 1/sC_2).$$

   There is a much better way.

   (d) Use the ABCD (Transmission) matrix approach to find the transmission (ABCD) matrix for circuit *A*.

   Solution: Circuit A

   $$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & sL_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/sC_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/sL_2 & 1 \end{bmatrix} \begin{bmatrix} P_3 \\ -U_3 \end{bmatrix}$$

   To find the input impedance, do the matrix algebra, set $U_2 = 0$ and then take the ratio of the two equations. In this case all the algebra is confined, even easily implemented via Matlab/Octave complex matrix 2x2 matrix algebra manipulations.

Solution: From the lower equation we see that $V_1 = sCF_2 - (s^2 MC/2 + 1)V_2$. Recall that $F_2 = 0$, thus

$$\frac{V_2}{V_1} = \frac{-1}{s^2 MC/2 + 1} = \left(\frac{c_+}{s - s_+} + \frac{c_-}{s - s_-} \right).$$

with $s_\pm = \pm \jmath \sqrt{\frac{2}{MC}}$ and $c_\pm = \pm \jmath / \sqrt{2MC}$.

3. Find $h_{21}(t)$, the inverse Laplace transform of $H_{21}(s)$. Solution:

$$h(t) = \oint_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s^2 MC/2 + 1} \frac{ds}{2\pi j} = c_+ e^{-s_+ t} u(t) + c_- e^{-s_- t} u(t).$$

The integral follows from the CRT. The poles are at $s_\pm = \pm \jmath \sqrt{\frac{2}{MC}}$ and the residues are $c_\pm = \pm \jmath / \sqrt{2MC}$.

4. What is the input impedance $Z_2 = F_2/V_2$ if $F_3 = -r_0 V_3$? Solution:

Starting from T calculated above, find Z_2

$$Z_2(s) = \frac{F_2}{V_2} = T \begin{bmatrix} F_3 \\ -V_3 \end{bmatrix} = \frac{-(1 + s^2 CM/2) r_0 \cancel{V_3} - (sM/2)(2 + s^2 CM/2) \cancel{V_3}}{-sCr_0 \cancel{V_3} - (1 + s^2 CM/2) \cancel{V_3}}$$

5. Simplify the expression for Z_2 with $N \to \infty$ by assuming that:
1) $F_3 = -r_0 V_3$ (i.e., V_3 cancels), 2) $s^2 MC << 1$: 3) $r_0 = \sqrt{M/C}$ Solution:

$$Z_2(s) = \frac{(1 + s^2 CM/2) r_0 + (sM/2)(2 + s^2 CM/2)}{sCr_0 + (1 + s^2 CM/2)} \approx \frac{r_0 + sM}{sCr_0 + 1}$$

Assumption 3 gives

$$Z_2 = \frac{sM + \sqrt{\frac{M}{C}}}{sC\sqrt{\frac{M}{C}} + 1} = r_0 \frac{\cancel{1 + s\sqrt{MC}}}{\cancel{1 + s\sqrt{MC}}}$$

The constant r_0 is called the *characteristic impedance*.This shows that below the cutoff frequency (approximation 2), the system approximates a transmission line.

6. State the ABCD matrix relationship between the first and Nth node, in terms of of the cell matrix. Solution:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T}^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}$$

7. Given a $\boldsymbol{T}$ (ABCD) transmission matrix, the eigenvalues are and vectors are given in Appendix C of the Notes (p. 143), repeated here.

Eigenvalues:

$$\begin{bmatrix} \lambda_+ \\ \lambda_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (A + D) - \sqrt{(A - D)^2 + 4BC} \\ (A + D) + \sqrt{(A - D)^2 + 4BC} \end{bmatrix}$$

Due to symmetry, $A = D$, this simplifies to $\lambda_\pm = A \mp \sqrt{BC}$ so that the eigen matrix is

$$\Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix}$$

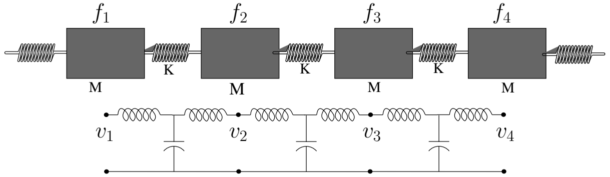


Figure 4: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $v_n(t)$.*

3.2 Transfer functions of a transmission line

In this problem, we will look at the transfer function of a two-port network, shown in Fig. 4. We wish to model the dynamics of a freight-train having N such cars. The model of the train consists of masses connected by springs.

The velocity *transfer function* for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity V_1 and each car responding with a velocity of V_n. Then

$$H(s) = \frac{V_N(s)}{V_1(s)}$$

is the frequency domain ratio of the last car having velocity V_N to V_1, the velocity of the engine, at the left most spring (i.e., coupler).

To do: Use the ABCD method to find the matrix representation of Fig. 4. Consistent with the figure, break the model into cells each consisting of three elements: a series inductor representing half the mass ($L = M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$). Each cell is symmetric, making the model a cascade of identical cells.

At each node define the force $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$ at junction n.

1. Write the ABCD matrix $\boldsymbol{T}$ for a single cell, composed of series mass $M/2$, shunt compliance C and series mass $M/2$, that relates the input node 1 to node 2 where

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} F_2(\omega) \\ -V_2(\omega) \end{bmatrix}$$

Note that here the mechanical force F is analogous to electrical voltage, and the mechanical velocity V is analogous to electrical current.

Solution:

$$\boldsymbol{T} = \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + s^2MC/2 & (sM/2)(2 + s^2MC/2) \\ sC & 1 + s^2MC/2 \end{bmatrix} \tag{16}$$

2. Assuming that $N = 2$ and that $F_2 = 0$ (two mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\boldsymbol{T}$ matrix you determined above, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

2. Analyize circuit B, with a transmission line terminated at each end with a impedance $z_l(s)$. Besides the TL *characteristic impedance* z_o, every acoustic TL has a wave speed $c = 345$ [m/s], a length L, and a cross-sectional area A_o. When sound enters at $x = 0$, it arrives with a delay T that depends on the length L. Since $c = L/T$, $T = L/c$.

At $x = L$ the wave is reflected with a reflection coefficient

$$\Gamma = \frac{z_o - z_l}{z_o + z_l}, \tag{1}$$

where $z_o = \rho_o c/A_o$ is the characteristic impedance of the TL and z_l is the acoustic load impedance. Can you prove this? Solution: Basic texts on TLs. It is more than we need to get into right now.

When the TL is terminated with its characteristic impedance, namely when $z_l = z_o$, $\Gamma = 0$, thus removing the reflections, and therefore standing waves. One way to achieve this is to let $L = \infty$, which assures that the load impedance is equal to z_o

(a) Assuming $U_3 = 0$ (open-circuit at output), find the input impedance: Solution:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh(\omega L/c) & z_0 \sinh(\omega L/c) \\ y_o \sinh(\omega L/c) & \cosh(\omega L/c) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y_0 & 1 \end{bmatrix} \begin{bmatrix} P_3 \\ U_3 = 0 \end{bmatrix}, \tag{2}$$

where the *characteristic admittance* is $y_o = 1/z_o$. As before, take the ratio of the two equations to find the impedance. The pressure P_3 cancels in this ratio.

(b) If the load impedance $z_l = 0$, describe transfer function, i.e., the poles and zeros of the input impedance of the transmission line. Hint: It is described by standing waves, due to the round-trip delay. Use Eq. 1. The zeros are determined by the numerator, and the poles by the denominator. Solution: The input impedance is given by

$$Z_{in}(s) = R_1 + z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L}.$$

One must solve this transcendental equation for the poles and zeros of $Z_{in}(s_k)$, at frequencies s_k in the complex s plane.

The poles of the input impedance are given by

$$Z_{in}(s_p) = R_1 + z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} = \infty$$

while the zeros are

$$Z_{in}(s_z) = R_1 + z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} = 0,$$

where

$$\Gamma_L = \frac{z_l - z_0}{z_l + z_0} e^{-s2L/c} = -e^{-s2L/c} \leftrightarrow -\delta(t - 2L/c),$$

since $z_l = 0$. Γ_L is called the *propagated reflectance*, defined as the ratio of reflected to incident wave at the input to the transmission line (on the left of the figure).

The zeros and poles are approximately given by the poles and zeros of the input impedance with $R_1 = 0$, namely

$$\frac{1 - e^{-s_k 2L/c}}{1 + e^{-s_k 2L/c}}.$$

Thus the zeros are approximately (assuming $R_1 = 0$) given by $e^{-s_z 2L/c} = 1$, and the poles by $e^{-s_p 2L/c} = -1$.

When $R_1 = 0$ it is clear that the poles and zero must alternate along the $j\omega$ axis. When $R_1 = z_o$, there are not reflections at the input. For $R_1 \neq 0$ the situation is more complex, due to reflections at $x = 0$.

2 2-port Network Measurements and Analysis (Lab 3)

This assignment will require making measurements on several different electrical 2-port networks, and then doing some simple analysis on them. You will need to learn how to measure the response using the computer system, and analyze the resulting files using simple Matlab tools.

Your results must be clearly explained in a short report, with figures and equations that shows your analysis and explains your conclusions.

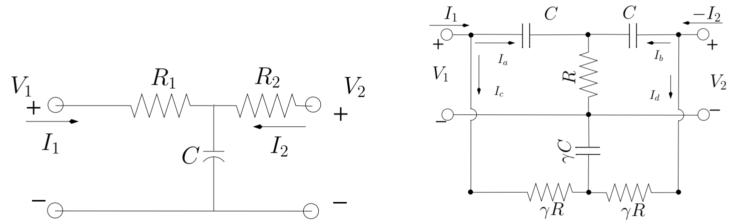


Figure 1: **Left:** *This RC-lowpass circuit is meant to be trivial to analyze. The resistors are in kilo-ohms. Let $Z_1 = 10$ kΩ and $C = 0.1$ μFd.* **Right:** *This 3-element (R and C) circuit is called a Twin-T. It is far more interesting than the first circuit. Try to figure out how it works. You can look it up on Google, or wikipedia for help. The values of C and R will be pick by the TA, and the circuit will be pre-configured for you. Do the analysis in terms of C and R. First analyze each of the parallel circuits, and then add the currents together (e.g., $I_1 = I_a + I_c$) to get the total current, and thus the final matrix representation. Factor γ is normally 2. The wires are only connected if a solid dot is indicated at their intersection point.*

For the circuit on the right, the following method is used:

1. Find the T matrix for the upper and lower circuits. Then convert them into an admittance matrix using the transformation given in the Vanvalkenberg handout

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{B} \begin{bmatrix} D & -\Delta_T \\ -1 & A \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{3}$$

   Cal this matrix Y_1.

2. Do the same for the lower circuit, and call this admittance matrix Y_2

$$\begin{bmatrix} I_a \\ I_c \end{bmatrix} = \frac{1}{B} \begin{bmatrix} D & -\Delta_T \\ -1 & A \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \tag{4}$$

3. The addinging the two matrix equations together, adds the currents, giving the desired result.

To Do:

1. Evaluate the impedance matrix and the ABCD matrix for the two circuits of Fig. 1. A Matlab/Octave script has been provide below to plot the solution, so you may check your answer.[1] Solution: First use the method I taught you to find the ABCD matrix. Then convert this into an impedance matrix.

   Solution Analysis: One may also write the system in chain (ABCD) matrix form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{5}$$

[1] `https://web.engr.illinois.edu/~jontalle/uploads/403/EvalResp.m`

5. Find the roots of $D(s)$ Solution: There is a simple pole at $s = 0$, thus the residue form is simply

$$Z(s) = \frac{K}{s} + R + Ms$$

   thus $c_0 = R$ and $c_1 = K$ and $c_{-1} = M$ (there is a pole at $s = \infty$).

6. Assuming that $M = R = K = 1$, find the residue form of the admittance $Y(s) = 1/Z(s)$ (e.g. Eq. 14) in terms of the roots $s_\pm$. You may check your answer with the Matlab's `residue` command. Solution: First find the roots of the numerator of $Z(s)$ (the denominator of $Y(s)$):

$$s_\pm^2 + s_\pm + 1 = (s_\pm + 1/2)^2 + 3/4 = 0,$$

   which is

$$s_\pm = \frac{-1 \pm \jmath\sqrt{3}}{2}.$$

7. Second form a *partial fraction expansion* (PFA)

$$\frac{s}{1+s+s^2} = c_0 + \frac{c_+}{s-s_+} + \frac{c_-}{s-s_-} = \frac{s(c_+ + c_-) - (c_+ s_- + c_- s_+)}{1+s+s^2}. \tag{15}$$

   For this to apply, the degree of polynomial $N(s)$ must be less than the degree of polynomial $D(s)$. If this is not the case, then one must reduce it by dividing $D(s)$ into $N(s)$.

   To verify that you understand how this expansion works, do some examples using the Matlab/Octave command `[R,P,K]=residue(N,D)`, where vector R contains the residues, P the poles, and K the *direct terms* (the polynomial that results from dividing A into B, so that $\text{Deg}(N) < \text{Deg}(D)$. Then do these examples by hand. Start with simple cases where you are sure you know the answer. You can start with the residues and poles and find $N(s), D(s)$ with the command `[N,D]=residue(R,P,K)` command, i.e., with three arguments. Use the Matlab/Octave `help residue` or `doc residue`, for a detailed explaination of how this works.

   Comparing the two sides shows that $c_0 = 0$. We also have two equations for the residues $c_+ + c_- = 1$ and $c_+ s_- + c_- s_+ = 0$. The best way to solve this is to set up a matrix relation and take the inverse

$$\begin{bmatrix} 1 & 1 \\ s_- & s_+ \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{thus:} \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{s_+ - s_-} \begin{bmatrix} s_+ & -1 \\ -s_- & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

   which gives $c_\pm = \pm \frac{s_\pm}{s_+ - s_-}$ The denominator is $s_+ - s_- = \jmath\sqrt{3}$ and the numerator is $\pm 1 + \jmath\sqrt{3}$. Thus

$$c_\pm = \pm \frac{s_\pm}{s_+ - s_-} = \frac{1}{2}\left(1 \pm \frac{\jmath}{\sqrt{3}}\right).$$

   As always, finding the coefficients is always the most difficult part. Using 2x2 matrix algebra can really simplify up the process, and will be more likely to get the right answer.

8. Use the residue form method (Eq. ??) of the expression that you derived in the previous exercise. Solution:

$$z(t) = \frac{1}{2\pi\jmath} \oint_{\mathcal{C}} Z(s) e^{st} ds.$$

   were $\mathcal{C}$ is the Laplace contour which encloses the entire left-half s plane. Applying the CRT

$$z(t) = c_+ e^{s_+ t} + c_- e^{s_- t}.$$

   where $s_\pm = -1/2 \pm \jmath\sqrt{3}/2$ and $c_\pm = 1/2 \pm \jmath/(2\sqrt{3})$.

1. Find the Laplace transform ($\mathcal{L}$) of the three force relations in terms of the force $F(s)$ and the velocity V(s), along with the electrical equivalent impedance:

   (a) Hooke's Law $f(t) = Kx(t)$. Solution: Taking the $\mathcal{L}$ gives

   $$F(s) = KX(s) = KV(s)/s,$$

   since

   $$v(t) = \frac{d}{dt}x(t) \leftrightarrow sX(s).$$

   Thus the impedance of the spring is

   $$Z_s(s) = \frac{K}{s},$$

   which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as $C = 1/K$.

   (b) Dash-pot resistance $f(t) = Rv(t)$. Solution: From the $\mathcal{L}$ this becomes

   $$F(s) = RV(s)$$

   and the impedance of the dash-pot is then

   $$Z_r = R,$$

   analogous to that of an electrical resistor.

   (c) Newton's Law for Mass $f(t) = Mdv(t)/dt$. Solution: Taking the $\mathcal{L}$ gives

   $$F(s) = sMV(s)$$

   thus

   $$Z_m(s) = sM,$$

   analogous to an electrical inductor.

2. Take the Laplace transform ($\mathcal{L}$) of Eq. 13, and find the total impedance $Z(s)$ of the mechanical circuit. Solution: From the properties of the $\mathcal{L}$ that $dx/dt \leftrightarrow sX(s)$, we find

   $$Ms^2X(s) + RsX(s) + KX(s) = F(s).$$

   In terms of velocity this is $(Ms + R + K/s)V(s) = F(s)$. Thus the circuit impedance is

   $$Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s}.$$

3. The impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in *residue form* as

   $$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}. \qquad (14)$$

4. What are $N(s)$ and $D(s)$ Eq. 14 for the LT of Eq. 13 Solution: $D(s) = s$ and $N(s) = K + Rs + Ms^2$.

Next we let each of the impedances have the value $Z_1 = 1$[K Ohms], $Z_2 = 1/s$ (C=1), $Y_2 = s$, $Z_3 = 3$[K Ohms], and then we multiply out the three matrices.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} s+1 & 1 \\ s & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} s+1 & 3s+4 \\ s & 3s+1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \qquad (6)$$

To verify this is reciprocal, take the determinate of the last matrix Δ_T, and it must be 1.

Finally we may convert this to an impedance matrix using the formula

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & \Delta_T \\ 1 & D \end{bmatrix} = \frac{1}{s} \begin{bmatrix} s+1 & 1 \\ 1 & 3s+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \qquad (7)$$

This may easily checked by comparing it to the circuit. The input impedance with $I_2 = 0$ is $R_1 + 1/sC$. Don't confuse the $C(s)$ of the ABCD matrix with the capacitor C of the circuit of Fig. 1.

2. For the two circuits: Using the network analyizer (MU-box) provided in the lab, measure the input impedance and transfer function, in both directions, and then from your measurements, compute the impedance matrix elements. This is the basis of Lab 2.

   Solution: Figure 1: For this example you really don't need the computer, just a VOM (volt-ohm meter). Of course that would not teach you much about how to analyze Fig. 1 (right). Learning how to use the computer to measure the impedance and transfer functions, will be taught in class, and you will need to practice doing this on your own. Some elementary Matlab is necessary to compute and plot the frequency response of the 2-port elements.

   Figure 1 (right) is the same to measure as the circuit on the left, but a lot harder to calculate. Rather than finding the impedance matrix, one must find the two admittance matrices of the top and bottom half of each circuit. Then by adding the currents, and recognizing that the voltage outputs are the same, one may show that the final admittance matrix is the sum of the two individual matrices.

   Here is how to proceed: From the upper half of the circuit

   $$\begin{bmatrix} V_1 \\ I_a \end{bmatrix} = \begin{bmatrix} 1 & 1/sC \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/R & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/sC \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_b \end{bmatrix} = \frac{1}{s\tau} \begin{bmatrix} s\tau+1 & 2R+1/sC \\ sC & s\tau \end{bmatrix} \begin{bmatrix} V_2 \\ -I_b \end{bmatrix} \qquad (8)$$

   with the time constant of $\tau = RC$. For the lower circuit

   $$\begin{bmatrix} V_1 \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & \gamma R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s\gamma C & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_d \end{bmatrix} = s\tau \begin{bmatrix} 1/s\tau + \gamma^2 & \gamma R(2/s\tau + \gamma^2) \\ \gamma/R & 1/s\tau \end{bmatrix} \begin{bmatrix} V_2 \\ -I_d \end{bmatrix} \qquad (9)$$

   Next these T matrices need to be converted to admittance matrices. This is done with a transformation given in the handout. For the first matrix we find:

   $$\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \frac{1}{B} \begin{bmatrix} D & -1 \\ -1 & A \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \frac{sC}{s2RC+1} \begin{bmatrix} sRC & -1 \\ -1 & 1+sRC \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} \qquad (10)$$

   For the second matrix this is

   $$\begin{bmatrix} I_c \\ I_d \end{bmatrix} = \frac{1}{\gamma R(2+s\gamma^2 RC)} \begin{bmatrix} 1 & -1 \\ -1 & 1+s\gamma^2 RC \end{bmatrix} \begin{bmatrix} V_c \\ V_d \end{bmatrix} \qquad (11)$$

   If we add these two matrices together, we find the total input current $I_1 = I_a + I_c$ and $I_2 = I_c + I_d$ as the sums of the two branches. Note that the voltages on the right are the same, so they may be factored out. Thus

   $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \left\{ \frac{sC}{s2RC+1} \begin{bmatrix} sRC & -1 \\ -1 & 1+sRC \end{bmatrix} + \frac{1}{\gamma R(2+s\gamma^2 RC)} \begin{bmatrix} 1 & -1 \\ -1 & 1+s\gamma^2 RC \end{bmatrix} \right\} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (12)$$

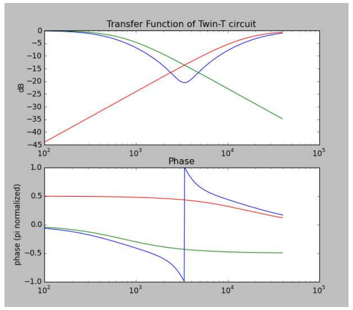


Figure 2: Twin-T frequency response. The red curve is the high-pass response while the blue curve is the lowpass response. The composite curve shows the null around 3.5 [kHz]. The lower curve shows the corresponding unwrapped phase response.

Note that there is no need to add these two matrices together, as that is just busy work that makes the answer much more complex.

We turn to Matlab for the final analysis.

```
% Matlab code to evaluate the RC lowpass filter response of the first 2-port problem
% HWb, Feb 7, Section 2 Fig 1, left.

NFT=1024; NF=1+NFT/2;%number of non-negative frequencies
Fmax=1e4; Fs=2*Fmax; Fmin=Fs/NFT;
 f=0:Fmin:Fmax; %sweep frequency from 0 to Fmax [Hz] in steps of Fmin [Hz]

R1=1e4; R2=1e4; %in [Ohms]
C=0.1e-6; %[Farads]
T=zeros(NF,2,2); %Init output array

for k=1:NF;
s=2*pi*f(k)*1j; %Define complex frequency

%Define three matrices for the three elements in the network
T1=[1 R1;0 1];
T2=[1 0; s*C 1];
T3=[1 R2;0 1];
T(k,:,:)=T1*T2*T3; %avoid all the messy algebra

end %end for freq loop

%Save T matrix elements
kf=2:(NF-1); %freq limits; avoid ends
A=T(kf,1,1); B=T(kf,1,2);
C=T(kf,2,1); D=T(kf,2,2);

%compute Z matrix Z1, Z2, Z12
Z1=D./C; Z12=1./C;
Z2=A./C; Z21=12; % Assumes reciprocity
```


3 Laplace transforms

Given a *Laplace transform* ($\mathcal{L}$) pair $f(t) \leftrightarrow F(s)$, the frequency domain will always be written in an upper-case font [e.g. $F(s)$], while the time domain is written in lower case font [$f(t)$]. The definition of the forward transform ($f(t) \to F(s)$) is

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt,$$

where $s = \sigma + j\omega$ is the complex Laplace frequency in [radians] and t is time in [seconds]. Note that the Laplace transform the time function must be causal (i.e., $f(t < 0) = 0$).

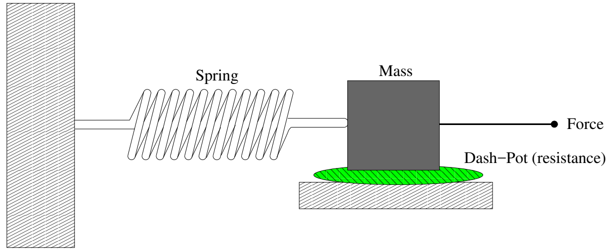


Figure 3: *This three element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).*

An example: Hooke's Law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dash-pot is $f(t) = Rv(t)$, and Newton's famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for constant M is $f(t) = Mdv/dt$ (Newton's second Law of motion).

The equation of motion for the mechanical oscillator in Fig. 3 is given by Newton's third law (the sum of the forces must balance to zero)[2]

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t). \tag{13}$$

These three constants, the mass M, resistance R and stiffness K are all real and positive.. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

Newton's second law (1687) is the mechanical equivalent of Kirchhoff's (1845) voltage law (KCL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$).

Equation 13 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

3.1 Impedance

For each of the three mechanical items, the Mass, the dash-pot and the spring, define their impedance, defined as the ratio of the force over the velocity.

[2]It may be helpful to point out that Newton's laws of motion (c1687) have much in common with Kirchhoff's (1845) current and voltage laws of electricity.