Topic of this homework: Loudspeaker Impedance; Analytic power series; Acoustic Signal processing Acoustics; Fourier Transform; Signal processing;

Deliverable: Show your work.

If you hand it in late, you will get zero credit (I will be handing out my solution at that time). You will only get credit for what you hand in. I want a paper copy, with your name on it. Please **no** files.doc.

No matter how limited your results, on the due date submit what ever you have. Some credit is better than NO credit.

Note: This homework will be discussed by the entire class on Disc: 2/28. You need to be there. Each person is to do there own final writeup, but obviously you can discuss it as much as you like between yourselves.

1 Model of a loudspeaker

The attached figure shows the equivalent circuit for an electro-dynamic earphone.

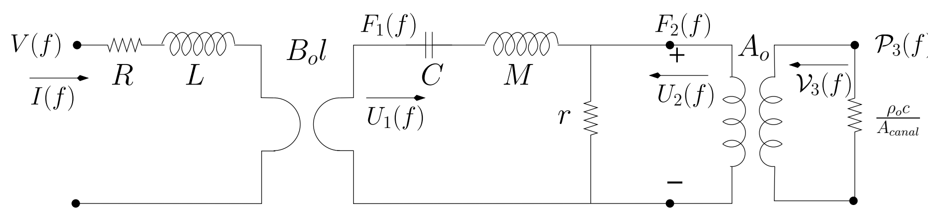

Figure 1: *Equivalent circuit for an earphone (?, p. 109). This model is taken from ?. There are three sections, the electrical input (left), the mechanical response (center), and the acoustic output (right). The electrical input in in terms of the voltage $V(f)$ and current $I(f)$. There are two elements, the coil resistance R and its inductance L. The center section corresponds to the mechanical components, with a compliance C (spring), mass M and mechanical damping r. The mechanical force $F_2(f)$ and a velocity $U_2(f)$ are the input to the transformer which converts the force into a pressure. The diaphragm has an area A, which results in a pressure $\mathcal{P}_2(f) = F_1/A_0$, and a volume velocity $\mathcal{V}_\in(f) = A_0 U(f)$ at the right.*

1. Based on the parameters given in **?**, the following segment of Matlab/Octave will model the loudspeaker

```
NFT=1024; NF=1+NFT/2;%number of non-negative frequencies
Fmax=1e4; Fs=2*Fmax; Fmin=Fs/NFT;
f=0:Fmin:Fmax; %sweep frequency from 0 to Fmax [Hz] in steps of Fmin [Hz]

%Define basic parameters based on Kim and Allen
R=195; L=9e-3; %electrical section
G = 7.5; %Gryator Bo l: Electrical to Mechanical transformation
C=1.25e-3; M=4.3e-6;r=3e-3; %Mechanical section
A = 2.4e-6; %Mechanical to Acoustical transformation

for k=1:NF;
s=2*pi*f(k)*1j; %Define complex frequency

%Define three matrices for the three elements in the network
Te=[1 R+s*L;0 1]; %electrical
Tem = [0 G; 1/G 0]; %gyrator electrical to mechanics
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Tm = [1 0; s*C 1]*[1 s*M; 0 1]*[1 r; 0 1]; %Mechanical section
Tma = [1/A 0; 0 A]; %coupling of mechanics to acoustics

T(k,:,:)=Te*Tem*Tm*Tma; %avoids a nightmare of algebra

end %end for freq loop
```

2. Explain how this code works. For full credit you must be clear. Identify the key lines of code, and explain how they work.

3. Plot out the transfer function between an input of one volt $V(f) = 1$ and

   (a) The current into the gyrator $I(f)/V(f)$ with the speaker motor blocked ($U_1 = 0$)

   (b) $F_1(f)/V(f)$ with $U_1 = 0$ (blocked)

   (c) $U_1(f)/V(f)$ with $F_1(f) = 0$ (open circuit)

   (d) $F_2(f)/V(f)$ with $U_2 = 0$

   (e) $U_2(f)/V(f)$ with $F_2(f) = 0$

   (f) $\mathcal{P}(f)/V(f)$ with $V_3(f) = 0$, expressed in Pascals [Pa].

   In each of the above response, discuss the bandwidth and properties of each transfer function in terms of the circuit elements.

4. In a final summary figure, assume the earphone is term in an infinitely long tube of area A_{canal}, having a radiation impedance of $\rho_o c/A_{canal}$. In this figure plot the following *magnitudes* of the following variables as a function of frequency: generalized forces $F_1, F_2, \mathcal{P}_\ni$, and in a second plot the generalized flows $I, F_1, U_1, U_2 calV_3$. Normalize all the plots to a single value at 1 kHz, so that the dynamic range of the plot is not more than a factor of 100 (40 [dB]), again, so that the figure is readable. Explain what each curve is telling you (identify resonances, and explain their source). For example, the two elements C, M form a resonance. Identify that on your charts.

   Be sure the curve is labeled properly, so that the different variables are easily distinguished.

2 Filter classes

In the following let $s = \sigma + i\omega$ be the Laplace (complex) frequency. *Filters* are causal functions (one sided in time) that modify a *signal* (any function of time) into another signal. For example if $h(t)$ is a filter, and $s(t)$ a signal then

$$y(t) = h(t) \star s(t) \equiv \int_{\infty}^{t} h(t-\tau)x(\tau)d\tau$$

where $\star$ defines convolution.

Both *transfer functions* and *impedances* are in the class of filters.

An important property of a filter is that it is causal, and has a *Laplace transform.* However, a signal has a Fourier transform and may NOT be causal. A signal can be causal, but that is not the norm. An physical real-world filter is always causal. Mathematically one may easily define a non-causal filter (e.g., $(u(-t))$, but it is hard (i.e., impossible) to understand exactly what that would mean in practice.

Background:

1. A *causal* filter $h(t) \leftrightarrow H(s)$ is one that is zero for negative time. It necessarily has a Laplace transform.

2. An *finite impulse response* (FIR) filter has finite duration, namely if $f(t)$ is FIR, then it is zero for $t < 0$ (it is causal) and for $t > T$ where T is positive constant (time). FIR filters only have *zeros* (they do not have poles).

3. An *Infinite impulse response* (IIR) filter is one that is non-zero in magnitude as $t \to \infty$, bu it is still causal ($h(t < 0) = 0$). All IIR filters have poles (as well as zeros), namely if $h(t) \leftrightarrow H(s)$ then $H(s)$ has poles in the region $\sigma \le 0$.

4. A *Minimum Phase* filter $m(t) \leftrightarrow M(s)$ is a filter (it must be causal) having the smallest phase (i.e., $\angle M(j\omega)$) of any filter with magnitude $|M(\omega)|$, on the $j\omega$ axis. A minimum phase filter also satisfies the very special condition

$$\aleph(t) \leftrightarrow N(s) \equiv \frac{1}{M(s)},$$

namely the inverse of $m(t)$ is causal. Thus $m(t) \star \aleph(t) = \delta(t)$ where $\star$ represents convolution. All impedances are minimum phase (every impedance $Z(s)$ has a corresponding admittance $Y(s) \equiv 1/Z(s)$.

5. An *all-pass* filter modifies the *phase* but not the *magnitude* of a signal; namely if $a(t)$ is a causal impulse response of a causal all-pass filter, having Fourier Transform $A(\omega) \equiv |A(\omega)|e^{i\phi(\omega)}$, then $|A(\omega)| = 1$. Thus the phase $\phi(\omega)$ completely specifies the all-pass filter. The group delay is defined as

$$\tau_g(\omega) \equiv -\frac{\partial \phi(\omega)}{\partial \omega},$$

Since

$$\phi(\omega) = \int_0^\omega \tau_g(\omega) d\omega,$$

the group delay also may be used as the definition of an all-pass (i.e., the filter can bederived the group delay).

A causal all-pass filter having a real time response ($a(t)$ real and causal), must have its poles and zeros symmetrically located across both the σ and $j\omega$ axes. For example a pole at $s_p = -1 + j$ and a zero at $s_z = 1 + j$

$$\tilde{A}(s) = \frac{s - 1 - j}{s + 1 - j}$$

would produce an all-pass response. This is because $|\tilde{A}|_{s=j\omega} = 1$ (verify this for yourself). The inverse Laplace transform $a(t) \leftrightarrow A(s)$ is be complex because the conjugate poles and zeros have been ignored in this example. To repair this (to force $a(t)$ to be real), the full filter must be

$$A(\omega) \equiv \tilde{A}(\omega) \cdot \tilde{A}^*(\omega) = \frac{s - 1 - j}{s + 1 - j} \cdot \frac{s - 1 + j}{s + 1 - j}.$$

6. A *positive real* (PR) filter $z(t) \leftrightarrow Z(s) = R(s) + iX(s)$ is both minimum phase, and has a positive real part in the right half s plane, namely

$$R(\sigma > 0) > 0$$

that is, for $\sigma > 0$ $\Re Z > 0$.

Every impedance $z(t) \leftrightarrow Z(s)$ is PR. Since impedance is used in the definition of power, it represents a *positive definite* operator (a fancy name for a filter). For example, if one convolves a current $i(t)$ with an impedance, a voltage results. Namely $v(t) = z(t) \star i(t)$. Since power is voltage times current, the complex power is $\mathcal{P}(t) \equiv v(t)i(t)$. The *time average power* is the time average of $\mathcal{P}(t)$, $\overline{\mathcal{P}(t)} \equiv \int_t \mathcal{P}(t)dt$.

To do: Prove (or discuss in detail) each of the following:

1. The relation $h(t) \star g(t) \leftrightarrow H(\omega)G(s)$.
   (a) Start by writing out the formula for the convolution of $h(t)$ and $g(t)$, denoted $h(t) \star g(t)$.
   (b) Then show that the inverse transform of a product of filters is a convolution.
   (c) The point of this example is that one function is of ω while the other a function of s. How does this impact the convolution?
2. Is an all-pass filter minimum phase? Explain.
3. Prove that $\delta(t-5)$ is all-pass.
4. Is $\delta(t+5)$ all-pass?
5. Is e^{-t} all-pass or minimum phase? Justify your answer.
6. When is $F(s) = \frac{s-a}{s+b}$ all-pass?
7. When is $F(s) = \frac{s-a}{s+b}$ minimum phase?
8. Does $F(s) = \frac{s+j}{s-j}$ have a real impulse response?
9. In the continuous time domain, a pure delay by T [s] may be written as $\delta(t-T) \leftrightarrow e^{-i\omega T}$. Find the expression for z^{-N}, a pure delay of N samples in the discrete time domain.
10. Where are the poles and zeros for
    (a) a Stable filter
    (b) an all-pass filter
    (c) a Minimum phase filter
    (d) an Impedance (PR function)
11. Describe the mathematical relationship between $i(t)$ and $v(t)$ if they are related by the Laplace transform via Ohm's law
$$V(\omega) = Z(s)I(\omega).$$

3 Name that transform

1. You are given a specification of the time and frequency properties of some signals and you are asked to name the Fourier type transform that would be used to analyze these signals
   (a) The time response is zero for $t < 0$ and the frequency response is a function of the radian frequency $\omega = 2\pi f$
   (b) The time response is zero for $t > 0$ and the frequency response is a function of the radian frequency $\omega = 2\pi f$

(c) The time response is given at points $t_n = nT$, where $T = 1/F_s$ with $F_s = 44100$ kHz, and the frequency response is specified outside the unit circle.

(d) The time response is given at times $t[n] = nT$ for integer n and constant T, and the frequencies are given at $f[k] = k/T$

2. Find the Fourier series expansion of the periodic function

$$f((t))_T \equiv \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Give the formula for $F[k]$. Show your work. Explain how you get the solution.

3. Find the Laplace transform of $h(t) = 3e^{-t/\tau}U(t)$. Give an example of an electrical circuit that has this impulse response.

4. If the impulse response of some system is $h(t) = 3e^{t/\tau}U(t+1)$, describe the interesting things about the system.

5. What is the basic idea behind an *analytic function?* Give an example of a function that is analytic, and one that is not.

6. Laplace vs. Fourier

   (a) When do you use a Laplace transform and when do you use the Fourier transform?

   (b) Give an example where you can use both.

   (c) Give an example where you cannot use the Laplace transform.

7. Given the transform pair $f(t) \leftrightarrow F(\omega)$ one may prove that $F^*(t) \leftrightarrow 2\pi f^*(\omega)$.

   Apply this relationship to the following transform pairs, to derive new transform pairs (I worked out the first question, as an example):

   (a) $\delta(t) \leftrightarrow 1$
   Solution: $f(t) = \delta(t) \leftrightarrow F(\omega) = 1$. Thus applying the above relationship we find that if the time function is 1 then the transform is $2\pi\delta(\omega)$.

   (b) $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

   (c) $U(t) \leftrightarrow \pi\delta(\omega) + 1/j\omega$
   All of the above solutions need a careful checking!

4 History

1. Describe some interesting things about Pythagoras. Be sure to include when, where, and why. What might this have to do with Audio Engineering?

2. Give a few reasons that Newton might be relevant to Audio Engineering.

3. What year did Fourier work out his analysis of heat transfer? How did he do it?

References