ECE 403 **HWd – Version 1.16 January 4, 2019** **Spring 2019**

Univ. of Illinois **Lec 16, Disc: Lec 17, Due: Lec 20** **Prof. Allen**

Topic of this homework: Webster Horns: 1D, 3D & Exp Horns; Reflectance; Thermal noise; Hilbert Transform

Deliverable: Show your work.

If you hand it in late, you will get zero credit. I would like a paper copy, with your name on it. **No** doc files.

Some credit is better than NO credit.

1 Wave equation

1.1 History of the wave equation

1. What year did d'Alembert derive his solution to the wave equation?
2. What is the form of D'Alembert's solution?
3. Who was the first person to calculate the speed of sound, and what was the result?

1.2 The Webster wave equation:

The *Webster Horn equation* may be written in the time domain as 1D transmission line equation:

$$\frac{\partial}{\partial x}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix} = -\begin{bmatrix} 0 & \frac{\rho_o}{A(x)} \\ \frac{A(x)}{\gamma P_o} & 0 \end{bmatrix} \frac{\partial}{\partial t}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix}, \tag{1}$$

where $\nu(x,t) = A(x)u(x,t)$ is the *volume velocity*, more generally defined as the integral over the normal component of the particle velocity $u(x,t)$, over the cross-sectional area $A(x)$ of the tube. Transforming to the frequency domain we have

$$\frac{d}{dx}\begin{bmatrix} P(x,\omega) \\ V(x,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & Z_s(s,x) \\ Y_s(s,x) & 0 \end{bmatrix} \begin{bmatrix} P(x,\omega) \\ V(x,\omega). \end{bmatrix} \tag{2}$$

Here we use the complex *Laplace frequency* s when referring to the *per-unit impedance*

$$Z_s(s,x) \equiv s\frac{\rho_o}{A(x)} = sM(x) \tag{3}$$

and *per-unit admittance*

$$Y_s(s,x) \equiv s\frac{A(x)}{\gamma P_o} = sC(x), \tag{4}$$

where $M(x) = \rho_o/A(x)$ is the horn's *per-unit-length mass*, and $C(x) = A(x)/\gamma P_o$ *per-unit-length compliance*, to remind ourselves that these functions must be *causal*, and except at their poles, *analytic* in s.

The horn of a loudspeaker cone is either conical $A(x) = A_0(x/x_0)^2$, or in the shape of an exponential.

1.3 To Do:

1. Assuming a *conical horn*, having area $A(x) = A_o(x/x_o)^2$ with $A_o \leq 4\pi$, rewrite these equations as a second order equation solely in terms of the pressure P (remove U), and thereby find the frequency domain solutions $P_\pm(x, s)$ for the conical horn equation.

2. Starting from the transmission line equations given above, and assuming an exponential area function
$$A(x) = A_0 e^{2mx}$$
(m is a positive constant, called the *horn flair parameter*), derive the exponential horn equation for the pressure
$$\frac{\partial^2 p(x,t)}{\partial x^2} + 2m\frac{\partial p(x,t)}{\partial x} = \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial^2 t} \tag{5}$$

3. In general the soluton to a wave equation is of the form
$$p(x,t) = P^+(\kappa, s)e^{\kappa x}e^{st} + P^-(\kappa, s)e^{-\kappa x}e^{st}$$
where $s = \sigma + \jmath\omega$ is the Laplace frequency and $\kappa(s)$ is the complex "wave number." What is $\kappa(s)$ for the
   (a) conical horn?
   (b) exponential horn?
   (c) What is the significance of $\kappa(s)$?
   (d) Why is it a function of s?
   (e) What is the role of $P^\pm(\kappa, s)$?

4. Show that the solution to Eq. 5 is of the form
$$P^\pm(x, s) = e^{-mx}e^{\mp\sqrt{m^2+(s/c)^2}\,x}.$$

   Discussion: This solution is loss-less under all conditions. For high frequencies, when $\omega > mc$, the pressure changes by e^{-mx} as it propagates with a frequency dependent delay, making it an acoustic transformer. At very high frequencies, when m can be ignored with respect to ω/c, the wave propagates without dispersion, but still with a decay. It is still loss-less. The decay in amplitude is due to the change in area.

   At very low frequencies, the wave solution is still causal, but no longer decays spacial [the exp() term is canceled out]. To see this requires taking the inverse Laplace transform of $P(x, s)$.

   Note the interesting and seemingly related relations (s is the Laplace frequency and $\leftrightarrow$ represents the Laplace transform)
$$\sinh^{-1}(s) = log_e[s + \sqrt{s^2+1}] \tag{6}$$
   (see Matlab's `doc asinh`) and the low-pass *sinc()*-like relation
$$\frac{J_1(t)}{t}u(t) \leftrightarrow \sqrt{s^2+1} - s \tag{7}$$

1.4 Reflectance:

1. Find (derive) the formula for the "input" impedance of a transmission line, having characteristic impedance $z_0(x, s)$, in terms of the reflectance. Define all the terms. Hint, I did this in class several times.

2. Find the formula for the reflectance $R(s)$ in terms of the load impedance $Z_L(s)$ and the characteristic impedance z_0 if:[1]

   (a) $Z_L(x, s) = r$ [Nt-s/m^5]

   (b) $Z_L(x, s) = 1/sC$ [Nt-s/m^5]

   (c) $Z_L(x, s) = r||sM$ [Nt-s/m^5]

   (d) Two transmission lines are in cascade, the first one having an area of 1 [cm^2] and a second having an area of 2 [cm^2], with lengths L_1 and L_2 respectively, terminated with a resistor $r = \rho c/A$, where $A = 2 \times 10^{-4}$ [m^2]. Find $R(x = 0, s)$.

   (e) What is the inverse Laplace transform of

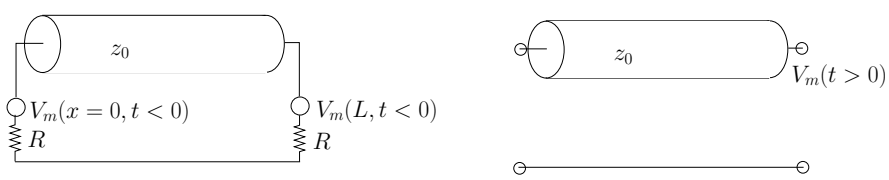
      i. $H(s) = 1/(s + 1)$? Find $h(t)$.

      ii. $R(s) = \frac{Z-z_0}{Z+z_0}$ where $L = 1$, $Z = 1$ and $z_0 = 2$? Find $r(t)$ at the input.

      iii. $H(s) = s/(s + 1)$?

2 Nyquist Thm on Thermal noise

The purpose of this problem is to do a simulation of Harry Nyquist's famous result on the noise of a resistor. The experiment is to use a Thevenin equivalent model of a resistor as a resistance R in series with a voltage source. A stub of transmission line having characteristic impedance z_0 is terminated in each end with this Thevenin model, with $R = z_0$. Then at $t = 0$, the resistance is short or open circuited. If open circuited you may watch the voltage at one end, and if short circuited, you may watch the current. Let's monitor the voltage with an open circuit. This setup is shown in the figure.



The transmission line stores the voltage at $t = 0$ once the switch is opened removing the resistors (and also the Thevenin source). At that point voltage $V_m(L, t)$ becomes periodic with a period of $2L/c$, where L is the length of the line and c is the speed of the wave.

Define an array that has a duration of the period, and load it with thermal random samples [`x=randn(1,N)` with N the number of noise samples]. Set the RMS to 1. The RMS may be computed using `std(x,1)`. Use a sampling period $T = 1/f_s$ with $f_s = 10 \times 10^3$ Hz for this experiment, and let $c = 345$ [m/s] be the speed of sound (an acoustic transmission line), with $L = 10$ [m].

1. What is the fundamental period of the noise?

2. Every periodic signal has a *Fourier series*. If the period is T [s], what are the Fourier series frequecies? You know the answer to the question, but you may need to think about it. This is not difficult.

[1] Note that r, C and M represent an acoustic resistance, compliance and mass. Namely they are positive constants.

3. Plot two periods of the time domain signal. Using `fft()`, find the spectrum of the periodic noise, for 1, 4 and 10 periods. Be sure to properly label all axes (with units)! For each of the FFT plots, show one figure of the full FFT (from 0 Hz up to the half-sample-rate), and one figure zoomed in on the range 0 to ~300 Hz (linear axis, not log axis). Comment/explain any observations.

   Note: Let's say your array length is N. When creating 10 periods of random noise, don't use `randn(1,10*N)`. Use `randn(1,N)` to randomly populate 1 period, and then repeat that noise 10 times, so that it is identical. Remember, its in a delay line.

4. Why won't the values of your spectral peaks be the same as your fellow students? Average the values of your spectral peaks over many noise samples (that is, many initializations of the `randn(1,N)` array), and plot the resulting spectral average. Just do this for the case of 10 periods.

5. Given $T = 300$ degrees Kelvin (300-273 = 27 degrees C) and $k = 1.38 \times 10^{-23}$ [J/degree K], what would the RMS value of the voltage be? Justify your answer. Hint: this is also know as "Johnson/Nyquist Noise". By sampling we inherently band-limit the signal.

3 Hilbert transform

In all parts of this problem $h(t) \leftrightarrow H(s)$ and $H(\omega) = H(s)|_{s=j\omega}$

Analyze the real impulse response

$$h(t) = e^{-t/\tau_0} u(t),$$

with $\tau_0 = 10$ [ms], in terms of its Hilbert transform (integral) relations.

1. Find $H(s)$, the Laplace transform of $h(t)$.
2. Where are the poles of $H(s) \leftrightarrow e^{-t/\tau_0} u(t)$?
3. Evaluate the following:
   (a) Determine $g(t) \equiv h(t) * \delta(t)$. ($*$ represents convolution.)
   (b) Determine $h(t)\delta(t-1)$.
   (c) Determine $h(0)$.
4. Find the real and imaginary parts of $H(\omega) \equiv H(s)|_{s=j\omega}$.
5. Write out the symmetric $h_e(t)$ and antisymmetric $h_o(t)$ functions.
6. Find the Fourier transforms of $h_e(t) \leftrightarrow H_e(\omega)$ and $h_o(t) \leftrightarrow H_o(\omega)$.
7. Find the Hilbert (integral) relations between $H_r \equiv \Re H(\omega)$ (real part) and $H_i \equiv \Im H(\omega)$ (imag part) of $H(\omega)$.