Absolute Measurement of Sound Without a Primary Standard

W. R. MacLean
101 Lafayette Avenue, Brooklyn, New York
(Received April 23, 1940)

An application of the principle of reciprocity applied to electroacoustic transducers led W. Schottky some time ago to derive a universal relationship between the action of a reversible unit as a microphone and as a loudspeaker.[1] His results, properly used, can lead to the absolute measurement of sound by means of purely electrical observations. Such absolute measurement in its practical form amounts to the calibration of a microphone. (Or in its less practical form, to the calibration of a loudspeaker.) It is the purpose of this paper to outline the theoretical basis for this type of measurement. The microphone undergoing calibration does not itself have to be reversible.

A first application of Schottky's results leads to a free field calibration in terms of either open-circuit voltage or short circuit current. A slightly different application, carried out in a chamber instead of in free space, leads to a pressure calibration in terms of the same units. These two methods are sketched out below.

First Consequences of Reciprocity

The results of this section, in part the same as those obtained by Schottky, are based upon the assumption of the validity for this case of one of the laws of reciprocity, which may be stated as follows: In a passive, linear 4-pole, (1) the two short circuit transfer impedances are equal, (2) the two open-circuit transfer admittances are equal, and (3) the open-circuit voltage ratio in one direction is equal to the short circuit current ratio in the other direction. For both electrical and mechanical cases involving a finite number of degrees of freedom, this follows from the symmetry of the system determinant. However, two reversible electroacoustic transducers coupled to the same medium and accessible only through the two pairs of electrical terminals form a 4-pole which is neither wholly mechanical nor electrical. A fully general and satisfactory proof of the validity of the law has not, to the knowledge of the writer, been given for this case. For that reason, its applicability is herein assumed; however, in an appendix, the scope of the assumption is reduced by proving the theorem from a more elementary assumption. In the above, the two transducers and the surroundings have been taken as linear and passive, but otherwise, quite general.

This generality may now be restricted. The surroundings are assumed to be free space, one of the transducers, U, may be general, but the other, U', is a pulsating sphere connected through a conservative mechanism to an electromechanical transducer of one degree of freedom on each side. The mechanical system is assumed to be highly damped, and the radius of the sphere always negligibly small compared with the wave-length. Two points, O and D, separated a distance d, are chosen in free space. U' is placed at D. A point on, in, or near U, but fixed with respect to it, is chosen, as well as a direction also fixed with respect to this unit. The point is called the center, and the direction is called the axis of U, although they are arbitrarily chosen. This transducer is placed with its center coincident with O and with its axis pointing toward D.

With this configuration fixed, the following quantities are defined for U:

M_o: Free field sensitivity as a microphone in open-circuit abvolts per bar
M_s: Free field sensitivity as a microphone in short circuit abamperes per bar
S_o: Free space calibration as a speaker in bars at distance d, per abampere driving current (1)
S_s: Free space calibration as a speaker in bars at distance d, per terminal abvolt driving.

S is in terms of pressure on the (arbitrarily chosen) axis at distance d, from the (arbitrarily chosen) center. M is in terms of f.f. pressure at the center for waves arriving along the axis,

[1] Schottky, "Tiefempfangsgesetz," Zeits. f. Physik **36**, 689 ff (1926).

having at the center a radius of curvature d. Similar primed quantities apply to the transducer U', except that the center is the center of the sphere, and the question of an axis is eliminated by symmetry.

If U is now driven with 1 abampere, the f.f. pressure at D will be S_o. The voltage generated in U' by this field will therefore be $S_o M_o'$.

If the 4-pole is driven in the other direction with the same driving current, the same generated voltage will result. Similarly by using all four of the laws of reciprocity, the following relations are found:

$$\begin{cases} S_o M_o' = S_o' M_o, \\ S_s M_s' = S_s' M_s, \\ S_o M_s' = S_s' M_o, \\ S_s M_o' = S_o' M_s. \end{cases} \tag{2}$$

These reduce to the three equations:

$$\frac{S_0}{M_0} = \frac{S_0'}{M_0'} = \frac{S_s}{M_s} = \frac{S_s'}{M_s'}, \tag{3}$$

which means that for any unit the relation:

$$\frac{M_0}{S_0} = \frac{M_s}{S_s} = H \tag{3'}$$

holds, and that this ratio is independent of design. It remains only to determine the value of H. Since it is independent of design, the ratio can be obtained by computing it for the idealized spherical transducer.

For this purpose, the mechanism of U' is assumed to be devoid of mass, stiffness, and friction, with the exception of a large mechanical impedance Z_m, imposed upon the one degree of freedom coupling with the electrical side. The mechano-electrical transducer is likewise free of impedances, both mechanical and electrical. Electrodynamic coupling is chosen with a force factor, k. The mechanical advantage of the mechanism is so chosen that:

$$Q = \dot{x}, \tag{4}$$

where Q is the volume current of the sphere, and $\dot{x}$ is the velocity of the mechanical coordinate. The mechanism as a 4-pole connecting volume current and linear velocity acts as an ideal transformer with unit turns ratio, i.e., as a direct connection. The impedance Z_m is then in series on the mechanical side. The electromechanical coupler itself has a circuit matrix:

$$\begin{vmatrix} 0 & k \\ -k & 0 \end{vmatrix}. \tag{5}$$

The radiation impedance of a pulsating sphere in terms of volume current and pressure is known to be:

$$Z_r = \frac{r}{4\pi a^2 - j2a\lambda} \doteq j\frac{r}{2a\lambda} \cdots 2\pi a \ll \lambda, \tag{6}$$

where r is the characteristic resistance of the medium, a is the radius of the sphere, and λ is the wave-length. Putting (4), (5) and (6) together, it is seen that the equivalent circuit is as shown in Fig. 1.

The box marked $\boldsymbol{k}$ is the electromechanical transducer. No equivalent circuit can be shown for this since its matrix is not symmetrical. The matrix of the entire 4-pole shown in Fig. 1 is:

$$\begin{vmatrix} (Z_r + Z_m) & k \\ -k & 0 \end{vmatrix} \tag{7}$$

wherein the first diagonal element applies to the acoustical end, and the second to the electrical. The applied pressure at the acoustical end is the free field pressure. The microphone sensitivity in terms of short circuit current is seen to be the reciprocal of the short circuit transfer impedance of Fig. 1 when driven acoustically. This amounts to:

$$M_s' = \frac{k}{k^2} = \frac{1}{k}. \tag{8}$$

The reciprocal of the s.c. transfer impedance when driven electrically is the volume current of the sphere for unit applied voltage, which is numerically the negative of the above value. Multiplication by the radiation resistance of the sphere gives the pressure at the surface of the sphere:

$$P_0 = -\frac{1}{k} Z_r. \tag{9}$$

(9) reduced for divergence and phase at distance d is the free-space voltage calibration of the unit as a speaker:

$$S_s' = -\frac{1}{k} Z_r \frac{a}{d} e^{-j((2\pi/\lambda)d)}. \tag{10}$$

Whence the value for H becomes:

$$H=\frac{M_s'}{S_s'}=-\frac{1}{Z_r}\frac{d}{a}e^{j((2\pi/\lambda)d)}=\frac{2d\lambda}{r}e^{j((2\pi/\lambda)d+\pi/2)}. \quad (11)$$

Ignoring the phase factor:

$$H=\frac{2d\lambda}{r}. \quad (12)$$

By means of (12) the characteristic of a unit as a microphone may be computed in absolute units if its characteristic as a speaker is known, and vice versa. It shows that if any unit is flat as a speaker operated by $\frac{\text{const. current}}{\text{const. voltage}}$ it rises on the low end as an $\frac{\text{o.c. voltage}}{\text{s.c. current}}$ free field microphone at the rate of 20 db per decade (approx. 6 db per octave). This law was called by Schottky the "Tiefempfangsgesetz."

Equation (12) is exact; however, the conditions under which M and S are defined must be kept in mind. The result is independent of the choice of center and axis, but each coordinated pair of M and S are associated with *one* choice. Let the greatest linear physical dimension of U be called l (the arbitrary center is made part of the physical configuration for the purpose of determining l). (12) is independent of the relative magnitudes of l, d and λ. However, if $l \ll d$, then for a *pressure* microphone, M will be practically the same as the plane wave f.f. calibration. For a velocity microphone, this will also be true so long as $\lambda \ll d$, but as soon as λ becomes comparable with d, the calibration in terms of pressure for waves of radius d (which is M) is appreciably larger than the plane wave value. The two values are related by the factor $(1+(c/\omega d)^2)^{\frac{1}{2}}$ where c = speed of propagation and $\omega = 2\pi$ frequency; hence the plane wave calibration can be obtained from M by dividing by this factor.

Analogous but different results are obtained if the procedure is carried out in a small confined chamber instead of in free space. In this case, the chamber is assumed to be so small with respect to the wave-length that the pressure is the same everywhere within. In this space are placed an arbitrary reversible unit and also the same special spherical unit as before. Fig. 1 is now changed in that the free space radiation impedance of the sphere is replaced by the impedance of the chamber. An equivalent circuit of the entire system is shown in Fig. 2. U is again the arbitrary transducer. Quantities analogous to (1) are defined, but using the symbols T instead of M, and R instead of S. T is the calibration of the unit as a transmitter in terms of pressure *on* the diaphragm. (In this case, only pressure type units can be used.) R is the calibration of the unit as a receiver in terms of volume displacement when operating into zero external impedance. The arbitrary impedance added to the mechanical side of U' is made so large that the volume admittance of the sphere vanishes compared to that of the chamber. Z_o and Z_s are the acoustical impedances of the unit U with electrical terminals open and shorted. By examining Fig. 2, it is seen that a set of equations analogous to (2) can be written down. The first would be:

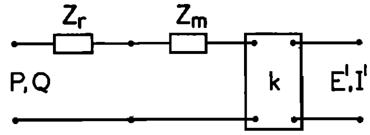

FIG. 1. Equivalent circuit of idealized spherical transducer.

$$j\omega R_o\frac{Z_0Z_c}{Z_0+Z_c}T_o'=j\omega R_o'\frac{Z_0Z_c}{Z_0+Z_c}T_o.$$

It is seen that in all four,

$$\text{either } j\omega\frac{Z_0Z_c}{Z_0+Z_c} \text{ or } j\omega\frac{Z_sZ_c}{Z_s+Z_c} \quad (13)$$

enter on both sides as factors and cancel out. Analogies to (3) and (3′) now follow, the latter being:

$$\frac{T_o}{R_o}=\frac{T_s}{R_s}=G. \quad (14)$$

The ratio G, independent of design, is found as before by computing it from U', the idealized spherical transducer; the equivalent circuit is the same as in Fig. 1 without the Z_r, so that its matrix is (7), also without the Z_r. T_s is the reciprocal of the s.c. transfer impedance when driven acoustically, and $j\omega R_s$ of the s.c. transfer impedance when driven electrically. These two

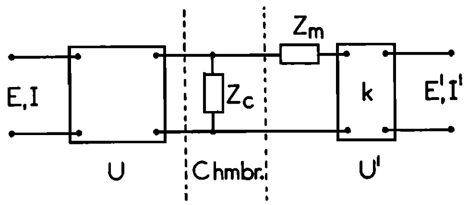

FIG. 2. Equivalent circuit of idealized and general transducers coupled to a chamber.

values are the negative of each other, hence:

$$G = -j\omega = \omega e^{-i(\pi/2)}. \tag{15}$$

Or, dropping the phase factor:

$$\mathbf{G} = \boldsymbol{\omega}. \tag{16}$$

From (16) the absolute calibrations of a unit as a receiver (R) and as a pressure microphone (T) can be computed from one another.

Taking R as a measure of the pressure sensitivity of the unit as a receiver—which will actually be the case in terms of relative pressure generated if it operates into a pure compliance soft compared with the diaphragm (the ear cavity)—the following law is deduced from (14) and (16): If a unit is flat as a receiver operated by $\frac{\text{constant current}}{\text{constant voltage}}$, it rises on the high end as an $\frac{\text{o.c. voltage}}{\text{s.c. current}}$ pressure microphone at the rate of 20 db per decade (approx. 6 db per octave). If there is a "Tiefempfangsgesetz" for free space, as Schottky found, there is also a "Hochempfangsgesetz" for a chamber.[2]

The Absolute Measurement of Sound—Free Field

The results given by (12) and (16) can be used to obtain the absolute calibration of a microphone in terms of free field or applied pressure. The pressure method suffers from some of the difficulties common to all chamber measurements, such as cooling of the walls, which, however, has been compensated for in the past in work on the thermophone and pistonphone. The free field problem which will be treated first suffers from the difficulty of simulating free space conditions; however these are likewise met with in Rayleigh disk measurements.

In the experiment leading to the f.f. calibration of the microphone there enter: (a) the microphone to be calibrated, U, (b) a reversible unit, U_x, and (c) a sound generator, U_o. Since it is desired ultimately to obtain the plane wave f.f. calibration in terms of pressure, and since the curvature of the wave affects the calibration of a velocity microphone at low frequencies, a somewhat more strictly specified experiment is needed in this case than for a pressure microphone. This more restricted experiment will be described since it includes both cases.

The sound generator U_o should be a zero order radiator, i.e., one which behaves as a point source at sufficiently low frequencies. On, in, or near U, U_x, and U_o arbitrary centers are chosen. Similarly, three axes are chosen. Let l be the greatest physical dimension of any of the three devices (the centers are included as part of physical configuration).

U and U_x are now placed one after the other at the same spot in the sound field of U_o. More specifically, U is placed co-axially with U_o, and their centers are separated by a distance d. This distance is chosen so that $l \ll d$. Without changing U_o in any way, U_x is substituted for U. The voltages generated by each, E and E_x, are recorded. If M_o and M_o^x are the o.c. free field calibrations of U and U_x for waves of radius d, then since the sound field was the same in both cases:

$$\frac{E}{M_o} = \frac{E_x}{M_o^x} \quad \text{whence:} \; \frac{M_o}{M_o^x} = \frac{E}{E_x}. \tag{17}$$

The use in (17) of M (which is for waves of radius d) must be justified, since U_o does not necessarily emit spherical waves at all frequencies. For high frequencies with $\lambda \ll d$, the values for response of U and U_x in plane, spherical, or actual field would be nearly identical since $l \ll d$. For a velocity microphone, the curvature effect becomes apparent for $d \sim \lambda$ or $d \ll \lambda$. But then we have also $l \ll \lambda$ and therefore U_o behaves as a spherical radiator. Consequently the use of M is justified.

U_o is now removed, and the two units placed in juxtaposition with their axes coincident but

[2] The words "receiver" and "transmitter" have opposite meanings in the German and English literature.

opposed, and their centers separated distance d. U_x is driven with a current I'. The f.f. pressure at U's center is $I'S_o^x$, where, as before, S_o^x is the calibration of U_x as a speaker. If U now puts out the o.c. voltage E', then:

$$E'=I'S_o^xM_o, \tag{18}$$

where the use of M is similarly justified. But now from (12) we have the value for the ratio of M_o^x to S_o^x. So from (18) and (12):

$$M_oM_o^x=\frac{E'}{I'}\frac{2d\lambda}{r}, \tag{19}$$

which together with (17) gives:

$$\boldsymbol{M_o}=\left(\frac{E'E}{I'E_x}\frac{2d\lambda}{r}\right)^{\frac{1}{2}}. \tag{20}$$

(20) is the absolute calibration of the given microphone in o.c. abvolts per f.f. bar for waves of radius d. Quite similar results are obtained for s.c. abamperes per f.f. bar, and analogous results for the calibration of a unit as a speaker, by applying (12) to the above, for instance.

If the microphone being tested is one of the pressure type, M is practically equal to the plane wave value; if the microphone is a velocity type, and the plane wave value is desired, the correction factor $\{1+(c/\omega d)^2\}^{-\frac{1}{2}}$ must be applied.

The Absolute Measurement of Sound—Chamber Pressure

To apply the method to obtain the pressure calibration of a given microphone, there enter the same three devices as before, except that this time they would want to be of such physical size and otherwise so chosen as to be suitable for work in a chamber. Coupling all three to a small chamber and driving U_o, the analog of (17) is obtained:

$$\frac{T_o}{T_o^x}=\frac{E}{E_x}. \tag{17'}$$

For the second experiment, U_o may or may not be left in the chamber. If it is, its diaphragm impedance must be added in parallel to the chamber impedance, the former taken with the electrical connections in the condition in which they are left. Calling Z_c, Z_o, Z_o^x the impedance of the chamber, the o.c. diaphragm volume impedance of U, and the same for U_x, the pressure generated in the chamber by virtue of a current I' in U_x is:

$$P=I'R_o^xj\omega Z, \tag{21}$$

where Z is the parallel combination of the chamber and the two diaphragm impedances. If E' is now the o.c. voltage in U, then the analog of (18) is:

$$E'=j\omega I'R_o^xT_oZ, \tag{18'}$$

which together with (16) gives:

$$T_oT_o^x=\frac{-jE'}{I'Z}. \tag{19'}$$

Whence (19′) and (17′) lead to:

$$\boldsymbol{T_o}=\left(\frac{E'E}{I'E_x}\frac{1}{Z}\right)^{\frac{1}{2}} \tag{20'}$$

after dropping the phase factor. This is the absolute calibration of the given microphone in o.c. abvolts per bar applied pressure. Instead, the s.c. calibration could have been obtained, or even the calibration of a unit as a receiver. The application of this formula requires a knowledge of the impedance of the chamber as modified by wall cooling and by the units coupled to it, these taken with their electrical terminals o.c. in the given case, or s.c. for the s.c. case.

The derivation of the principle results given by (12), (16), (20) and (20′) was carried out with c.g.s. electromagnetic units. This was done to facilitate the computation of the idealized spherical transducer which used magnetic coupling. However, electrostatic coupling could have been used and the work carried through with e.s. units. Nevertheless, the four principle equations would remain unaltered but giving M_o, for instance, in statvolts per bar rather than in abvolts per bar.

It is customary to express M and S in terms of bars on the one hand and volts or amperes on the other. It is therefore neither c.g.s. nor m.k.s., but mixed. If d, λ, ω, Z, and r are in c.g.s. units, and E and I in international units, then the four equations become:

$$H=10^{-7}\frac{2d\lambda}{r}, \qquad (12a)$$

$$G=10^{-7}\omega, \qquad (16a)$$

$$M_o=(10^{-7})^{\frac{1}{2}}\left(\frac{E'E}{I'E_x}\frac{2d\lambda}{r}\right)^{\frac{1}{2}}, \qquad (20a)$$

$$T_o=(10^{-7})^{\frac{1}{2}}\left(\frac{E'E}{I'E_x}\frac{1}{Z}\right)^{\frac{1}{2}} \qquad (20'a)$$

with M, S, T, R in bars and volts or amperes. (H and G are ratios of these quantities.)

In all methods previously employed to obtain absolute measurements, a primary standard was necessary. This primary standard was, in a word, a sound detecting or generating device whose construction was so simple that it could be reliably computed. In other words, a computable standard was used. It would seem impossible to obtain absolute measurements without such a standard. In the present case, such a computable standard, namely the spherical transducer, also enters into the picture, but it remains only hypothetical, i.e., it has to be used in the theoretical work, but it does not have to be built.

Appendix

To prove the law of reciprocity as it has been used, it suffices to assume that the 4-pole formed by two acoustically coupled transducers can be represented as the limiting case of an infinite sequence of electromechanical systems of a finite but indefinitely increasing number of degrees of freedom. Such a proof carries considerable weight, although it is not as satisfying as a real wave solution.

The proof is carried through for the case of electromagnetic and/or electrostatic couplers. A similar result can no doubt be obtained for piezoelectric couplers. Fig. 3 represents one of the systems in the infinite sequence of systems with increasing numbers of degrees of freedom whose limit is supposed to be the acoustical case. If reciprocity holds for them all, it presumably holds for the limit as well. The logic of this is not too precise.

N_m is a mechanical system with an arbitrary number of coordinates available on the left-hand side, and another arbitrary number available on the right-hand side. W_1 and W_2 are electromechanical couplers devoid of unnecessary mechanical or electrical impedances. Each side is homogeneous, that is, only one kind of coupling, static or magnetic is present on any one side of N_m. N_1 and N_2 are two general electrical networks. N_{12} represents eventual direct electric coupling between N_1 and N_2. 1 and n are the input and output meshes. It is noted that a single

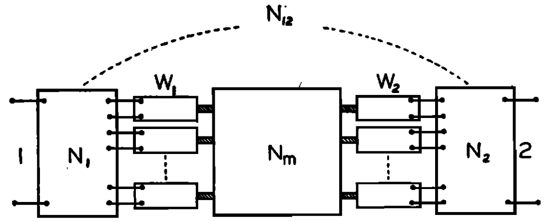

FIG. 3. Equivalent circuit of one out of the infinite sequence of 4-poles whose limit is the acoustically coupled 4-pole.

coupler on each side would not be sufficient, for that would imply that a single mechanical coordinate was imposed between the acoustical and electrical systems. Such might be the case with a cone speaker, but it is not the case for a velocity microphone with an undulating ribbon or with a condenser microphone with a flexible diaphragm. The entire system is assumed to be passive and linear.

For Fig. 3, a system matrix Δ can be set up. The entire principle of reciprocity rests merely upon the relation between the two minors Δ_{1n} and Δ_{n1} of this matrix. If their determinants are equal, reciprocity applies, otherwise not. They will certainly be equal if Δ is symmetrical. Since the two-rowed matrix of a simple electrostatic coupler, formed, say by a rigid piston facing a back plate, is symmetrical, one concludes that reciprocity applies if all couplers are electrostatic, for then the entire system matrix will also be symmetrical. The two-rowed matrix representing a simple magnetic coupler is, however, skew symmetric as shown in (5), so that it is necessary to demonstrate the validity of reciprocity in this case.

Both sides magnetic

If the couplers on both sides are magnetic, the system matrix would appear:

$$\Delta=\left|\begin{array}{c|c|c} N_1 & W_1 & N_{12} \\ \hline -W_1^T & N_m & W_2 \\ \hline N_{12}^T & -W_2^T & N_2 \end{array}\right| \qquad (22)$$

Δ is broken up into several component oblongs. The three on the main diagonal are square, and each would be the matrix of that part of the circuit alone which is indicated therein if all its accessible meshes were shorted (unconstrained). W_1 is not, in general, square. It represents half the coupling terms due to the like-named couplers. Were it not for the peculiar characteristic of magnetic coupling, the middle left oblong would be merely the transform of W_1. As it is, every term there has a minus sign. A similar situation obtains for W_2. N_{12} is also not square in general. This represents direct electric coupling. The lower left oblong is the transform of this with plus sign.

The two minors are derived from Δ as shown below:

$$(-)^n|\Delta_{1n}|=\left|\quad\right|, \quad (-)^n|\Delta_{n1}|=\left|\quad\right|. \qquad (23)$$

In each case, the determinant within the dotted line is evaluated to find the minor. In the first case, the first row and last column are stripped from Δ, and in the second case, the first column and last row are stripped. The second one can now be rotated on its main diagonal without changing its value. Then in both cases it will be the first row and last column that are stripped, but in the first case from Δ, and in the second case from Δ^T, i.e., from the transform of Δ. Δ and Δ^T differ only in that the sign of every element in the W sections is different. If in the second case every row and column contained in N_m is multiplied through by -1, the value of the minor determinant will not be changed, since an even number of negatives has been used. But now it is identical with the first minor, for every element in the W sections has changed back again, and the elements of N_m have all been multiplied twice and hence are unchanged Therefore if W_1 and W_2 are both magnetic, the law of reciprocity applies.

W_1 magnetic and W_2 static

This case differs from the above in that only the W_1 sections have opposite signs. In this case we proceed as before, except that instead of multiplying only those rows and columns contained in N_m by -1, we include those contained in N_2 as well. As a result of this, the W_1 sections change their signs back again; N_m, N_2, and the W_2 sections get multiplied twice and remain as they were, but the two N_{12} sections are multiplied only once and hence change signs. The two minor determinants are now identical, except that in the second one, the N_{12} sections are negative. Unless all the elements of N_{12} are zero, the two are definitely unequal, and the law of reciprocity does not hold. Since the last column was multiplied by -1 but removed to form the minor, the minor determinant has been multiplied by an odd number of negatives; so if all elements of N_{12} vanish, reciprocity holds, but with *phase reversal.* Hence for magnetic coupling on one side and static on the other, reciprocity holds only when there is no direct electric coupling between the two electrical sides of the system. In the calibration problem, this was not the case, and hence the assumption was justified, but the example of such cases where reciprocity does not apply means that a certain amount of care must be observed in using it.