1 Problems NS1

Topic of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function.

Deliverable: Answers to problems

Note: The term 'analytic' is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Problem *# 1: A two-port network application for the Laplace transform*

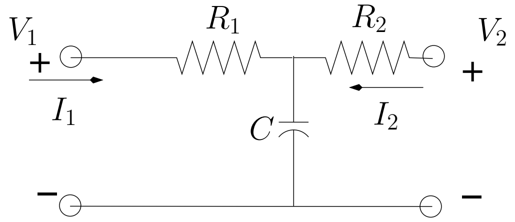

Figure 1: *This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$. It is convenient to define the dimensionless ratio $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.*

– 1.1: Find the 2×2 ABCD matrix representation of Fig. 1. Express the results in terms of the dimentionless ratio s/s_c where $s_c = 1/\tau$ is the cutoff frequency and $\tau = RC$ is the time constant.

SOL:

$$\begin{aligned}
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} &= \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & R \\ sC & 1+sRC \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \\
&= \begin{bmatrix} (1+sRC) & R(2+sRC) \\ sC & (1+sRC) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \\
&= \begin{bmatrix} (1+\frac{s}{s_c}) & R(2+\frac{s}{s_c}) \\ sC & (1+\frac{s}{s_c}) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}
\end{aligned}$$

– 1.2: Find the eigenvalues of the 2×2 matrix. The eigenvalues $\lambda_\pm$ of the 2×2 matrix $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ *are* $\lambda_\pm = \frac{1}{2}\begin{bmatrix} (\mathcal{A}+\mathcal{D}) - \sqrt{(\mathcal{A}-\mathcal{D})^2 + 4\mathcal{BC}} \\ (\mathcal{A}+\mathcal{D}) + \sqrt{(\mathcal{A}-\mathcal{D})^2 + 4\mathcal{BC}} \end{bmatrix}$. SOL: Since $\mathcal{A} = \mathcal{D}$, this simplifies to

$$\lambda_\pm = \begin{pmatrix} 1 + s/s_c - \sqrt{\mathcal{BC}} \\ 1 + s/s_c + \sqrt{\mathcal{BC}} \end{pmatrix} \tag{NS-1.1}$$

where $\sqrt{\mathcal{BC}} = \sqrt{\frac{s}{s_c}\left(2 + \frac{s}{s_c}\right)} = \sqrt{\frac{s}{s_c}}\sqrt{\cancelto{\text{zero}}{\frac{s}{s_c}} + 2} \approx \sqrt{2\frac{s}{s_c}}$ since $s/s_c \ll 2$.

In summary, the eigen-values are (recall that $0 \le s/s_c < 1$, thus not negative).

$$\lambda_\pm = 1 + \frac{s}{s_c} \pm \sqrt{2\frac{s}{s_c}} = \frac{1}{2}\left(1 \pm \sqrt{\frac{2s}{s_c}}\right)^2 + \frac{1}{2} = \left(\frac{1}{\sqrt{2}} \pm \sqrt{\frac{s}{s_c}}\right)^2 + \frac{1}{2}.$$

When $s \to 0$ both eigen values approach 1. As $s \to s_c$ the eigen values approach $3/2 \pm 0.707$).

In the above expression we have completed the square to simplify the result. This step is easily verified by evaluating the squared term.

– *1.3: Assuming that $I_2 = 0$, find the transfer function $H(s) \equiv V_2/V_1$.*

SOL: Since $I_2 = 0$ the upper row of the ABCD matrix gives the relationship between V_1 and V_2 as

$$V_1 = (1 + s/s_c)V_2$$

– *1.4: Find the pole and* residue *of $H(s)$?*

SOL: If we rewrite $H(s)$ in the standard form, the pole s_p and residue A may be identified:

$$H(s) = \frac{A}{s - s_p} = \frac{s_c}{s + s_c}$$

Thus the pole is $s_p = -s_c$ and the residue is $A = s_c$.

– *1.5: Find $h(t)$, the inverse Laplace transform of $H(s)$.*

SOL:

$$h(t) = \oint_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{s_c e^{st}}{(s + s_c)} \frac{ds}{2\pi j} = s_c e^{-s_c t} u(t),$$

where $u(t)$ is the step function. There are two poles, at $s_p = -s_c^\pm$ and two residues, at $s_c^\pm$.

The integral follows from the *Residue Theorem* (See Allen's book for details).

– *1.6: Assuming that $V_2 = 0$, find $Y_{12}(s) \equiv I_2/V_1$.*

SOL: Setting $V_2 = 0$ we may read off the requested function as

$$V_1 = -R(2 + RCs)I_2$$

thus

$$Y_{12}(s) = -\frac{1}{R(2 + s/s_c)} = \frac{A}{s - s_p}$$

with residue $A = s_c/R$ and pole $s_p = -2s_c$.

– *1.7: Find the input impedance to the* right-hand side *of the system, $Z_{22}(s) \equiv V_2/I_2$ for two cases:*

1. $I_1 = 0$

   SOL: When $I_1 = 0$, $Z_{22} = R + 1/sC$. When $V_1 = 0$

$$\begin{aligned} Z_{22}(s) &= 2R || \frac{1}{sC} \\ &= R\frac{2 + s/s_c}{1 + s/s_c} \\ &= R\frac{s + 2s_c}{s + s_c} \end{aligned}$$

2. $V_1 = 0$

   SOL: In this case
$$0 = (1 + 1/s_c)V_2 - R(2 + s/s_c)I_2.$$
   Solving for $Z_{22} = V_2/I_2$ gives requested result.

– *1.8: Find the determinant of the ABCD matrix.*

SOL: *Hint: It is always* 1
$$\begin{vmatrix} 1 + RCs & 2R + R^2Cs) \\ sC & 1 + RCs \end{vmatrix} =$$
$$1 + 2RCs + (RCs)^2 - 2RCs - (RCs)^2$$
$$= 1$$

History

Problem # 2: *Write a sentnece or two about each person.*

– *2.1: Provide a brief definition of the following properties:*

1. Ramon y Cajal. SOL: Spanish physiologist who drew amazing freehand micrographs of nerve cells (1890-1933).

2. Charles Scott Sherrington. SOL: Author of first book describing brain function, as functional groups (1906).

3. Rafael Lorente de No. SOL: Student of Cajal. Was not allowed to publish his views on feedback in the brain until 1934 once Cajal died.

4. Minsky and Papert (1969). SOL: Wrote book *Perceptrons* that predicted perceptrons had several limitations (p. 12) such as detecting embedded patterns, called *patterns in context*.

5. McCulloch and Pitts. SOL: Published a mathematical analysis of brain circuits that was the first to introduce model neural feedback (nets with circles) (1943). In one form their model may be expressed as
$$V_j(t - \tau) = H\left[\left(\sum_k \alpha_{jk} V_k(t - \tau_k)\right) - \Theta_j\right].$$

6. Albert Einstein. SOL: In 1905 he published a paper that provided the quantitative relation between the diffusion constant D and the mobility constant μ (p. 56, 59):
$$D = \frac{kT}{q}\mu,$$
   now known as the *Einstein relation*. This relation is equally important to the semiconductor industry.

7. Hodgkin and Huxley. SOL: H–H were the first to work out the mechanism between spike propagation in nerve fibers.

8. Hermann Helmholtz. SOL: Helmholtz was the first to measure the speed of a spike in a frog nerve, which he found to be 27 [m/s] (1850).

System Classification

Problem *# 3: Answer the following system classification questions about physical systems, in terms of the system postulates.*

– *3.1: Provide a brief definition of the following properties:*

L/NL : linear(L)/nonlinear(NL): SOL: Superposition and scaling hold

TI/TV : time-invariant(TI)/time varying(TV): SOL: The measurement time is irrelevant

P/A : passive(P)/active(A): SOL: An active system has a power source, a passive system does not.

C/NC : causal(C)/non-causal(NC): SOL: Responds only when driven for $t \geq 0$.

Re/Clx : real(Re)/complex(Clx): SOL: The time function is real (or complex).

– *3.2: Along the rows of the table, classify the following* systems*: In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:*

					Category		
#	Case:	Definition	L/NL	TI/TV	P/A	C/NC	Re/Clx
1	Conduction	$\boldsymbol{i}(t) = g_m \boldsymbol{E}(t)$	L	TI	P	C	Re
2	Diffusion	$i(t) = D\frac{d[Na]}{dx}$	L	TI	P	C	Re
3	Switch	$v(t) \equiv \begin{cases} 0 & t \leq 0 \\ V_0 & t > 0. \end{cases}$	L	TV	P	C	Re
5	Channel	$i(t) = g_m(v(t))$	NL	TV	A	C	Re
6	Membrane	$I_{out} = g_m(V_{in})$	NL	TI	P	C	Re
7	Nerve cell	Hogkin-Huxley Eqs.	NL	TI	A	C	Re
8	Nerve cell	Physical nerve cells	NL	TV	A	C	Re
9	Neural spike	$v(t,x) = \delta(t - x/c_o)$	L	TI	A	C	Re
10	Trans. Line	ABCD matrix	L	TI	P	C	Re