1 Problems NS2

Topic of this homework: Neuron and synapse terminology; Postulates of systems; Analysis of a diffusion transmission line.

Problem *# 1: Labeled sketch of the neuron and synapes.*

– *1.1: Sketch and fully label the drawing of the neuron (text Fig. 2.1).*

SOL:See Fig 2.1

– *1.2:Sketch and fully label the Synapes (text Fig. 2.2).*

SOL:See Fig 2.2

Problem # 2: *Using the system properties of networks, discuss the following properties*

– *2.1: Causality*
SOL:As we stated above, due to causality the negative properties (e.g., negative refractive index) must be limited in bandwidth, as a result of the Cauchy–Riemann conditions. However even causality needs to be extended to include the delay, as quantified by the d'Alembert solution to the wave equation, which means that the delay is proportional to the distance. Thus we generalize P1 to include the space dependent delay. This says that the delay must be proportional to the distance Δ [m], with delayed response $\delta(t - \Delta/c_o)$.

– *2.2: Reciprocity*
SOL:Reciprocity is defined in terms of the unloaded output voltage that results from an input current. Specifically)

$$\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} = \frac{1}{C(s)} \begin{bmatrix} A(s) & \Delta_T \\ 1 & D(s) \end{bmatrix}, \tag{NS-2.1}$$

where $\Delta_T = A(s)D(s) - B(s)C(s) = \pm 1$ for the reciprocal and antireciprocal systems respectively. This is best understood in term of Eq. NS-2.1. The off-diagonal coefficients $z_{12}(s)$ and $z_{21}(s)$ are defined as

$$z_{12}(s) = \left. \frac{\Phi_i}{U_l} \right|_{I_i=0} \qquad\qquad z_{21}(s) = \left. \frac{F_l}{I_i} \right|_{U_l=0}.$$

When these off-diagonal elements are equal $[z_{12}(s) = z_{21}(s)]$ the system is said to obey Rayleigh reciprocity. If they are opposite in sign $[z_{12}(s) = -z_{21}(s)]$, the system is said to be antireciprocal. If a network has neither reciprocal or antireciprocal characteristics, then we denote it as nonreciprocal. The most comprehensive discussion of reciprocity, even to this day, is that of Rayleigh (1896). The reciprocal case may be modeled as an ideal transformer while for the antireciprocal case the generalized force and flow are swapped across the two-port. An electromagnetic transducer (e.g., a moving coil loudspeaker or electrical motor) is antireciprocal.

– *2.3: Positive-real impedance*
SOL:When the input resistance of the impedance is real, the system is said to be passive, which means the system obeys conservation of energy.

Given any linear PR impedance $Z(s) = R(\sigma, \omega) + jX(\sigma, \omega)$, having real part $R(\sigma, \omega)$ and imaginary part $X(\sigma, \omega)$, the impedance is defined as being PR if and only if

$$\Re Z(s) = R(\sigma \geq 0, \omega) \geq 0. \tag{NS-2.2}$$

That is, the real part of any PR impedance is nonnegative everywhere in the right half-plane ($\sigma \geq 0$). This is a very strong condition on the complex analytic function $Z(s)$ of a complex variable s. This condition is equivalent to any of the following statements.

1. There are no poles or zeros in the right half-plane ($Z(s)$ may have poles and zeros on the $\sigma = 0$ axis).
2. If $Z(s)$ is PR, then its reciprocal $Y(s) = 1/Z(s)$ is PR (the poles and zeros swap).
3. If the impedance can be written as the ratio of two polynomials (a limited case related to the quasistatics approximation, Postulate P9) that have degrees N and L, then $N = L \pm 1$.
4. The angle of the impedance $\theta \equiv \angle Z$ lies within $[-\pi \leq \theta \leq \pi]$.
5. The impedance and its reciprocal are complex analytic in the right half-plane, thus they each obey the Cauchy–Riemann conditions there.

– *2.4: Explain why the postulates are important for the case of neurons and cells.*
SOL:Physical systems obey basic rules that follow from the physics. It provides intuition to summarize these restrictions as postulates presented in terms of a taxonomy, or categorization method, of the fundamental properties of physical systems.

– *2.5: In mathematical terms, define negative and positive feedback*
SOL:For both cases the output is feed back to the input, leading to poles in the input to output response function. For the case of negative feedback the system is stable, and the poles are in the left half plane. For positive feedback the system is unstable, leading to poles in the right half plane. See the discussion in the text on pages 13-14.

– *2.6: In a few paragraphs discuss the McCullock-Pitts model of a Neuron*
SOL:In their classic paper on the dynamics of a brain, McCulloch and Pitts assumed the most simple model for an individual neuron. Briefly, they supposed that the dendritic trees (see Figure 9.1) gathered a linear weighted sum of the incoming synaptic signals and compared this sum with a threshold level at the base (initial segment) of the axonal tree. In this model, if the sum of input signals exceeds the threshold, then an impulse is launched from the cell to the axon. Once launched, the impulse travels at a constant speed to the first axonal branching, where two impulses are generated that proceed down both secondary fibers. This process continues until most synapses at the tips of the tree have received impulses, with none being lost at axonal branchings.

Variables in this process include the velocity of the spike, number of branches, relative number and relative area of the branches, and the synaptic terminal strengths.

Since the late 1950s, several variations of the M–P neuron have been suggested for computer-based models of a brain, with diverse means for adjusting the (synaptic) interconnection weights during the course of neural activity.

Problem *# 3: Analysis of the diffusion equation:*
In the previous homework we analyzed the circuit of Fig. 1. By cascading many of these cells together we may find the solution of the diffusion equation:

$$\frac{\partial^2}{\partial x^2} v(x,t) = D_o \frac{\partial}{\partial t} v(x,t) \leftrightarrow s D_o V(s,x),$$

where D_o is the the diffusion parameter (constant) and $s = \sigma + \jmath\omega$ is the Laplace frequency. The double arrow symbol represents the Laplace Transform ($\mathcal{LT}$).

In this problem we seek the solution to a system composed of many of these cells cascaded together. The diffusion equation is used to model neural spike propagation. However the equation must be made nonlinear to emulate neural spikes. Here we study the linear fission equation.

We wish to cascade N cell we need to compute $\mathcal{T}^N$, which may be done using an eigenmatrix expansion. If we define the matrices

$$\mathcal{T}(s) = \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix}, \qquad \Gamma(s) = \begin{bmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{bmatrix} \qquad \boldsymbol{E}(s) = \begin{bmatrix} \boldsymbol{E}^+, & \boldsymbol{E}^- \end{bmatrix}$$

In terms of the two eigenvectors $\boldsymbol{E}^{\pm}_{2\times 1}$ and eigenvalues $\lambda_{\pm}$,

$$\mathcal{T}\boldsymbol{E} = \boldsymbol{E}\Lambda.$$

Post- and pre-multiplying by $\boldsymbol{E}^{-1}$ give the useful relations

$$\mathcal{T} = \boldsymbol{E}\Lambda\boldsymbol{E}^{-1} \quad \text{and} \quad \Lambda = \boldsymbol{E}^{-1}\mathcal{T}\boldsymbol{E}.$$

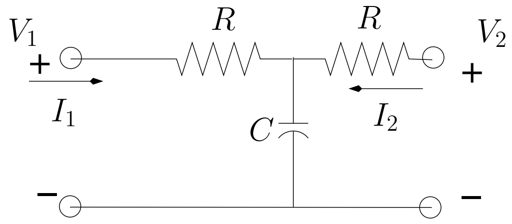

Figure 1: *This three-element electrical circuit that is a lumped element version of the diffusion equation. In the previous homework we defined the normalized frequency as $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.*

– 3.1: Find the 2×2 ABCD matrix representation of Fig. 1. Express the results in terms of the dimensionless ratio s/s_c where $s_c = 1/\tau$ is the cutoff frequency and $\tau = RC$ is the time constant.

SOL:The answer is the same as that for Problem #2:

$$\begin{aligned}\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} &= \begin{bmatrix} (1+RCs) & R(2+RCs) \\ sC & (1+RCs) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \\ &= \begin{bmatrix} (1+\frac{s}{s_c}) & R(2+\frac{s}{s_c}) \\ sC & (1+\frac{s}{s_c}) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \\ &= \mathcal{T} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}\end{aligned}$$

Reversible systems: When $\mathcal{A} = \mathcal{D}$ the transmission matrix is said to be reversible, and the properties greatly simplify. In this case the eigenmatrix $\boldsymbol{E}$ and eigenvalue matrix are:

$$\boldsymbol{E} = \begin{bmatrix} -\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} & +\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \mathcal{A} - \sqrt{\mathcal{BC}} & 0 \\ 0 & \mathcal{A} + \sqrt{\mathcal{BC}} \end{bmatrix}$$

– 3.2: Write out the eigenmatrix equation for the diffusion line as the product of $N = 2$ cells.

SOL:$\mathcal{T}^2$ is

$$\mathcal{T}^2 = \boldsymbol{E}\Lambda \cancelto{1}{(\boldsymbol{E}^{-1}\boldsymbol{E})} \Lambda \boldsymbol{E}^{-1} = \boldsymbol{E}\Lambda^2\boldsymbol{E}^{-1}$$

– 3.3: Find the eigenvalues of the 2×2 diffusion matrix.

SOL:Since the system is reversible ($\mathcal{A} = \mathcal{D} = 1 + s/s_c$), $\sqrt{\mathcal{BC}} = \sqrt{sRC(2+s/s_c)} = \sqrt{(2+s/s_c)s/s_c}$. When $|s/s_c| < 1$, $\sqrt{\mathcal{BC}} = \sqrt{2s/s_c}$. Thus

$$\lambda_\pm \approx (1 + s/s_c) \mp \sqrt{2s/s_c}.$$

– 3.4: Find the eigenvector matrix of the transmission matrix.

SOL:

$$\boldsymbol{E} = \begin{bmatrix} -\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} & +\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} \\ 1 & 1 \end{bmatrix}, \qquad \text{(NS-2.3)}$$

with $\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} = \sqrt{\frac{R}{C}}\sqrt{\frac{2+s/s_c}{s}}$

– 3.5: Find $\mathcal{T}^N$. Hint: Use the properties of the eigenequation expansion.

SOL:

$$\mathcal{T}^N = \boldsymbol{E}\Lambda \left(\boldsymbol{E}^{-1}\boldsymbol{E}\right)^{1} \cdots \Lambda \boldsymbol{E}^{-1} = \boldsymbol{E}\Lambda^N \boldsymbol{E}^{-1} \tag{NS-2.4}$$