1 Problems NS5

Topic of this homework:

Linear systems of equations; Gaussian elimination; Matrix permutations; Over-specified systems of equations; Analytic geometry; Ohm's law; Two-port networks

Deliverable: Answers to problems

Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle $x^2 + y^2 = 1$ may be solved for $y(x) = \pm\sqrt{1-x^2}$. If we let $z_+ = x + y\jmath = x + \jmath\sqrt{1-x^2} = e^{\theta\jmath}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of z as $|z|^2 = (x+y\jmath)(x-y\jmath) = 1$.

Problem *# 1: Give the curve defined by the equation:*

$$x^2 + xy + y^2 = 1$$

– *1.1: Find the function* $y(x)$*.*

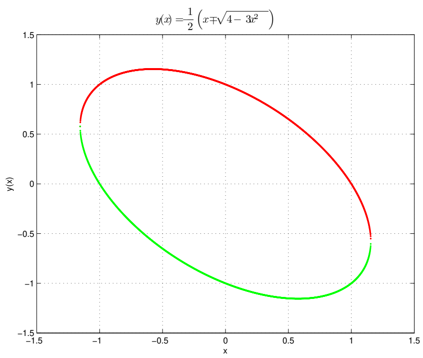
Sol: Completing the square in y and solve for $y(x)$:

$$\begin{aligned}(y + x/2)^2 - x^2/4 + x^2 = 1\\ (y+x/2)^2 = 1 - \frac{3}{4}x^2\\ y + x/2 = \pm\sqrt{\frac{4-3x^2}{4}}\\ y = \frac{1}{2}\left(\pm\sqrt{4-3x^2} - x\right)\end{aligned}$$

■

– *1.2: Using Matlab/Octave, plot* $y(x)$ *and describe the graph.*

Sol:



Thus we find the equation is a rotated ellipse. ■

– *1.3: What is the name of this curve?*
Sol: It is an ellipse, rotated by 45 degrees. ■

– *1.4: Find the solution (in x, p, and q) to these equations:*

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Sol: Solve the first equation for y as $y = p - x$, and then substitute it into the second equation

$$x(p-x) = -x^2 + px = q.$$

Thus we find the quadratic

$$x^2 - px + q = 0$$

having roots given by completing the square

$$(x - p/2)^2 = (p/2)^2 - q.$$

resulting in $x = p/2 \pm \sqrt{(p/2)^2 - q}$, $y = p - x$.

Summary: Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots. ■

– *1.5: Find an equation that is linear in y starting from equations that are quadratic (second-degree) in the two unknowns x and y:*

$$x^2 + xy + y^2 = 1 \tag{NS-5.1}$$
$$4x^2 + 3xy + 2y^2 = 3. \tag{NS-5.2}$$

Sol: The goal is to obtain a linear equation in y.
Method 1: remove xy term: Scale the upper equation by 3 and subtract from the lower:

$$\begin{aligned} 4x^2 + 3xy + 2y^2 &= 3 \\ 3x^2 + 3xy + 3y^2 &= 3 \end{aligned}$$

giving $x^2 - y^2 = 0$, or $x = \pm y$.
This results in the two equations

$$\begin{aligned} x^2 \qquad -y^2 &= 0 \\ x^2 + xy + y^2 &= 1 \end{aligned}$$

Adding these gives $2x^2 \pm x^2 = 1$, which is $3x^2 = 1$ and $x^2 = 1$. Thus the final solutions are $x = \pm y = \pm 1/\sqrt{3}$ and $x = \pm y = \pm 1$. ■

– *1.6: Compose the following two quadratic equations and describe the results.*

$$\begin{aligned} x^2 + xy + y^2 &= 1 \\ 2x^2 + xy \qquad &= 1 \end{aligned}$$

Sol: By isolating y from one of the two equations, we may remove it from the other equation, giving us a single 4^{th} degree equation in x:

$$x^2 + (1 - 2x^2) + (1 - 2x^2)^2/x^2 = 1$$

or

$$x^4 + x^2 - 2x^4 + 1 - 4x^2 + 4x^4 - x^2 = 0$$

Collecting terms

$$3x^4 - 4x^2 + 1 = 0$$

This is a quartic, but quadratic in x^2. Thus it may be solved for x^2 by the completion of squares

$$\begin{aligned} x^4 - \frac{4}{3}x^2 &= -\frac{1}{3} \\ \left(x^2 - \frac{2}{3}\right)^2 &= \frac{1}{3}\left(\frac{4}{3} - 1\right) \\ x^2 &= \frac{2}{3} \pm \frac{1}{3} \\ x &= \pm\frac{\sqrt{2 \pm 1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} \text{ and } \pm\frac{2}{\sqrt{3}} \end{aligned}$$

resulting in four roots. ■

Gaussian elimination (9 pt)

Problem # *2:(2pt) Gaussian elimination*

– *2.1:(1pt) Find the inverse of*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Sol:

$$A^{-1} = \frac{1}{3-8}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}.$$

■

– *2.2(1pt): Verify that* $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Sol: Multiply them to show this. ■

Problem # *3:(7pt) Find the solution to the following 3x3 matrix equation $Ax = b$ by Gaussian elimination. Show your intermediate steps.*

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

– *3.1:(2pt) Show (i.e., verify) that the first GE matrix G_1, which zeros out all entries in the first column, is given by*

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Sol: Operate with GE matrix on A

$$G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row. ∎

– *3.2:(3pt) Find a second GE matrix, G_2, to put G_1A in upper triangular form.*

Sol:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

which scales the second row by -1 and adds it to the third row. Thus we have

$$G_2G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

∎

– *3.3:(2pt) Solve for $\{x_1, x_2, x_3\}^T$.*

Sol: From the preceding problems, we see that $[x_1, x_2, x_3]^T = [3, -1, 1]^T$ ∎

Two linear equations (11 pt)

Problem # 4:*(6 pt) Given two linear equations in variables (x, y):*

$$y_1 = ax_1 + bx_2$$
$$y_2 = \alpha x_1 + \beta x_2.$$

– *4.1:(1pt) Write this as a 2x2 matrix equation*

Sol:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad \text{(NS-5.3)}$$

∎

– *4.2:(3pt) What does it mean, graphically, if these two linear equations have*

1. *a unique solution,*
2. *a non-unique solution, or*
3. *no solution?*

Sol: There are three possibilities:

1. When the two planes have different slopes, they intersect on a line. These lines cross where $y_1 = y_2$.
2. If the two planes lie on top of each other there are an infinite number of solutions.
3. If they have the same slope (are parallel to each other) but different intercepts (not touching) there are no solutions.

■

– 4.3:(2pt) Assuming the two equations have a unique solution, find the solution for x and y.

Sol: Since there must be one point where the two are equal, we may solve for that by setting the y values equal to each other and solving for x:

$$ax_1 + b = \alpha x_2 + \beta$$

Thus the ratio of slopes are

$$\frac{x_1}{x_2} = \frac{\beta - b}{a - \alpha}$$

■

Problem *# 5: (5 pt) The application of* linear functional relationships *between* two variables*:*

Figure 1 shows an example segment of a transmission line.

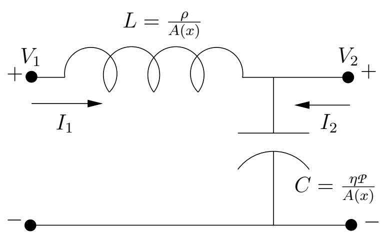


Figure 1: *This figure shows a cell from an LC transmission line. The input is on the left and output on the right.*

Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1, and the output (load) variables V_2 and I_2 of the transmission line.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

– 5.1:(3pt) Let the output (the load) be $V_2 = 1$ and $-I_2 = 2$ (i.e., $-V_2/I_2$ =1/2 $\{\Omega\}$). Find the input voltage and current, V_1 and I_1.

Sol: This case corresponds to

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\jmath + 2 \\ 1 - 2 \end{bmatrix}$$

Thus $V_1 = 2 + 1\jmath$ and $I_1 = -1$. ■

– 5.2:(2pt) Let the input (source) be $V_1 = 1$ and $I_1 = 2$. Find the output voltage and current V_2 and I_2.

Sol: With the input specified the two equations are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

To find the input we must invert the matrix ($\Delta = -j - 1$)

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \frac{1}{1+j} \begin{bmatrix} 1 & 1 \\ 1 & -\jmath \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Thus $V_2 = 3/(1+\jmath) = 3(1-\jmath)/2$, $-I_2 = (1-2\jmath)/(1+1\jmath) = -(1+3\jmath)/2$. One point of this exercise is that the two lines have a complex intersection point, not easily visualized. ∎

Linear equations with three unknowns (15 pt)

Problem # 6:*(7 pt) This problem is similar to the previous problem, except we consider 3 dimensions. Consider two linear equations in unknowns x, y, z, representing planes:*

$$a_1x + b_1y + z = c_1 \tag{NS-5.4}$$
$$a_2x + b_2y + z = c_2 \tag{NS-5.5}$$

– 6.1:(3pt) In terms of the geometry (i.e., think graphically), under what conditions do these two linear equations have (a) a unique solution, (b) a non-unique solution, or (c) no solution?

Sol: This problem is virtually identical to the previous problem, except the solution is for the intersection of two planes in three dimensions $z(x, y)$ rather than the intersection of two lines $y(x)$, in 2 dimensions. One might picture a plane as a line in two dimensions. That is, if you "sweep" a line along a third dimension, it forms a plane.

In this case, 2 equations with 3 unknowns, there is no unique solution in x, y, z. There are three possibilities:

1. There is no unique solution - we require a third plane to have a single point of intersection.
2. There are two cases: (1) When the two planes have different slopes, they meet at along a line. The solution of the problem (x, y) is then a line rather than a point. (non-unique). (2) If the two planes are identical (same slope and same intercept), all points on the plane(s) are solutions. (non-unique)
3. If the planes have the same slope, but different intercepts (are parallel to each other) there is no solution. (no solution)

∎

– 6.2:(2pt) Define the slope of a plane*?*

Sol: This is a concept that is natural in vector calculus. We have not gotten there yet, but this idea requires the concept of a gradient, which defines a vector perpendicular to the plane. We shall deal with this concept the third section of this course (i.e, stream 3). ∎

– 6.3:(2pt) Given 2 equations in 3 unknowns, the closest we can come to a 'unique' solution is a line (describing the intersection of the planes) rather than a single point. This line is an equation in (x, y), (y, z), or (x, z). Find a solution in terms of x and y by substituting one equation into the other.

Sol: $(a_1 - a_2)x + (b_1 - b_2)y + (c_2 - c_1) = 0$ ∎

Problem *# 7:(3pt) Now consider the intersection of the planes at some arbitrary constant height, $z = z_0$.*

– 7.1: Write the modified plane equations as a 2x2 matrix equation in the form $A\vec{x} = \vec{b}$ where $\vec{x} = \{x, y\}^T$, and find the unique solution in x and y using matrix operations.
Sol:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 - z_0 \\ c_2 - z_0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{a_1 b_2 - a_2 b_1} \begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix} \begin{bmatrix} c_1 - z_0 \\ c_2 - z_0 \end{bmatrix}$$

■

– 7.2:(2pt) Assuming the two equations have a unique solution, find the solution for x and y.
Sol: We are looking for the specification of a line, that is $x(y)$ or $y(x)$,that is determined by the two equations, which define two planes $(z_1(x, y), z_2(x, y))$. To find the solution solve for $z(x, y)$ and use *Gaussian elimination* (GE) on the system of equations. Setting ■

– 7.3:(2pt) When will this solution fail to exist (for what conditions on a_1, a_2, b_1, b_2, etc.)?
Sol: As stated above, if the planes have the same "slope," but different intercepts, there is no solution. The problem is that we don't know what the slope or intercept of the plane means. But we do know how to apply Gaussian elimination. We have shown before that if the determinant $\Delta = a_1 b_2 - a_2 b_1 \neq 0$, then the 'slopes' of the planes are not equal (the planes are not parallel to each other at height z_0). Thus the geometry has the same meaning as for the case of lines, but in three rather than two dimensions. To prove this we need to apply GE (or the equivalent). ■

– 7.4:(pt) Now, write the system of equations as a 3x3 matrix equation in x, y, z given the additional equation $z = z_0$ (e.g. put it in the form $A\vec{x} = \vec{b}$ where $\vec{x} = \{x, y, z\}^T$).
Sol:

$$\begin{bmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ z_0 \end{bmatrix}$$

■

Problem *# 8:(2pt) Show that the determinant of a 3x3 matrix is given by*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Hint: Expand using cofactors.

Problem *# 9:(3pt) Put the following system of equations in matrix form find (i) the determinant of the matrix, (ii) the matrix inverse, and (iii) the solution (x, y, z).* ***If it is not possible to complete (iii), state why.***

– 9.1:(3pt) Consider the equations

$$\begin{aligned} x + 3y + 2z &= 1 \\ x + 4y + z &= 1 \\ x + y &= 1 \end{aligned}$$

Sol:

$$\begin{bmatrix}1 & 3 & 2\\1 & 4 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

(i) The matrix determinant is -4. Expand in cofactors about the first column because it is all 1. (ii) Using Gaussian elimination (the numbers are easily manipulated for this case), the inverse matrix is

$$\frac{1}{4}\begin{bmatrix}1 & -2 & 5\\-1 & 2 & -1\\3 & -2 & -1\end{bmatrix}$$

or

$$\begin{bmatrix}.25 & -.5 & 1.25\\-.25 & .5 & -.25\\.75 & -.5 & -.25\end{bmatrix}$$

(iii) The solution is $[1,0,0]^T$ ∎

Ohm's Law (3 pt)

Problem *# 10:(3pt) In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage isthe 'force' and the current is the 'flow.' Ohm's law states that the voltage across and the current through acircuit element are related by the* impedance *of that element (which may be a function of frequency).*
Find the impedance for the following cases:

– *10.1:(1pt) A resistor with resistance R:*
Sol: $Z = R$ ∎

– *10.2:(1pt) An inductor with inductance L:*
Sol: $Z = sL$ with $s = \sigma + \omega\jmath$. Note the flux $\psi(t) = Li(t)$. The voltage $v(t)$ is the time derivative of the flux

$$v(t) = \frac{d\psi(t)}{dt} = L\frac{di(t)}{dt}.$$

∎

– *10.3:(1pt) A capacitor with capacitance C:*
Sol: $Z = 1/sC$. Note the charge $q(t) = Cv(t)$, thus the current $i(t)$ is the time derivative of the charge

$$i(t) = \frac{d}{dt}q(t) = C\frac{dv(t)}{dt}.$$

∎

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