1 Problems NS5

Topic of this homework:

Linear systems of equations; Gaussian elimination; Matrix permutations; Over-specified systems of equations; Analytic geometry; Ohm's law; Two-port networks

Deliverable: Answers to problems

Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle $x^2 + y^2 = 1$ may be solved for $y(x) = \pm\sqrt{1-x^2}$. If we let $z_+ = x + y\jmath = x + \jmath\sqrt{1-x^2} = e^{\theta\jmath}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of z as $|z|^2 = (x + y\jmath)(x - y\jmath) = 1$.

Problem *# 1: Give the curve defined by the equation:*

$$x^2 + xy + y^2 = 1$$

– 1.1: Find the function $y(x)$.

Sol:

– 1.2: Using Matlab/Octave, plot $y(x)$ and describe the graph.

Sol:

– 1.3: What is the name of this curve?

Sol:

– *1.4: Find the solution (in x, p, and q) to these equations:*

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Sol:

– *1.5: Find an equation that is linear in y starting from equations that are quadratic (second-degree) in the two unknowns x and y:*

$$x^2 + xy + y^2 = 1 \tag{NS-5.1}$$

$$4x^2 + 3xy + 2y^2 = 3. \tag{NS-5.2}$$

Sol:

– *1.6: Compose the following two quadratic equations and describe the results.*

$$\begin{aligned} x^2 + xy + y^2 &= 1 \\ 2x^2 + xy \quad &= 1 \end{aligned}$$

Sol:

Gaussian elimination (9 pt)

Problem *# 2:(2pt) Gaussian elimination*

– *2.1:(1pt) Find the inverse of*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Sol:

– *2.2(1pt): Verify that* $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Sol:

Problem *# 3:(7pt) Find the solution to the following 3x3 matrix equation* $Ax = b$ *by Gaussian elimination. Show your intermediate steps.*

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

– *3.1:(2pt) Show (i.e., verify) that the first GE matrix* G_1*, which zeros out all entries in the first column, is given by*

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Sol:

– *3.2:(3pt) Find a second GE matrix,* G_2*, to put* $G_1 A$ *in upper triangular form.*

Sol:

– *3.3:(2pt) Solve for* $\{x_1, x_2, x_3\}^T$.

Sol:

Two linear equations (11 pt)

Problem *# 4:(6 pt) Given two linear equations in variables* (x, y)*:*

$$y_1 = ax_1 + bx_2$$
$$y_2 = \alpha x_1 + \beta x_2.$$

– *4.1:(1pt) Write this as a 2x2 matrix equation*

Sol:

– *4.2:(3pt) What does it mean, graphically, if these two linear equations have*

1. *a unique solution,*
2. *a non-unique solution, or*
3. *no solution?*

Sol:

– *4.3:(2pt) Assuming the two equations have a unique solution, find the solution for* x *and* y.

Sol:

Problem *# 5: (5 pt) The application of* linear functional relationships *between* two variables*:*

Figure **??** *shows an example segment of a transmission line.*

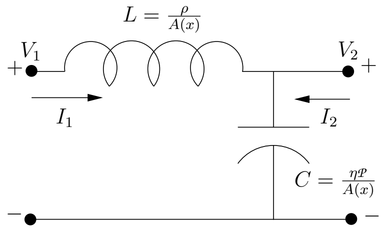

Figure 1: This figure shows a cell from an LC transmission line. The input is on the left and output on the right.

Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1, and the output (load) variables V_2 and I_2 of the transmission line.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

– 5.1:(3pt) Let the output (the load) be $V_2 = 1$ and $-I_2 = 2$ (i.e., $-V_2/I_2$ =1/2 $\{\Omega\}$). Find the input voltage and current, V_1 and I_1.

Sol:

– 5.2:(2pt) Let the input (source) be $V_1 = 1$ and $I_1 = 2$. Find the output voltage and current V_2 and I_2.

Sol:

Linear equations with three unknowns (15 pt)

Problem *# 6:(7 pt) This problem is similar to the previous problem, except we consider 3 dimensions. Consider two linear equations in unknowns x, y, z, representing planes:*

$$a_1 x + b_1 y + z = c_1 \qquad \text{(NS-5.3)}$$

$$a_2 x + b_2 y + z = c_2 \qquad \text{(NS-5.4)}$$

– *6.1:(3pt) In terms of the geometry (i.e., think graphically), under what conditions do these two linear equations have (a) a unique solution, (b) a non-unique solution, or (c) no solution?*
Sol:

– *6.2:(2pt) Define the* slope of a plane*?*
Sol:

– *6.3:(2pt) Given 2 equations in 3 unknowns, the closest we can come to a 'unique' solution is a line (describing the intersection of the planes) rather than a single point. This line is an equation in (x, y), (y, z), or (x, z). Find a solution in terms of x and y by substituting one equation into the other.*
Sol:

Problem *# 7:(3pt) Now consider the intersection of the planes at some arbitrary constant height, $z = z_0$.*

– *7.1: Write the modified plane equations as a 2x2 matrix equation in the form $A\vec{x} = \vec{b}$ where $\vec{x} = \{x, y\}^T$, and find the unique solution in x and y using matrix operations.*
Sol:

– *7.2:(2pt) Assuming the two equations have a unique solution, find the solution for* x *and* y.
Sol:

– *7.3:(2pt) When will this solution fail to exist (for what conditions on* a_1, a_2, b_1, b_2, *etc.)?*
Sol:

– *7.4:(pt) Now, write the system of equations as a 3x3 matrix equation in* x, y, z *given the additional equation* $z = z_0$ *(e.g. put it in the form* $A\vec{x} = \vec{b}$ *where* $\vec{x} = \{x, y, z\}^T$*).*
Sol:

Problem *# 8:(2pt) Show that the determinant of a 3x3 matrix is given by*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Hint: Expand using cofactors.

Problem *# 9:(3pt) Put the following system of equations in matrix form find (i) the determinant of the matrix, (ii) the matrix inverse, and (iii) the solution* (x, y, z). ***If it is not possible to complete (iii), state why.***

– *9.1:(3pt) Consider the equations*

$$\begin{aligned} x + 3y + 2z &= 1 \\ x + 4y + z &= 1 \\ x + y &= 1 \end{aligned}$$

Sol:

Ohm's Law (3 pt)

Problem *# 10:(3pt) In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage isthe 'force' and the current is the 'flow.' Ohm's law states that the voltage across and the current through acircuit element are related by the* impedance *of that element (which may be a function of frequency).*
Find the impedance for the following cases:

- *10.1:(1pt) A resistor with resistance R:*

Sol:

- *10.2:(1pt) An inductor with inductance L:*

Sol:

- *10.3:(1pt) A capacitor with capacitance C:*

Sol: