1 Problems NS6

Topic of this homework:

Linearization of a classic thermodynamic problem

Deliverable: A computer code (either Matlab/octave/Python) that generates the three graphs that confirm the analysis of the heating of water from the power lost by charging a capacitor.

Problem *# 1 TO DO: For the conditions given, write a matlab code that finds the temperature $T(t)$ of the bath the resistor sits in.*

A simple thermodynamics problem is proposed and solved using to different methods and their advantages and disadvantages are compared. This analysis demonstrates how to perform the two-port thermodynamic analysis and, compares it to traditional methods of thermodynamic analysis.

– 1.1: Your Matlab/Octave program should replicate Figs. 1, 3 and 5 from the analysis of thermodynamic modeling method, as described below.

1 Introduction

The purpose of this note is to investigate the possibility and usefulness of using techniques and methods found in modeling second order systems such as LRC circuits and spring-mass-damper systems for use in thermodynamic analysis. Specifically, this report will look at using the frequency domain to model thermodynamic systems and draw connections with components of electrical and mechanical systems to simplify thermodynamic analysis using a two-port transmission line analysis.

Traditionally, thermodynamics is analyzed in the time domain for its familiarity with our experiences. However, electrical analysis of LRC circuits have made large strides in modeling its system dynamics utilizing the frequency domain. This is primarily because LRC circuits are second order systems that benefit greatly from this sort of analysis. Though, the first order system of thermodynamics can be modeled as an RC circuit where the second order term is equal to zero.

1.1 Problem Statement

To show that the two-port transmission line analysis works with Thermodynamics, a simple and traditional thermodynamic problem is proposed and solved using 'traditional' methods as well as using a two-port analysis. The thermodynamics problem is as follows:

Assume an RC circuit with resistance (R), capacitance (C), Initial Voltage (V_0) and Initial Current (I_0) is placed in an incompressible fluid (such as water) with mass (m_w) and specific heat capacity (c_w) that is isolated from the environment. After the RC circuit has been turned on for a significant amount of time and the system has reached a steady state, what is the change in temperature of the fluid? Or stated another way, what is the time response of temperature of the fluid as the capacitor is charging up?

1.2 Traditional Solution

To determine the energy dissipated by the resistor into the fluid, the current passing through the resistor as a function of time must be known. In order to find the current, the RC circuit can be analyzed as a two port transmission line as seen below.

The two port can then be analyzed using a 2x2 representation as seen in the matrix equation below

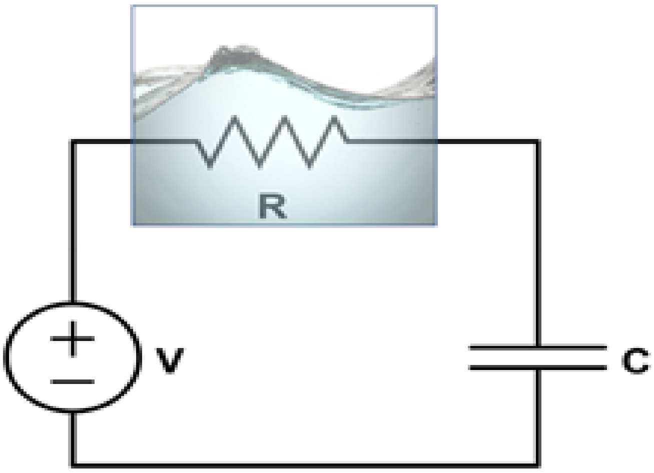

Figure 1: Thermodynamics Problem

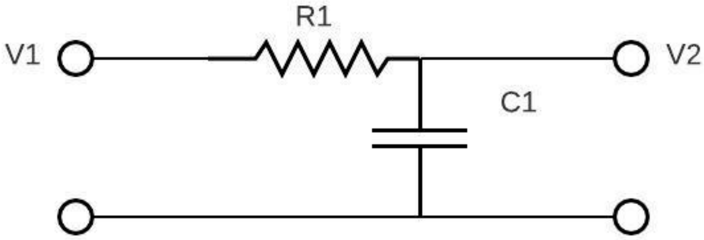

Figure 2: Transmission Line Representation of RC Circuit

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{NS-6.1}$$

From here, the current passing through the resistor (R_1) can then be solved through the following derivation

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 + sRC & R_1 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{NS-6.2}$$

Expanding out matrices

$$V_1 = 1 + sR_1C_1V_2 - RI_2 \tag{NS-6.3}$$

$$I_1 = sC_1V_2 - I_2 \tag{NS-6.4}$$

By the original formulation of the problem, it can be seen that there is $I_2 = 0$. Combining equation 3 and 4, the following equations can be derived

$$V_2 = \frac{V_1}{1 + sR_1C_1} \tag{NS-6.5}$$

$$I_1 = sC_1V_2 \tag{NS-6.6}$$

$$I_1 = \frac{sC_1V_1}{1+sR_1C_1} = sC_1V_1\frac{1}{1+sR_1C_1} = s\frac{V_1}{R_1}\frac{1}{s-1/R_1C_1} \tag{NS-6.7}$$

Also from the original problem, it is known that V_1 is turned on at $t = 0$ with voltage V_0. This is defined in the following equation.

$$V_1(t) = V_0u(t) \leftrightarrow V_0/s \tag{NS-6.8}$$

By substituting in V_1 in into equation 7 we get the following

$$I_1 = \frac{V_0}{R_1}\frac{1}{s-1/R_1C_1} \tag{NS-6.9}$$

Note that this equation is in the frequency domain. In order to convert this equation back to the time domain, the inverse Laplace transform must be taken. This is seen in the following equation.

$$\begin{gathered} I_1(t) = \frac{V_0}{R_1}\frac{1}{s-1/R_1C_1} \leftrightarrow \frac{V_0}{R_1}e^{-t/RC} = I_0e^{-t/\tau} \\ \text{where } \tau = RC \text{ and } I_0 = \frac{V_0}{R_1} \end{gathered} \tag{NS-6.10}$$

1.3 Thermodynamic relations

The power dissipated by the resistor at any given time can be calculated using the following equation

$$P(t) = I_0(t)^2R_1 = I_0^2e^{-2t/\tau}R_1 \tag{NS-6.11}$$

The energy dissipated by the resistor is the time integral of the power dissipation.

$$Q(t) = \int_0^t P(t)dt = \int_0^t I_0^2e^{-2t/RC}R_1dt = I_0^2R_1\frac{\tau}{2}(1-e^{-2t/\tau}) \tag{NS-6.12}$$

Assuming all the energy dissipated by the resistor is absorbed by the fluid, the relationship between the energy absorbed by the fluid and the change in temperature is the following.

$$Q = m_fc_f\Delta T \tag{NS-6.13}$$

Rearranging and substituting

$$\Delta T(t) = \frac{Q(t)}{m_fc_f} = \frac{I_0^2R_1\frac{\tau}{2}(1-e^{-2t/\tau})}{m_fc_f} \tag{NS-6.14}$$

Making the following assumptions:

Parameters:	Values:
Voltage (V_0)	10V
Resistance (R_1)	10Ω
Capacitance (C_1)	1F
Mass of Fluid (m_f)	.001kg
Specific Heat Capcity of Fluid(c_f)	4186

$$\tau = RC = 10 \cdot 1 = 10sec \tag{NS-6.15}$$

$$I_0 = \frac{V_0}{R_1} = \frac{10}{10} = 1Amp \tag{NS-6.16}$$

$$\Delta T(t) = \frac{I_0^2 R_1 \frac{\tau}{2}(1 - e^{-2t/\tau})}{m_f c_f} = \frac{1^2 \cdot \frac{10}{2} \cdot (1 - e^{-2t/10})}{0.001 \cdot 4186} \tag{NS-6.17}$$

The time response of the temperature can also be plotted as shown in Figure 3.

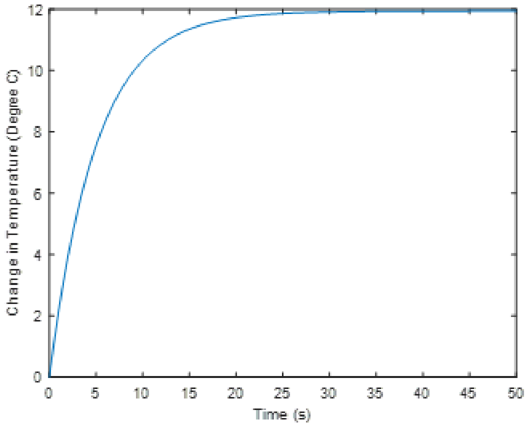

Figure 3: Time Response of Fluid

2 Two Port Analysis Method

The system above can be seen as a two-port transmission line with a resistor, 2 capacitors and an ideal transformer in the following configuration.

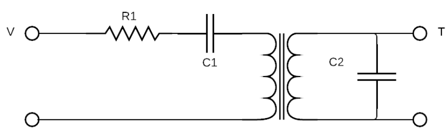

Figure 4: Two-Port Model

The transmission matrix would be defined as follows:

$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T \\ \dot{S} \end{bmatrix} = \tau \begin{bmatrix} T \\ \dot{S} \end{bmatrix} \tag{NS-6.18}$$

Where:

$$\tau = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_2 & 1 \end{bmatrix} \tag{NS-6.19}$$

This can then be simplified to:

$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} \frac{s + a^2C_1 + C_2}{aC_1} & \frac{sRC_1 + 1}{asC_1} \\ \frac{sC_2}{a} & \frac{1}{a} \end{bmatrix} \begin{bmatrix} T \\ \dot{S} \end{bmatrix} \tag{NS-6.20}$$

Since the system is isolated from the environment, the entropy flow ($\dot{S}$) is equal to zero. This allows us to show the relationship between the input voltage (V) and the temperature rise (T) in the frequency domain.

$$V = \frac{C_1C_2R_1s + a^2C_1 + C_2}{aC_1} \cdot T \tag{NS-6.21}$$

Since we also know that V is a unit step function as defined in equation 8, the following equation can be derived.

$$T(s) = \frac{V_0}{s}\frac{aC_1}{C_1C_2R_1s + a^2C_1 + C_2} = \frac{V_0}{s}\frac{a}{C_2R_1s + a^2 + C_2/C_1} \tag{NS-6.22}$$

This can then be rewritten in order to more easily take the inverse Laplace.

$$T(s) = \frac{1}{s}V_0a\frac{1}{C_2R_1s + a^2 + C_2/C_1} = \frac{1}{s}\frac{V_0a}{C_2R}\frac{1}{s + \dfrac{a^2 + C_2/C_1}{C_2R}} \tag{NS-6.23}$$

Now taking the inverse Laplace.

$$T(s) \leftrightarrow T(t) = \frac{V_0a}{C_2R}\int_0^t e^{\frac{-t(a^2+C_2/C_1)}{C_2R}} dt \tag{NS-6.24}$$

Simplifying

$$T(t) = \frac{V_0a}{C_2R}\left(\frac{-C_2R}{a^2 + C_2/C_1}\right)e^{\frac{-t(a^2+C_2/C_1)}{C_2R}} + Const. \tag{NS-6.25}$$

Since the temperature rise ΔT is of interest, the boundary condition $T(t = 0) = \Delta T(0) = 0$ can be use to solve for *Const.*. This can be seen in the following equation.

$$T(t) = \frac{V_0a}{C_2R}(-\frac{C_2R}{a^2 + C_2/C_1})e^{\dfrac{-t(a^2 + C_2/C_1)}{C_2R}} + \frac{V_0a}{C_2R}(\frac{C_2R}{a^2 + C_2/C_1}) \tag{NS-6.26}$$

Rearranging and simplifying

$$T(t) = \frac{V_0a}{a^2 + C_2/C_1}(1 - e^{\frac{-t(a^2+C_2/C_1)}{C_2R}}) \tag{NS-6.27}$$

Note that this is now of the same form as equation 14. From here it is a matter of determining the value of a, C_1, and $C2$. Assuming that C_1 is the same for both solutions (i.e. $C_1 = C_1$), a and C_2 can be determined in the following equations.

Given:

$$T(t) = \frac{V_0a}{a^2 + C_2/C_1}(1 - e^{\frac{-t(a^2+C_2/C_1)}{C_2R}}) = \frac{I_0^2R_1\frac{\tau}{2}(1 - e^{-2t/\tau})}{m_fc_f} \tag{NS-6.28}$$

Then:

$$\frac{V_0a}{a^2 + C_2/C_1} = \frac{I_0^2R_1\frac{\tau}{2}}{m_fc_f} \tag{NS-6.29}$$

And:

$$\frac{-(a^2 + C_2/C_1)}{C_2R} = \frac{-2}{\tau} = \frac{-2}{RC_1} \tag{NS-6.30}$$

Equation 30 can then be rearranged to be the following.

$$a^2RC_1 + RC_2 = 2RC_2 \longrightarrow C_2 = a^2C_1 \tag{NS-6.31}$$

Substituting equation 31 back into equation 29, we get the following.

$$\frac{V_0 a}{a^2 + (a^2 C_1)/C_1} = \frac{I_0^2 R_1 \dfrac{RC_1}{2}}{m_f c_f} \tag{NS-6.32}$$

Simplifying

$$\frac{V_0 a}{2a^2} = \frac{V_0^2 C_1}{2 m_f c_f} \tag{NS-6.33}$$

$$a = \frac{m_f c_f}{V_0 C_1} \tag{NS-6.34}$$

Substituting back into equation 31 and solving for C_2

$$C_2 = \frac{m_f^2 c_f^2}{C_1} \tag{NS-6.35}$$

Plotting equation 27 and comparing to equation 17, it can be seen that the same solution has been reached

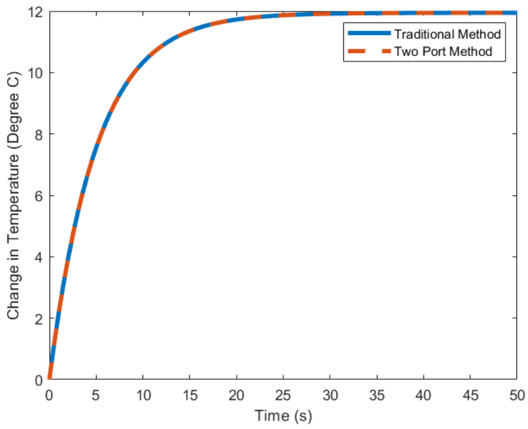

Figure 5: Time Response of Fluid with Both Methods

2.1 Conclusion

From the above demonstration of both methods, it can be seen that there are advantages to both the traditional method to thermodynamics as well as the two-port analysis of thermodynamics. For more trivial thermodynamics problems such as the one demonstrated above, it is often easier to use the traditional method of power and energy conversions. This method lends itself to a more instinctual understanding of the problem as most of the problem is solved in the time domain. However, the two-port representation is a more robust and algorithmic approach to solving the problem. It lends itself to more complex versions of the proposed problem where the voltage applied may not be a simple unit step function. This method would also be more useful in creating simulated environment algorithms that are more accurate and efficient compared to methods that are based around time integration such as modeling more complex thermodynamic phenomenons such as triple point and super cooling. By understanding this analysis and being able to apply this method to thermodynamics, it may open up new insights into the discipline of thermodynamics.