Sampling of continuous-time signals

Chap: 4.1-4.8; Pages: 140-201

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ECE-310

Periodic symmetry

- Every function $g(t)$ may be made T-periodic with an overlap and add OLA operation

$$\tilde{g}(t) = \sum_{n=-\infty}^{\infty} g(t - nT)$$

  n and integer, T the period
- Functions periodic in one domain (e.g., time) are "sampled" in the other domain (e.g., frequency)
- Convergence of this expression is an issue

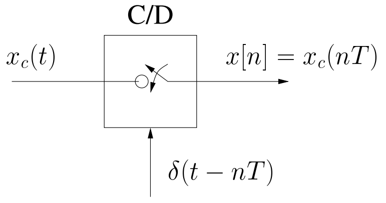
Review of nomenclature

FT type	time	limits	freq.	limits
DTFT 48 $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$	$h[n]$	$-\infty \le n \le \infty$	$H(e^{j\omega})$	$-\pi \le \omega \le \pi$
z Transform 95 $H(z) \equiv \sum_{n=-\infty}^{\infty} h[n]z^{-n}$	$h[n]$	$-\infty \le n \le \infty$	$H(z)$	z in ROC
Fourier Transform 143 $X_c(j\Omega) \equiv \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}dt$	$x_c(t)$	$-\infty \le t \le \infty$	$X_c(j\Omega)$	$-\infty \le \Omega \le \infty$
C/D Transform 143 $X_s(j\Omega) \equiv$ $\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\Omega - jk\Omega_s)$	$x_s(t) =$ $\sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT)$	$-\infty \le t \le \infty$ $t = nT$	$X_s(j\Omega)$ $F_s \equiv \frac{1}{T}$ $\Omega_s \equiv \frac{2\pi}{T}$	$-\frac{\Omega_s}{2} \le \Omega \le \frac{\Omega_s}{2}$ $F_{\max} \equiv \frac{\Omega_N}{2\pi}$ $\Omega_s > 2\Omega_N$

Periodic sampling 140

- Starting from a continuous-time signal $x_c(t)$, a sampler determines a discrete-time signal $x[n] \equiv x_c(t = nT)$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT).$$



- $x[n]$ is sampled $\Leftrightarrow$ periodic in frequency Ω

$$\tilde{X}_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\Omega - jk\frac{2\pi}{T}\right)$$

Periodic sampling

- Every periodic function $g(t) = g(t - nT)$ may be expanded in harmonics, at frequencies $\omega_k = 2\pi k/T$

$$g(t) = g(t - nT) = \sum_{k=-\infty}^{\infty} G_k e^{j2\pi f_k t} \underbrace{\left(e^{-j2\pi kn/T}\right)}_{1}$$

- From Fourier series formula:

$$G_k \equiv \frac{1}{T}\int_{t=0}^{T} f(t) e^{-j2\pi kt/T} dt$$

- Periodic impulses: page 143, Eq. 4.5

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j2\pi kt/T} \longleftrightarrow \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

Basic symmetry

- Periodic impulses 143, Eq. 4.5 and Munson notes 25.2

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j2\pi kt/T} \leftrightarrow \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

  - This is the Poisson Summation Formula PSF
  - PSF is a very important result
  - Based on Fourier series expansion of impulse-train

Applications of PSF

- Let $w(t) \leftrightarrow W(j\Omega)$.
  - Modulation formula:

$$\sum_{n} w(nT)\delta(t - nT) \leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty} W\left(j\Omega - jk\frac{2\pi}{T}\right)$$

    Multiply by $w(t)$ on left, convolve $W(j\Omega)$ on right
  - Overlap-add formula:

$$\sum_{n=-\infty}^{\infty} w(t - nT) \leftrightarrow \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} W\left(j\frac{2\pi k}{T}\right)\delta\left(\Omega - k\frac{2\pi}{T}\right)$$

    Convolve $w(t)$ on left, multiply $W(j\Omega)$ on right

Frequency domain nomenclature

- Details in working with Ω and ω:

$$e^{j\Omega_0 t} \longleftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

$$\sin(2\pi 500t) \longleftrightarrow j\pi\left[\delta(\Omega - 1000\pi) - \delta(\Omega + 1000\pi)\right]$$

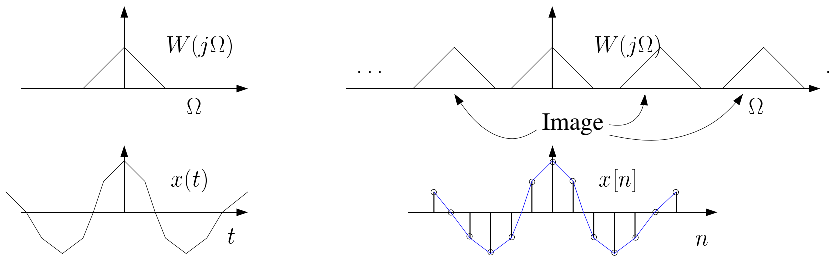
$$e^{j\Omega_0 nT} \longleftrightarrow \frac{2\pi}{T}\sum_{k} \delta\left(\Omega - \Omega_0 - k\frac{2\pi}{T}\right)$$

- A frequency of 200 Hz has a radian frequency of $\Omega_0 = 400\pi$, which corresponds to a normalized frequency of $\omega_0 = 400\pi/F_s = 400\pi T$.
- Delta scaling $|a|\delta(ax) = \delta(x)$ for any $a \neq 0$. Try $a = -j$ as an example. Example 4.1 148

Pulse train modulation

- General case of time modulation

$$\sum_n w(nT)\delta(t-nT) \leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty} W\left(j\Omega - jk\frac{2\pi}{T}\right)$$



Continuous time
Fourier Transform pair

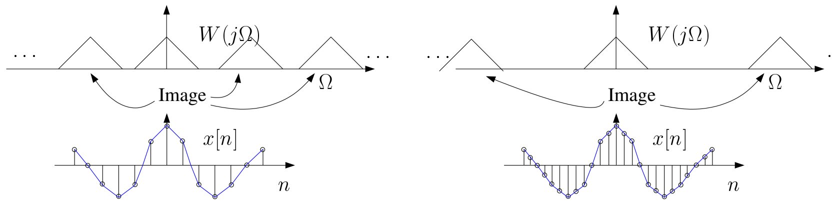
Discrete time
DTFT pair

Effect of increased sampling rate

- When T is halved ($T \to T/2$, F_s doubled,):

$$\sum_n w(nT)\delta(t-nT) \leftrightarrow \frac{1}{T}\sum_{k=-\infty}^{\infty} W\left(j\Omega - jk\frac{2\pi}{T}\right)$$

the images move out:



Discrete time
1X SAMPLING

Discrete time
2X SAMPLING

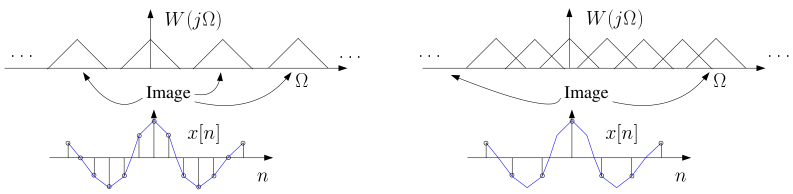
Effect of decreased sampling rate

- When T is doubled ($T \to 2T$, F_s halved):

$$\sum_{n=-\infty}^{\infty} w(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} W\left(j\frac{2\pi k}{T}\right) e^{j2\pi kt/T}$$

the images move in.

  - Overlap in the spectrum $W(\omega)$ is called aliasing



Discrete time
1X SAMPLING

Discrete time
0.5X SAMPLING

Harry Nyquist

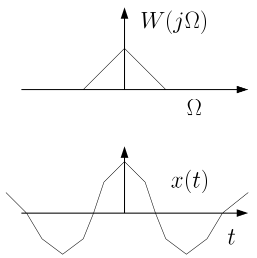
Born in Sweden;
Three famous theorems named after him:
The Nyquist

1. Sampling Thm., (Nyquist 1928)
2. Thermal noise Thm., (Nyquist 1932) and
3. Feedback stability Thm. (Nyquist 1934)

Windowing the images

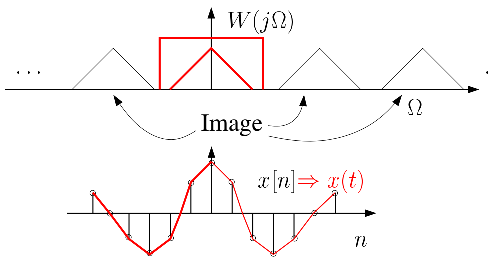
- What happens when images are removed by windowing in frequency?

$$x_c(t) = \frac{sin(\omega_c n)}{\pi n} \star \sum_{n=-\infty}^{\infty} x[T]\delta(t - nT)$$



Continuous time

Fourier Transform pair



Discrete time

DTFT pair

Nyquist sampling theorem 1928

- Any signal $x(t)$ may be uniquely represented by its samples $x[nT]$ if it is sampled at Ω_s, defined as more than twice its highest frequency Ω_N 146

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_N$$

- Note the somewhat confusing definitions in the book 146 regarding the terms Nyquist frequency $\equiv \Omega_N$, versus the Nyquist rate $\equiv 2\Omega_N$.

Some issues to think about

- The proof of the Sampling Theorem is based on convolution with $sin(\omega_c t)/\pi t$, namely the formula: 150

$$\hat{x}_{\text{reconstructed}}(t) = \frac{sin(\pi t/T)}{\pi t/T} \star x[n] \equiv \sum_{n=-\infty}^{\infty} x[n] \frac{sin[\pi(t/T - n)]}{\pi(t/T - n)}$$

- This low-pass reconstruction filter

$$\frac{sin(\pi t/T)}{\pi t/T}$$

is also called the interpolation filter, as it interpolates the signal between samples.

Some issues to think about

- In practice convolution by a “perfect” filter

$$\hat{x}_{\text{reconstructed}}(t) = \frac{sin[\pi t/T]}{\pi t/T} \star x[n]$$

is noncausal, and therefore cannot be implemented.

- A casual low-pass filter is used in practice.
- What are the practical implications of this?
- How will $\hat{x}(t)$ differ from the starting $x(t)$ at the input to the ideal C/D followed by D/C conversion process?
- Namely what is the RMS error going to look like?
- In practice this works because the ear cannot hear the phase distortion

Poisson Summation Formula

- Case of impulse:

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$$

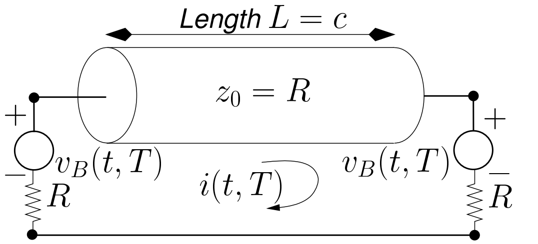
- General case of OLA comes from convolution on left by $w(t)$:

$$\sum_{n=-\infty}^{\infty} w(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} W\left(j\frac{2\pi k}{T}\right) e^{j2\pi kt/T}$$

  - $w(t)$ is a continuous time function
  - $W(j\frac{2\pi k}{T})$ is $W(j\Omega)\leftrightarrow w(t)$, sampled at $\Omega_k \equiv 2\pi k/T$

Nyquist's famous problem

- Find the Johnson thermal noise
- Transmission line terminated in resistors

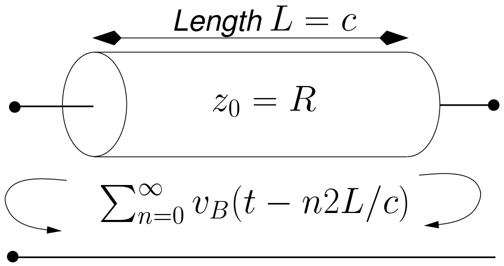


- Stored energy

$$E_{total} = \frac{1}{z_0}\int_{x=0}^{L} v_B^2(x-ct) + v_B^2(x+ct)dx$$

Nyquist's 2^d famous problem

- At $t=0$, remove the resistors



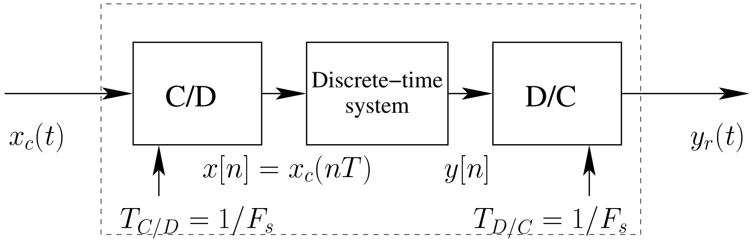
- Stored energy

$$E_{total} = \frac{1}{z_0}\sum_{n=0}^{\infty} v_B^2(t-n2L/c) \longrightarrow \frac{1}{2\pi z_0}\sum_{k=-\infty}^{\infty}\left|V_B\left(k2\pi\frac{c}{2L}\right)\right|^2$$

- Nyquist's Johnson-noise formula follows: $V_B^2 = 4kTR$

DT processing of CT signals 4.4

- Basic model of C/D→D/C processing ↔ C/D/C 153-154



- Ideal reconstruction (antialias) filter in D/C

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(n/T-n)]}{\pi(t/T-n)}$$

- This basic structure describes almost every telephone, for more than 30 years

Two basic type of C/D/C systems

- There are two basic categories of C/D/C systems:
- Real-time processing:
  - Any application where 1 sample in gives 1 sample out is a real-time method
- Non-real-time processing:
  - non-real-time I applications are those where input time and output time are different,

    or
  - non-real-time II where the computation time takes so much time that the processing cannot keep up

Real-time (RT) processing examples

- Today there are a great many examples of real-time systems:
  - cell phones, CD players, hearing aids, video conferencing over a wide-band channel (i.e., the Internet)
- Sometimes we find these systems to fall out of real time (i.e., cell phones and video)

Non-real-time (NRT) processing examples

- Cases of non-real-time applications where the input and output rates differ:
- Examples: pagers, fax machines, ...
- These make you wait

Non-real-time processing examples II

- Cases of non-real-time applications where the computation time is greater than real-time
  - video-conferencing over phone lines, cell-3G (3^d generation cell), some classes of speech-De-noising and music encoding such as MPEG audio and video

OLA processing

- For many C/D/C processing schemes RT and NRT, OLA frequency domain processing is used
  - This method is based on the OLA formula

$$\sum_{n=-\infty}^{\infty} w(t-nR) = \frac{1}{R}\sum_{k=-\infty}^{\infty} W\left(j\frac{2\pi k}{R}\right) e^{j2\pi kt/R}$$

  - Let
    - $w(n)$ be a low-pass filter.
    - R small such that $W(2\pi k/R) \approx 0$ for $k \geq 1$.
    - $|W(\Omega)|_{\Omega=0} \equiv R$
  - Under these conditions

$$\sum_{n=-\infty}^{\infty} w(t-nR) = \frac{1}{R} W(0) = 1$$

OLA formula

- From the last slide:

$$1 = \sum_{n=-\infty}^{\infty} w(t-nR)$$

- Typically $R \leq L/2$, where L is the length of window $w(t)$,
- Expand signal $s(t)$ into smooth OLA blocks

$$s(t) = \sum_{n=-\infty}^{\infty} w(t-nR)s(t) \equiv \sum_{n=-\infty}^{\infty} s_n(t)$$

- Define the windowed signal as

$$s_n(t) \equiv w(t-nR)s(t)$$

From frequency to time by OLA

- Expand signal $s(t)$ into smoothed OLA blocks

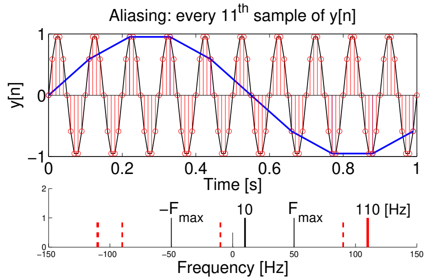
$$s(t) = \sum_{n=-\infty}^{\infty} w(t-nR)s(t) \equiv \sum_{n=-\infty}^{\infty} s_n(t)$$

- As before:
  - $s_n(t) \equiv w(t-nR)s(t)$
  - $S_n(j\Omega) \equiv \mathcal{F}\{s_n(t)\}$ where $\mathcal{F}\{\ \}$: Fourier Transform
  - $\Rightarrow s_n(t) \equiv \mathcal{F}^{-1}\{S_n(j\Omega)\}$
  - Thus

$$s(t) = \sum_{n=-\infty}^{\infty} s_n(t) = \sum_{n=-\infty}^{\infty} \mathcal{F}^{-1}\{S_n(j\Omega)\}$$

Aliasing 4.1-4.3 147-149

- Example of aliasing of a 110 Hz tone:

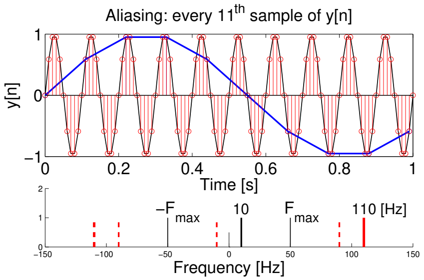


- Sample period $T_s = 0.01$ [s] $\Rightarrow F_s = 1/T_s = 100$ Hz;
  - $f = 10$ [Hz]: $y[n] = \sin(2\pi f\, nT_s) = \sin(2\pi 10n/100)$
  - $f = 110$ [Hz] stems:

$$y[n] = \sin\left(2\pi\frac{110n}{100}\right) = \sin\left(2\pi\frac{100+10n}{100}\right) = \sin\left(2\pi\frac{10n}{100}\right)$$

Aliasing 4.1-4.3 147-149

- Example of decimation-aliasing of a tone:



- Sample period $T_s = 0.01$ [s] $\Rightarrow$ $F_s = 1/T_s = 100$ Hz;
  - $f = 10$ [Hz] $\Rightarrow$ $y[n] = \sin(2\pi f n T_s) = \sin(2\pi 10 n/100)$
  - For the blue curve $T \to 11T$.

$$\sin(2\pi f nT) = \sin(2\pi f' nT') = \sin(2\pi f' n 11T)$$

  - Thus $11f' = f$, $f' = f/11$.

Down-sampling 158

- Suppose we cut the bandwidth by 2 in the frequency domain with an ideal low-pass filter
- We may then reduce $F_s = 1/T$ at the output by 2x,

$$\hat{S}_n(e^{j\omega}) = S_n(e^{j\omega}) \begin{cases} 1, & \omega < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

without aliasing, thus

$$T_{D/C} = 2T_{C/D}$$

- Alternate samples are dropped in this processing, called down-sampling

Ideal differentiator 158

- Suppose we wish to differentiate a continuous input signal

$$y_c(t) = \frac{dx_c(t)}{dt}$$

- This causal frequency response corresponds to

$$H_c(j\Omega) = j\Omega$$

- Thus

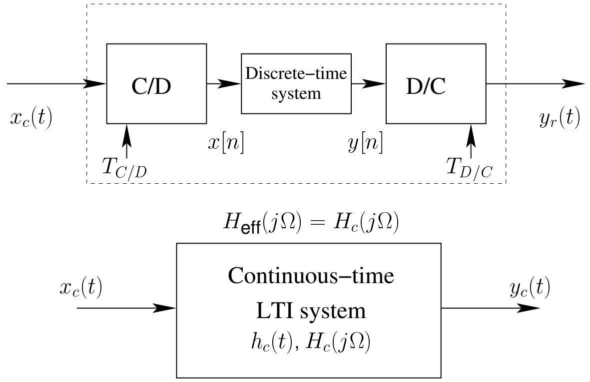
$$H_{\text{eff}}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

- It may be shown that note the noncausal nature of $h[n]$

$$h[n] = \begin{cases} 0, & n = 0 \\ \frac{\cos \pi n}{nT}, & n \neq 0. \end{cases}$$

Impulse-invariance 160

- Relate $h_c(t)$ and $H_c(j\Omega)$ to $H(e^{j\omega})$



- With impulse invariance the mapping from discrete to the continuous domain is define by

$$h[n] = Th_c(nT)$$

Example of impulse-invariance 162

- A common continuous-time impulse response, and corresponding Laplace Transform:

$$h_c(t) = e^{s_0 t}u(t) \longleftrightarrow \frac{1}{s - s_0} = H_c(s)$$

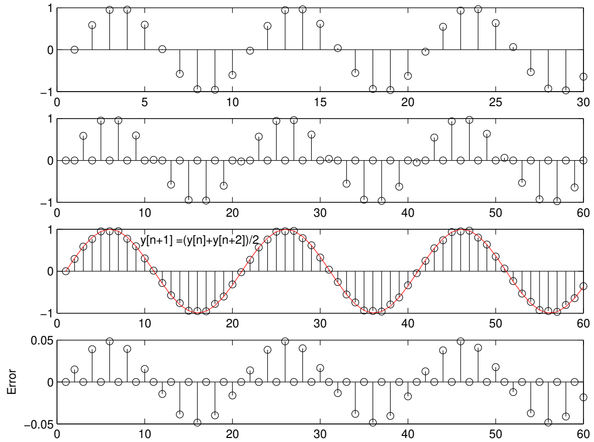
- From impulse-invariance note error in book 4.8 162

$$h[n] = Th_c(nT) = ATe^{s_0 Tn}u(n) \longleftrightarrow \frac{AT}{1 - e^{s_0 T}z^{-1}} \equiv H(z)$$

- This common example must alias since $H(z)$ is not bandlimited

Upsampling by linear interpolation I

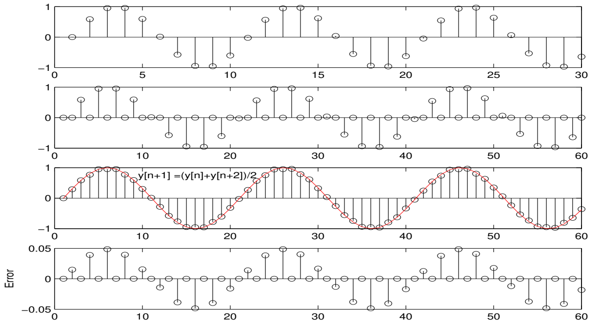
- When upsampling, we need to interpolate the new samples (Matlab help upsample, interp)



- This may be done by linear interpolation, but at a cost.

Upsampling by linear interpolation II

- Frequency response of a linear interpolator

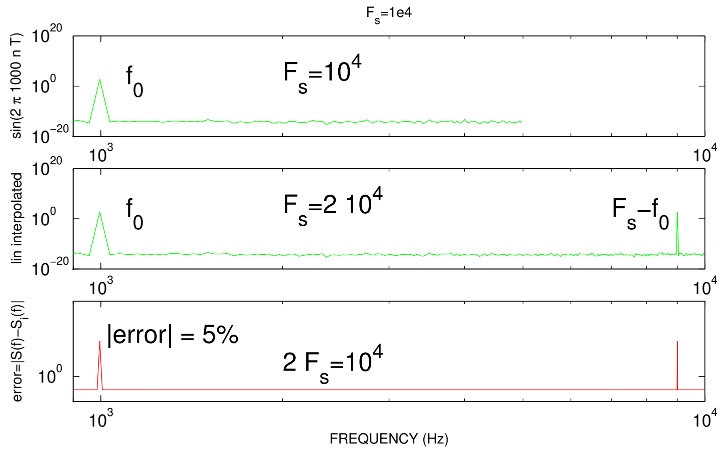


  - Even "stems" from linear interpolation Red curve is the exact;
  - The error is of the form (note $-1^n = e^{-j\pi n}$):

$$[1 + (-1)^n]\, e^{j\omega_0 n}/2 \longleftrightarrow;\ 2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega - \omega_N - \omega_0)$$

Upsampling by linear interpolation III

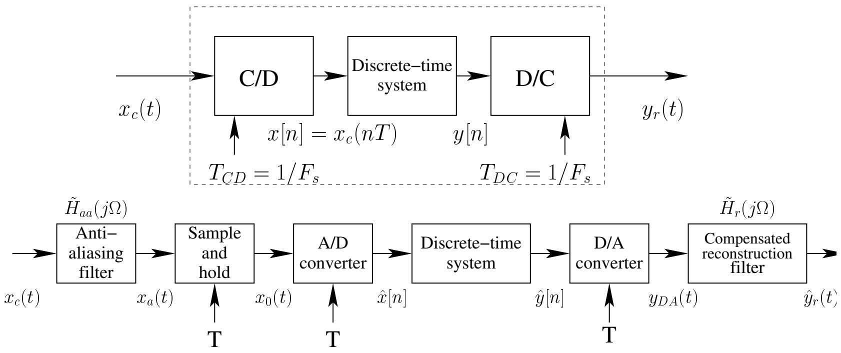
- Distortion is audible



- The error due to interpolation is 5% at f_0
- There is an unwanted tone at $F_s/2 - f_0$

DT processing of analog signals 4.8 185

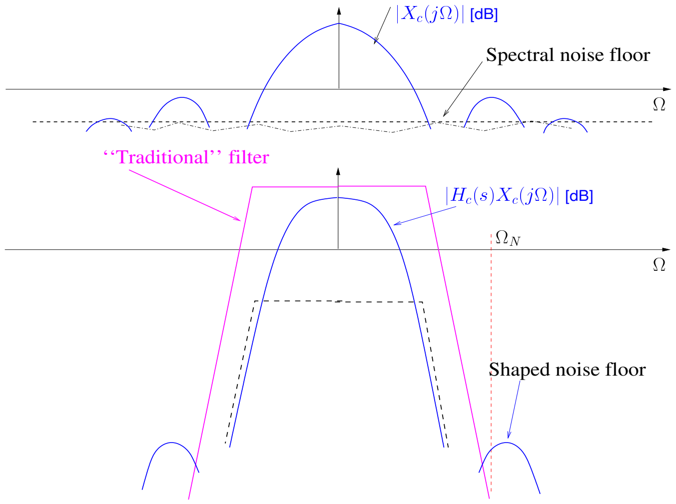
- DT processing with A/D and D/A



- Basic signal definitions

Traditional C/D conversion

- Traditional converter requires a high order filter

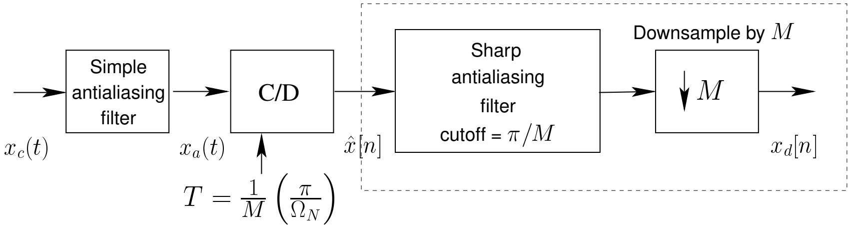


Some issues

- Analog prefilter is very expensive, large, requires laser trimmed resistors
- Phase response distorts the waveform (not necessarily a problem)
- Needs to be a very large order (i.e., >100 dB/oct)
- Multibit converters have linearity and "glitches"
- The oversampled sigma-delta Σ-Δ converter solved all these problems, plus others

Architecture of oversampled A/D 187

- Basic definitions:

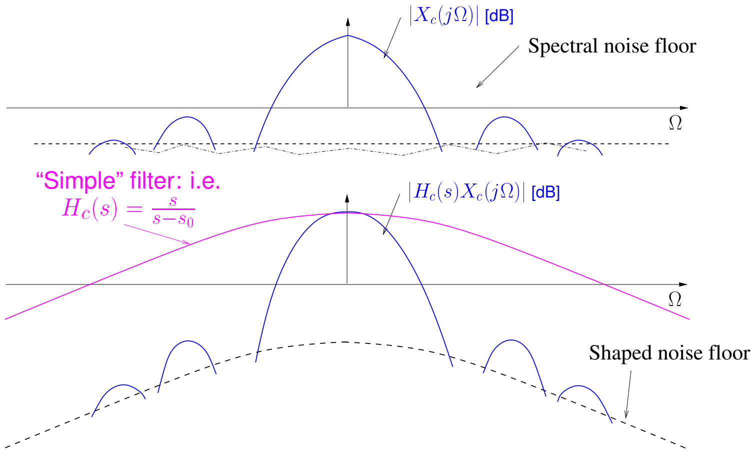


  - The simple antialiasing filter has a gradual i.e., $1/f$ or $1/f^2$ lowpass rolloff
  - Sampling is done at very high rate e.g., $M = 1000$
  - The steep antialiasing filter is then implemented in the DT domain
  - DT downsampled by M gives $x_d[n]$

Oversampling C/D conversion I

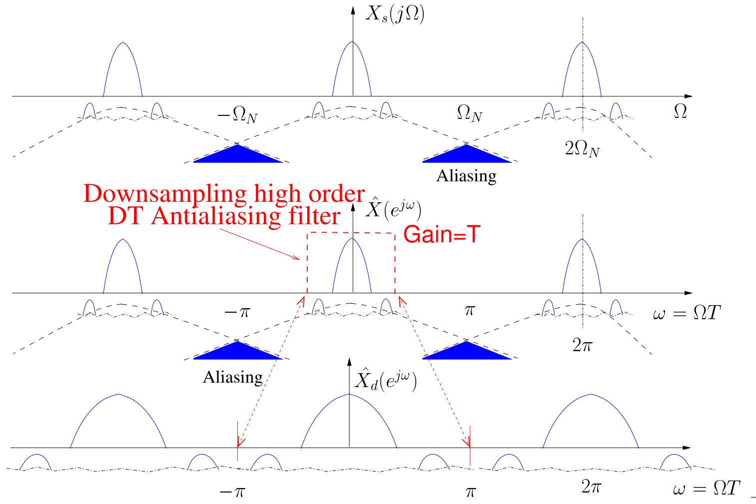
- Modern C/D conversion: 768x (3×2^8) oversampled Σ-Δ

http://courses.ece.uiuc.edu/ece310/Allen/sigmadelta.html



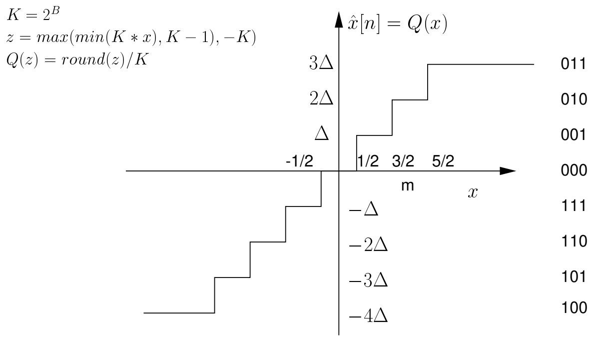
Oversampling C/D conversion II

- After sampling: $X(j\Omega) \longrightarrow X_s(j\Omega)$



Amplitude Quantizer

- Digital signals are both discrete in time and amplitude
- Values $\hat{x}[n]$ are two's complement integer "fraction" with value $-1^{a_0} \cdot \left(\sum_{n=1}^{B} a_n 2^{-n}\right)$ Example: 1.01011 = -(1/4+1/16+1/32)

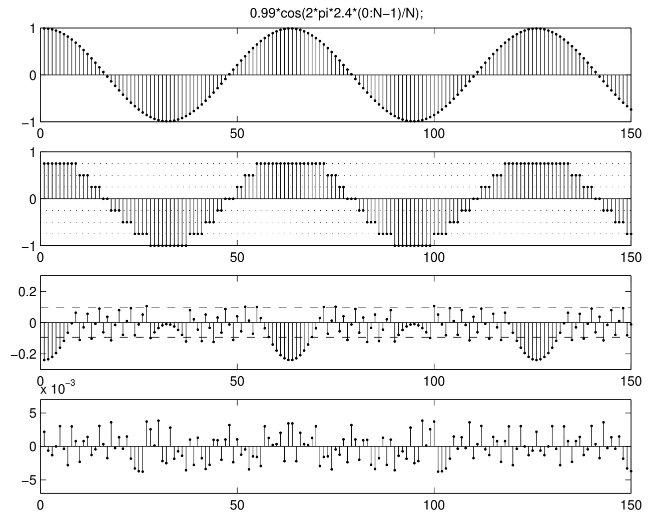


2's complement numbers $-\hat{x} = 1 + \overline{|\hat{x}|}$

Example: 111 = $\overline{001}$+1 = 110+1

Quantization noise 4.51 195

- Error for a 3 and 8 level quantizer



Analog Devices AD-1835A

- 5 V Stereo Audio 3.3 V Tolerant Digital Interface
- Differential Output
- Up to 192 kHz Sample Rates
- 256x, 512x, and 768x F_s Mode Clocks
- 16-20-24 Bit Word Lengths
- Σ-Δ Modulators with "Perfect Differential Linearity"
  ADCs:-95 dB THD+N, 105 dB SNR+Dynamic Range
  DACs:-95 dB THD+N, 108 dB SNR+Dynamic Range
- 4 DAC's and 1 ADC (Stereo) on 1 52-pin package

References

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