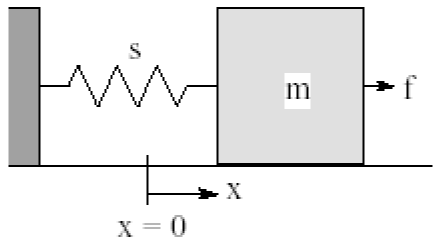
Chapter 1 - Fundamentals of Vibration

(1.2) Simple Harmonics Oscillator (SHO): A simple model for all wave phenomena.



If a mass, *m*, is attached to a spring and is displaced slightly from its rest position (conveniently chosen as x = 0), the mass will vibrate. The vibration is due to:

Restoring force – The force that tends to restore a stretched spring to the equilibrium position.

Inertia – This is a property of the mass in that the mass will tend to persist in whatever state of motion it is in.

In the case of a spring, the force exerted on the mass is a function of the displacement.

$$f = f(0) + \left(\frac{df}{dx}\right)_{x=0} x + \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x=0} x^2 + \ldots$$

Now, $f(0) = 0$ at equilibrium and assuming small displacements, so that nonlinear terms are negligible, then:

$$f \approx \left(\frac{df}{dx}\right)_{x=0} x = -sx \qquad \text{(Hooke's Law)}$$

where *f* is the restoring force (N), *s* is the stiffness (or spring) constant (N/m) and *x* is the displacement (m).

Applying Newton's 2nd Law:

$$f = m\frac{d^2x}{dt^2} = -sx$$

where $\frac{d^2x}{dt^2}$ is the acceleration of the mass.

(Actually, Newton's 2nd Law is $f = \frac{d(mv)}{dt}$, but for constant mass the above is true).

Rearranging the terms gives the Equation of Motion:

$$\frac{d^2x}{dt^2} + \frac{s}{m}x = 0 \qquad \text{(No loss)}$$

Setting the natural angular frequency w_0 (rad/s) as $w_0 = \sqrt{\frac{s}{m}}$ yields $\frac{d^2x}{dt^2} + w_0 x = 0$.

A simple solution to this equation can be expressed as:

$$x(t) = A\cos(w_0 t) + B\sin(w_0 t) = C\cos(w_0 t + f)$$

where $C = \sqrt{A^2 + B^2}$ and $f = \tan^{-1}\left(\frac{B}{A}\right)$.

The natural frequency is $f_0 = \frac{w_0}{2p}$ (Hz) which is dependent on the <u>intrinsic properties</u>, s and m (w_0 is itself an intrinsic property).

The constants A, B, C and f are called <u>extrinsic properties</u> and depend on the initial conditions (1.3). If the position (x_0) and velocity (u_0) are specified at one time (usually at t = 0), then A and B, or C and f can be determined:

$$x_{t=0} = \left[A\cos(w_0 t) + B\sin(w_0 t)\right]_{t=0} = A = x_0$$

and

$$\left[\frac{dx}{dt}\right]_{t=0} = \left[-w_0 A\sin(w_0 t) + w_0 B\cos(w_0 t)\right]_{t=0} = w_0 B = u_0$$

so that

$$x(t) = x_0\cos(w_0 t) + \left(\frac{u_0}{w_0}\right)\sin(w_0 t).$$

(1.5) An alternative way of expressing the solution to Equation of Motion is the complex exponential form:

Letting $\tilde{x} = \tilde{A}e^{gt}$ where the "~" above the variable denotes it as complex.

Substituting into the Equation of Motion:

$$\frac{d^2}{dt^2}\tilde{A}e^{gt} + \frac{s}{m}\tilde{A}e^{gt} = 0$$

$$g^2\tilde{A}e^{gt} + \frac{s}{m}\tilde{A}e^{gt} = 0$$

so that

$$g^2 + \frac{s}{m} = 0 \qquad \text{giving} \qquad g^2 = -\frac{s}{m}$$

This has two possible solution g for the ± roots giving

$$g^2 = \pm j\sqrt{\frac{s}{m}} = \pm jw_0$$ where j is defined as $\sqrt{-1}$ (for engineers)

This yields a solution of :

$$\tilde{x} = \tilde{A}_1 e^{jw_0 t} + \tilde{A}_2 e^{-jw_0 t}$$

If we apply the initial conditions: $\tilde{x}(0) = x_0, \tilde{u}(0) = \frac{d\tilde{x}(0)}{dt} = u_0$

$$\tilde{x}(0) = \tilde{A}_1 + \tilde{A}_2 = x_0$$

and

$$\frac{d\tilde{x}}{dt} = jw_0\tilde{A}_1 e^{jw_0 t} - jw_0\tilde{A}_2 e^{-jw_0 t},$$

$$\frac{d\tilde{x}(0)}{dt} = jw_0\tilde{A}_1 - jw_0\tilde{A}_2 = u_0 .$$

We have two equations and two unknowns, so lets solve for $\tilde{A}_1$ and $\tilde{A}_2$:

$$\tilde{A}_1 = \frac{1}{2}\left(x_0 - j\frac{u_0}{w_0}\right) \qquad \text{and} \qquad \tilde{A}_2 = \frac{1}{2}\left(x_0 + j\frac{u_0}{w_0}\right)$$

Interesting and important is the fact that $\tilde{A}_1$ and $\tilde{A}_2$ are complex conjugates of each other. This allows us to group the terms into a useful form:

$$\tilde{x} = \frac{1}{2}\left(x_0 - j\frac{u_0}{w_0}\right)e^{jw_0 t} + \frac{1}{2}\left(x_0 + j\frac{u_0}{w_0}\right)e^{-jw_0 t}$$

Rearranging

$$\tilde{x} = \frac{x_0}{2}\left(e^{jw_0 t} + e^{-jw_0 t}\right) - j\frac{1}{2}\frac{u_0}{w_0}\left(e^{jw_0 t} - e^{-jw_0 t}\right).$$

Important relations you need to know and memorize: Euler's identities

$$e^{jq} = \cos q + j\sin q \qquad \text{or} \qquad e^{-jq} = \cos q - j\sin q$$

Adding the first relation to the second gives:

$$\cos q = \frac{e^{jq} + e^{-jq}}{2}$$

Subtracting the second from the first gives:

$$\sin q = \frac{e^{jq} - e^{-jq}}{2j}$$

Making the substitution of the Euler identities gives for the position:

$$\tilde{x} = x_0\cos(w_0 t) + \frac{u_0}{w_0}\sin(w_0 t)$$

(1.4) Energy of Vibration:

Total mechanical energy = potential energy + kinetic energy ($E = E_p + E_k$).

E_p is the work done to distort the spring from its equilibrium position and E_k is the kinetic energy possessed by the mass. The potential energy is related to the force of the spring by:

$$E_p = -\int_0^x f\,\mathrm{dx} = \int_0^x sx\,\mathrm{dx} = \frac{1}{2}sx^2$$

The kinetic energy is given by:

$$E_k = \frac{1}{2}mu^2$$

Recall that $x = C\cos(w_0 t + f)$ and $u = \frac{dx}{dt} = -w_0 C\sin(w_0 t + f)$ so that

$$E = \frac{1}{2}sx^2 + \frac{1}{2}mu^2 = \frac{1}{2}sC^2\cos^2(w_0 t + f) + \frac{1}{2}mw_0^2 C^2\sin^2(w_0 t + f)$$

From $w_0 = \sqrt{\frac{s}{m}}$, $s = mw_0^2$ and

$$E = \frac{1}{2}mw_0^2C^2\cos^2\left(w_0t + f\right) + \frac{1}{2}mw_0^2C^2\sin^2\left(w_0t + f\right) = \frac{1}{2}mw_0^2C^2$$

Thus, E is constant with time and is equal to:

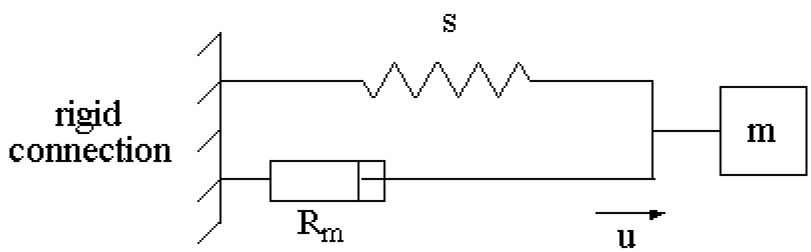
the maximum value of E_p (m is at greatest displacement and is not moving, $E_k = 0$) and the maximum value of E_k (when m passes through x = 0 and is moving with maximum speed, $E_p = 0$)

(1.6) Damped Harmonic Oscillator:

The way that we have set up our SHO, the mass will continue to oscillate forever with the same amplitude and frequency. Realistically, there is no such thing as perpetual motion or oscillation. In reality there are dissipative forces (like friction) acting on the oscillator which causes the oscillations to die down.

Viscous friction: A force proportional to the speed of the mass and is directed opposite the motion.

We graphically represent this as:



where the new force is represented by a dashpot or shock absorber. The viscous force is given by

$$f_r = -R_m \frac{dx}{dt}$$

where R_m is a positive constant with the unit $\left(\frac{N \bullet s}{m}\right)$ and is the called the mechanical resistance of the system.

If we now substitute this new force into our Equation of Motion we have:

$$\frac{d^2x}{dt^2} + \frac{R_m}{m}\frac{dx}{dt} + \frac{s}{m}x = 0$$

To find a solution to this Equation of Motion we try the complex exponential form:

$$\tilde{x} = \tilde{A}e^{gt}$$

Substitution yields:

$$\frac{d^2}{dt^2}\tilde{A}e^{gt} + \frac{R_m}{m}\frac{d}{dt}\tilde{A}e^{gt} + \frac{s}{m}\tilde{A}e^{gt} = 0$$

$$g^2 + \frac{R_m}{m}g + \frac{s}{m} = 0$$

Letting $b = \frac{R_m}{2m}$ and $w_0^2 = \sqrt{\frac{s}{m}}$ then solving for g gives:

$$g = -b \pm \sqrt{b^2 - w_0^2} \qquad \text{(solved by quadratic formula)}$$

There are three possible cases for our solution:

Case I: $b > w_0$ - Overdamped

Letting $h = \sqrt{b^2 - w_0^2}$ then our solution is:

$$\tilde{x} = \tilde{A}_1 e^{-(b+h)t} + \tilde{A}_2 e^{-(b-h)t}$$

An overdamped system has no oscillations (no e^{jwt} term), the mass asymptotically approaches it equilibrium point.

Case II: $b = w_0$ - Critically damped

The solution is simply:

$$\tilde{x} = \tilde{A} e^{-bt}$$

The critically damped system also has no oscillations and asymptotically approaches its equilibrium point. A critically damped system is just at the point between no oscillations and decaying oscillations.

Case III: $b < w_0$ - Underdamped (and the most interesting case)

Define: $w_d = \sqrt{w_0^2 - b^2}$ then $g = -b \pm jw_d$ which gives us a solution

$$\tilde{x} = e^{-bt}\left(\tilde{A}_1 e^{jw_d t} + \tilde{A}_2 e^{-jw_d t}\right)$$

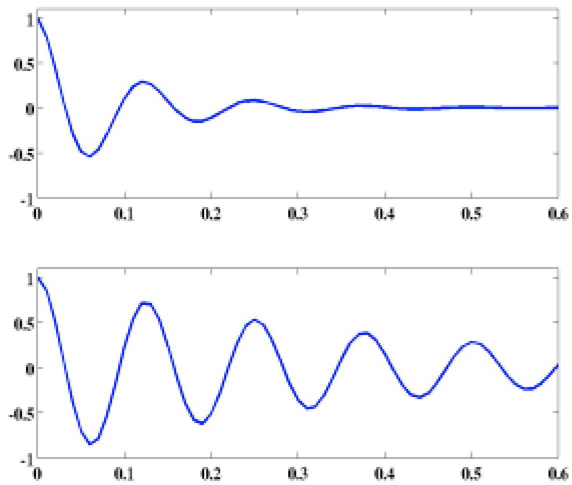
Note: the natural frequency of the damped harmonic oscillator is w_d.

If we look at the real part of our solution (which is itself a complete general solution, pg 6), we find

$$x(t) = \mathrm{Re}\{\tilde{x}\} = Ae^{-bt}\cos(w_d t + f) \quad \text{(Homework problem)}$$

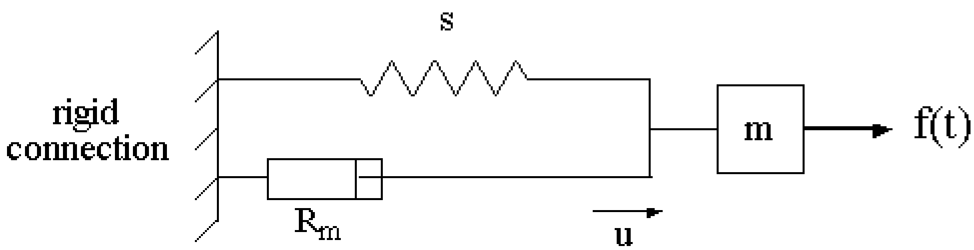
The amplitude of our oscillation is Ae^{-bt} and it decays with time. b is called the temporal absorption coefficient. We define the decay time (where the amplitude decays to 1/e of its initial value) as $t = \frac{1}{b} = \frac{2m}{R_m}$.

A graphical example of the influence of the temporal absorption coefficient:



Top: Absorption coefficient of b, Bottom: Absorption coefficient of $\frac{b}{4}$.

(1.7) Forced Oscillations:



Consider a system driven by an external force, $f(t)$. The Equation of Motion now becomes:

$$m\frac{d^2x}{dt^2}+R_m\frac{dx}{dt}+sx=f(t).$$

If $f(t)$ is harmonic [$f(t)=Fe^{jwt}$] then

$$m\frac{d^2x}{dt^2}+R_m\frac{dx}{dt}+sx=Fe^{jwt}.$$

Now, there will be a total solution to this equation, which is the sum of the steady state solution from the external applied force and the transient solution (the solution to the Equation of Motion when $F=0$). The transient solution will decay (due to the absorption) and we will be left with the steady state solution.

In order to solve for the total solution we must solve for the steady state solution and then use our initial conditions to get our constants.

For the steady state we assume a solution of the form:

$$\tilde{x}=\tilde{A}e^{jwt} \qquad \text{(not } w_0 \text{ or } w_d\text{)}$$

Note: the solution has the same angular frequency as the driving force

This yields a solution of:

$$-w^2m\tilde{A}+jwR_m\tilde{A}+s\tilde{A}=F$$

Solving for $\tilde{A}$ gives:

$$\tilde{A} = \frac{F}{jw\left[R_m + j\left(wm - \frac{s}{w}\right)\right]}$$

so that

$$\tilde{x} = \tilde{A}e^{jwt} = \frac{Fe^{jwt}}{jw\left[R_m + j\left(wm - \frac{s}{w}\right)\right]}$$

and

$$\tilde{u} = \frac{d\tilde{x}}{dt} = jw\tilde{x} = \frac{Fe^{jwt}}{R_m + j\left(wm - \frac{s}{w}\right)}.$$

From these relations we can define the complex mechanical impedance:

$\tilde{Z}_m = \frac{\tilde{f}}{\tilde{u}}$ with units $\left(\frac{N \cdot s}{m}\right)$ (like V/I for circuits, i.e. "mechanical" ohms)

Substituting in for $\tilde{f}$ and $\tilde{u}$ gives:

$$\tilde{Z}_m = \frac{Fe^{jwt}}{\left[\frac{Fe^{jwt}}{R_m + j\left(wm - \frac{s}{w}\right)}\right]} = R_m + j\left(wm - \frac{s}{w}\right)$$

We can define the real and imaginary parts of the mechanical impedance as: $\tilde{Z}_m = R_m + jX_m$

<u>Mechanical Resistance</u>: R_m

<u>Mechanical Reactance</u>: $X_m = wm - \frac{s}{w}$