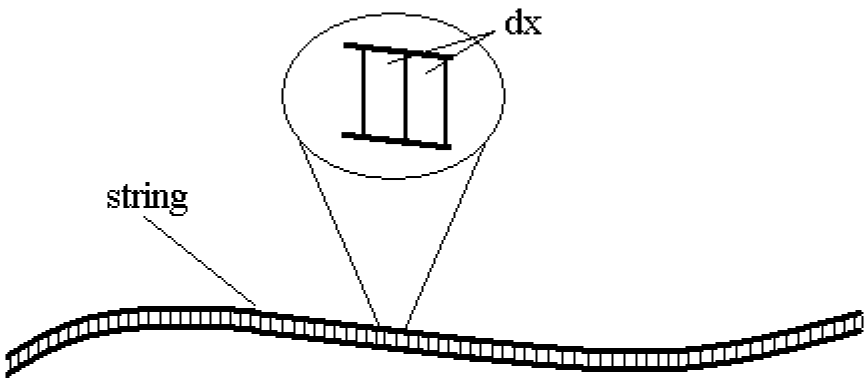
Chapter 2 – The Vibrating String

(2.1) For simple harmonic oscillator (one mass), the goal was to find the single function x(t) that would describe the entire history of the motion.

For a finite number of N masses connected by various springs, $x_1(t)$, $x_2(t)$,..., $x_N(t)$ functions would be needed.

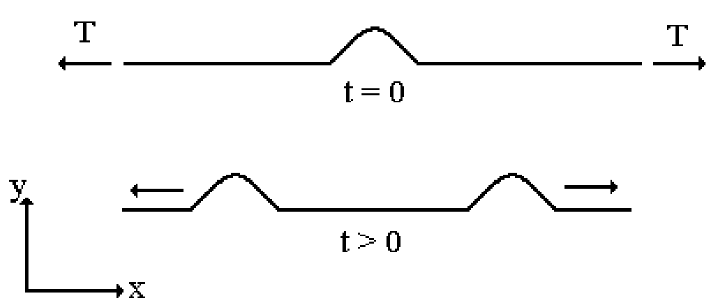
The vibrating string is a system that has an infinite number of masses (infinitely many parts), each of which may move in a different way.

(2.2 – 2.3) Each section of the string is an infinitesimally small mass taking up an infinitesimally small segment, dx.



Consider a long, uniform string:
total mass: m_s (kg)
length: L (m)
mass per unit length: $r_L = \frac{m_s}{L}$ (kg/m) -linear mass density
This means that each dx element will have a total mass of $r_L dx$.

Let's assume that the string is stretched tight with a tension, *T*, and then is plucked or displaced in the middle. A vibration or small disturbance will travel down the string in both directions.



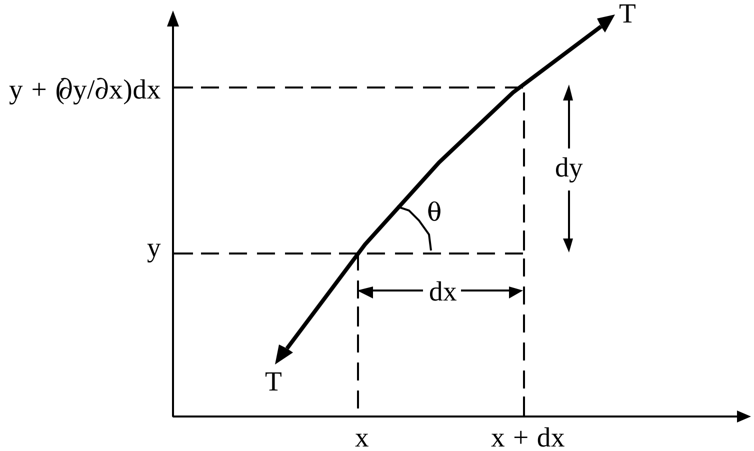
The tension, *T*, acts as the restoring force (N) for the displacement, $y(x,t)$, which is a function of both the position along the string and time (see above figure).

So, let's work out the Equation of motion for the displacement of the string:
1st some assumptions:

(1) T is large enough so that we can neglect gravity
(2) No loss (due to friction or acoustic radiation)
(3) Neglect stiffness (transverse motion allowed - like floppy string not wire)
(4) Displacement is small enough that T is still approx. constant along length

Now, let us examine the forces exerted on a particular dx segment. When the string is at rest (no displacement) the tensions at x and x + dx are equal in magnitude and opposite in direction. The net force is zero.

When the string has curvature, the tension acts in slightly different directions at x and x + dx, thus pulling the dx segment in such a direction to try to straighten it out.



Looking at the vertical component of the incremental force on the dx segment:

$$\Delta f_y = T\sin\theta_{x+dx} - T\sin\theta_x$$

If we apply a Taylor series expansion of $f_y(x+dx) = T\sin\theta_{x+dx}$ we have:

$$f_y(x+dx) = f_y(x) + \left(\frac{\partial f}{\partial x}\right)_x dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_x dx^2 + \ldots$$

$$f_y(x+dx) = T\sin\theta_x + T\frac{\partial \sin\theta}{\partial x}dx + \ldots$$

so that

$$\Delta f_y = T\frac{\partial \sin\theta}{\partial x}dx.$$

For small θ, $\cos\theta \approx 1$ so that

$$\tan\theta = \frac{\partial y}{\partial x} = \frac{\sin\theta}{\cos\theta} \approx \sin\theta$$

and

$$\Delta f_y = T\frac{\partial}{\partial x}\frac{\partial y}{\partial x}dx = T\frac{\partial^2 y}{\partial x^2}dx.$$

Now, using Newton's second law for constant mass (f=ma) then

$$m = r_L dx \qquad \text{and} \qquad a = \frac{\partial^2 y}{\partial t^2}.$$

Equating the two forces gives:

$$T\frac{\partial^2 y}{\partial x^2}dx = r_L dx \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad \text{where } c^2 = \frac{T}{r_L}$$

This is called the <u>one-dimensional wave equation</u>.

(2.4) General Solution of the Wave Equation:

A general solution to the 1D wave equation is:

$$y(x,t) = y_1(ct - x) + y_2(ct + x).$$

(2.5) What does this mean physically?
Well, let's assume that the phase portions of our functions are constant ($ct \pm x = const$).
If we differentiate our phase then

$$cdt \pm dx$$

or

$$c = \pm\frac{dx}{dt}.$$

In other words, the '+' phase represents a wave traveling in the positive direction and the '-' phase represents a wave traveling in the negative direction both with a <u>speed</u> *c*.

(2.6) Initial Values and Boundary Conditions

The particular functions $y_1(ct-x)$ and $y_2(ct+x)$ are determined by the type of excitation

For example, with stringed instruments you can have the following excitations

(a) striking - piano
(b) plucking - harp or guitar
(c) bowing - violin or cello

In the real world the strings are held at both ends so we must deal with boundary conditions that affect the waves that will exist on the string.

Also, note that in the case of a driven system the steady-state response is at the driving frequency.

(2.7) Reflection at a Boundary

If we have a string that is rigidly supported at the boundary (x = 0) then the total displacement in the y-direction must be zero. Looking at the general solution then reveals:

$$y(0,t) = y_1(ct-0) + y_2(ct+0) = 0\,.$$

This means that a wave is reflected from the boundary and that wave must sum with the incident wave to zero. In other words, if the incident wave has positive displacement, then the reflected wave must have displacement equal and opposite in magnitude to the incident wave.

Let's look at another case: A finite-length string supported freely at x = 0.
In this case the force at the end of the string will be zero so $f(x=0) = T\sin q = 0$.

Since $\sin q \approx \dfrac{\partial y}{\partial x}$ then if we let $w = ct - x$ and $v = ct + x$, using the chain gives

$$\frac{\partial y}{\partial x} = \left[\frac{\partial y_1}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial y_2}{\partial v}\frac{\partial v}{\partial x}\right]_{x=0} = \left[-\frac{\partial y_1}{\partial w} + \frac{\partial y_2}{\partial v}\right]_{x=0} = 0$$

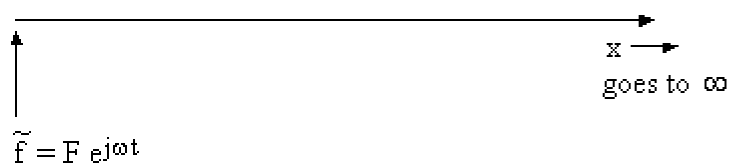
so

$$\left[\frac{\partial y_1}{\partial w}\right]_{x=0} = \left[\frac{\partial y_2}{\partial v}\right]_{x=0}$$

but at $x = 0$, $w = v$ so

$$\partial y_1 = \partial y_2 \qquad \text{which implies} \qquad y_1 = y_2$$

(2.8) Forced Vibration of an Infinite String



Next, let's consider a string of infinite length driven at one end by an oscillator. This means there is no boundary at the other end so we have a solution for a positive going wave only (no reflected wave). In other words:

$$y(x,t) = y_1(ct - x)$$

so that at $x = 0$,

$$y(0,t) = \tilde{A}e^{jwt}\,.$$

Since our solution is of the form $y_1(ct-x)$ this means that

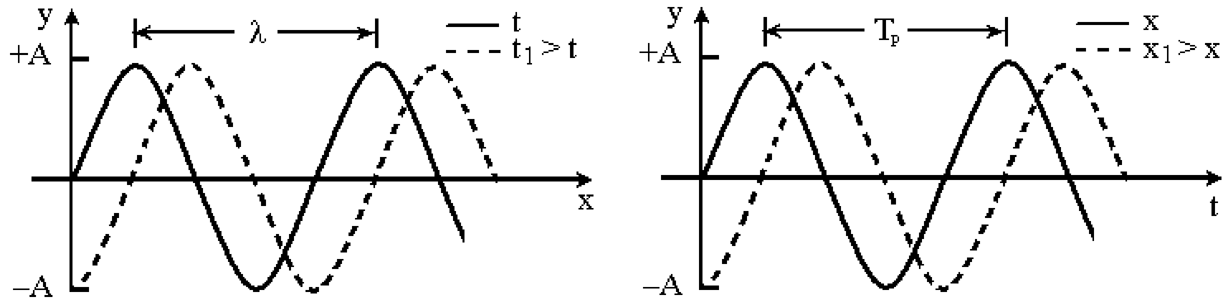
$$y_1(ct-0) = \tilde{A}e^{jk(ct)} = \tilde{A}e^{jwt}$$

where $k = \dfrac{w}{c} = \dfrac{2p}{l}$ and is called the wavenumber or propagation constant and is inversely proportional to the wavelength, l. Therefore,

$$y(x,t) = y_1(ct-x) = \tilde{A}e^{jk(ct-x)} = \tilde{A}e^{j(wt-kx)}\,.$$

TIME OUT:

Some important definitions about wave motion:



The wave propagates in the x direction as viewed in the panel on the left.
The <u>wavelength</u>, l, is defined as the distance over which the sinusoid varies by 2π or one cycle.
Temporally, the <u>period</u>, T_p, is related to the wavelength as is defined as the time required for the sinusoid to vary by 2π or one cycle.
Since distance = speed x time, then, $l = cT_p$

Recall <u>speed</u>: $$c = \sqrt{\frac{T}{r_L}}.$$

The <u>frequency</u> (Hz) is defined as the number of cycles per second:

$$f = \frac{1}{T_p} \rightarrow \frac{c}{f}.$$

The <u>radian frequency</u>, w, (radians/sec):

$$w = 2p\, f \rightarrow \frac{2p\, c}{w}.$$

The <u>wavenumber</u>, k, (radians/meter):

$$k = \frac{w}{c} = \frac{2p}{l}.$$

TIME IN: (Back to the driven string)

The applied driving force on the string must be equal to the transverse component of the tension force so that:

$$\tilde{f} = -\left[T \sin q\right]_{x=0} = -T\left[\frac{\partial y}{\partial x}\right]_{x=0}$$

so

$$\tilde{f} = Fe^{jwt} = -T\left[\frac{\partial}{\partial x}\tilde{A}e^{j(wt-kx)}\right]_{x=0}$$

$$\tilde{f} = Fe^{jwt} = jkT\tilde{A}e^{jwt}$$

or

$$\tilde{A} = \frac{F}{jkT}$$

Our solutions are then:

$$y(x,t)=\frac{Fe^{j(wt-kx)}}{jkT}$$

and

$$u(x,t)=\frac{dy}{dt}=\frac{wF}{kT}e^{j(wt-kx)}.$$

Recall that,

$$c=\frac{w}{k} \qquad \text{and} \qquad T=r_L c^2$$

so

$$u(x,t)=\frac{F}{r_L c}e^{j(wt-kx)}.$$

Let's discuss the concept of cause-effect.

Cause: Voltage	E Field	Force
Effect: Current	H Field	Velocity

Impedance $\rightarrow \frac{Cause}{Effect}$

<u>Input mechanical impedance</u>:

$$\tilde{Z}_{m0}=\frac{\tilde{f}}{\tilde{u}(0,t)}$$

$$\tilde{Z}_{m0}=\frac{Fe^{jwt}}{\frac{F}{r_L c}e^{j(wt-k0)}}=r_L c$$

Note: the mechanical impedance of the infinite string is a real quantity, which means it is purely a resistive load.

Power, Energy, etc. $\rightarrow$ *Cause* x *Effect*

<u>Instantaneous Input Power</u>:

$$\Pi_i=\mathrm{Re}\left\{\tilde{f}\right\}\mathrm{Re}\left\{\tilde{u}(o,t)\right\}$$

$$\Pi_i=\mathrm{Re}\left\{Fe^{jwt}\right\}\mathrm{Re}\left\{\frac{F}{r_L c}e^{j(wt-k0)}\right\}=\frac{F^2}{r_L c}\cos^2 wt$$

$$\text{Units:}\left(\frac{N^2}{N\cdot\frac{s}{m}}=N\cdot\frac{m}{s}=\frac{J}{s}=W\right)$$

Average Input Power:

$$\Pi = \frac{1}{T}\int_0^T \Pi_i dt \quad \text{(Take the average over one cycle).}$$

$$\Pi = \frac{1}{T}\int_0^T \frac{F^2}{r_L c}\cos^2 wt dt = \frac{1}{2}\frac{F^2}{r_L c}.$$

Now

$$u(0,t) = \frac{F}{r_L c} e^{j(wt-k0)}$$

so that

$$U_0 = |u(0,t)| = \frac{F}{r_L c}$$

which gives

$$\Pi = \frac{1}{2} r_L c U_0^2$$

(2.9) Forced Vibration of a Finite String

Case I.

Let's consider a finite-length string fixed at one end ($x = L$) and driven at $x = 0$ under steady-state conditions by Fe^{jwt}.

What kind of wave is going to reflect off of the fixed end?

In this case we are going to have reflections at both ends so we need a solution with both positive and negative going waves:

$$\tilde{y}(x,t) = \tilde{A}e^{j(wt-kx)} + \tilde{B}e^{j(wt+kx)}.$$

We want to use our boundary conditions to solve for the 2 unknowns, $\tilde{A}$ and $\tilde{B}$. At the end $x = 0$, the driving force must = the vertical tension force so,

$$\tilde{f}_y = -T\left(\frac{\partial y}{\partial x}\right)_{x=0}$$

Substituting our solution gives

$$Fe^{jwt} = -T\left(-jk\tilde{A}e^{j(wt-kx)} + jk\tilde{B}e^{j(wt+kx)}\right)_{x=0}$$

$$F = jkT\left(\tilde{A} - \tilde{B}\right)$$

The boundary condition at the fixed end is simply $y(L,t) = 0$ at all times so that (wlg)

$$\tilde{A}e^{-jkL} + \tilde{B}e^{jkL} = 0$$

So now we have 2 equations and 2 unknowns, let's solve:

$$\tilde{A} = \frac{Fe^{jkL}}{2jkT\cos kL}$$

$$\tilde{B} = -\frac{Fe^{-jkL}}{2jkT\cos kL}$$

This gives us a solution:

$$\tilde{y}(x,t) = \frac{Fe^{jkL}}{2\,jkT\cos kL}e^{j(wt-kx)} + \frac{-Fe^{-jkL}}{2\,jkT\cos kL}e^{j(wt+kx)}$$

$$= \frac{F}{2\,jkT\cos kL}\left(e^{j(wt+k(L-x))} - e^{j(wt-k(L-x))}\right)$$

$$= \frac{F}{kT}\,\frac{\sin[k(L-x)]}{\cos kL}\,e^{jwt}$$

The solution above is called a <u>standing wave</u>. What we have is the string as a non-propagating sine-wave that oscillates its amplitude according to e^{jwt}. We have two distinct features of the standing wave called the <u>node</u> and <u>antinode</u>.

The node is defined where the amplitude is always zero:

$$\sin[k(L-x)] = 0$$

or $$k(L-x) = qp \qquad \text{where } q = 0,1,2,\ldots \le \frac{kL}{p}$$

so that the nodes are defined at positions

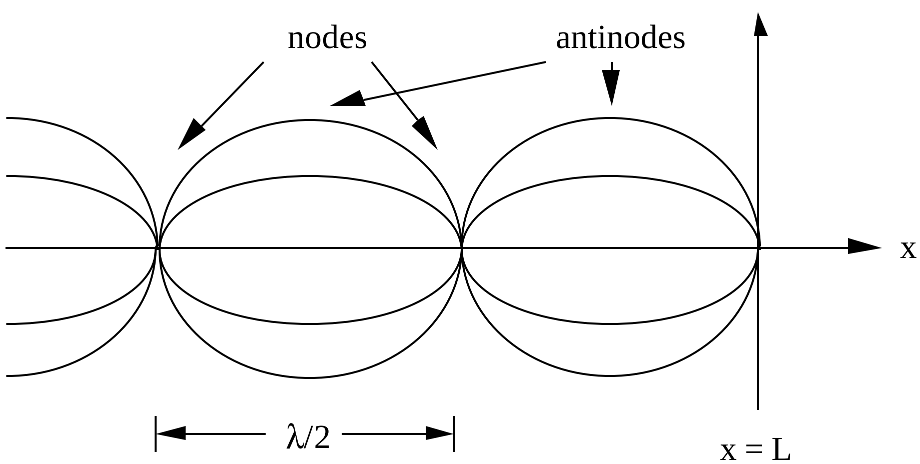
$$x_q = L - \frac{qp}{k}$$

but recall that $k = \dfrac{2p}{l}$ so $x_q = L - \dfrac{q\,l}{2}$.

The antinode is defined where the amplitude is at a maximum:

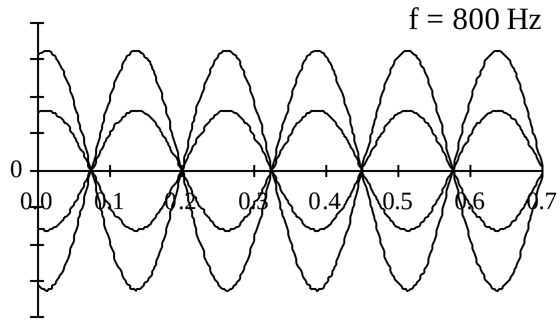
$$\sin[k(L-x)] = 1.$$

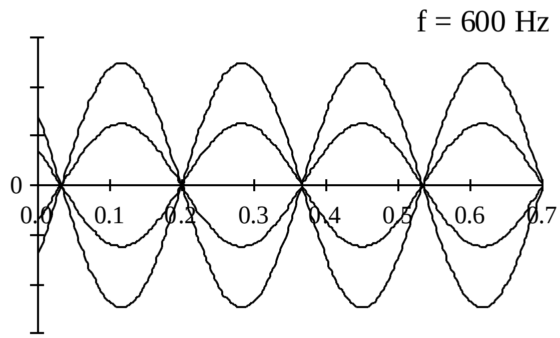
Below is a plot of the standing wave:

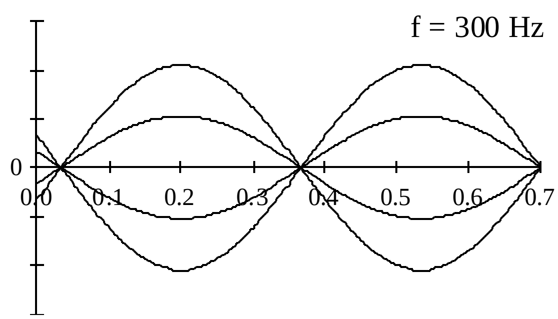


The nodes are separated by $\dfrac{l}{2}$ as are the antinodes. The antinodes are located at $x = L - \dfrac{l}{4} - n\dfrac{l}{2}$.

If we change the frequency of the driver the positions of the nodes and antinodes will change (when we change the frequency, w, we change the wavelength, l. This is illustrated in the figures below:







The amplitude of the antinodes can is also a function of the frequency of the driver and has some very interesting effects at the <u>resonant frequency</u>. Recall that:

$$\tilde{y}(x,t) = \frac{F}{kT}\,\frac{\sin[k(L-x)]}{\cos kL}\,e^{j\omega t}$$

the amplitude of the standing wave is mediated by the $\cos kL$ term. When

$$kL = (2n-1)\frac{\pi}{2} \qquad \text{for } n = 1,2,3,\ldots \text{ (odd multiples of } \tfrac{\pi}{2}\text{)}$$

then

$$\cos kL = 0.$$

The resonant frequency, f_r, is defined as:

$$\frac{2\pi f_r L}{c} = (2n-1)\frac{\pi}{2}$$

$$f_r = \frac{2\pi f_r L}{c} = \frac{(2n-1)c}{4L}$$

At the resonance the amplitude theoretically approaches infinity (of course in reality, losses would prevent any system from having infinite amplitude).

We can also define an antiresonant frequency where the amplitude of the standing wave is at a minimum. In that case,

$$\cos kL = \pm 1$$

so that

$$kL = n\pi \qquad \text{for } n = 1,2,3,\ldots$$

giving the antiresonance frequency:

$$f_{ar} = \frac{nc}{2L}$$

The input mechanical impedance:

$$\tilde{Z}_{m0} = \frac{\tilde{f}}{\tilde{u}(0,t)}$$

where

$$\tilde{u}(0,t) = \left(\frac{d\tilde{y}}{dt}\right)_{x=0} = j\omega \frac{F}{kT} \frac{\sin kL}{\cos kL} e^{j\omega t}$$

$$\tilde{u}(0,t) = j\omega \frac{F}{kT} \tan kL \; e^{j\omega t}$$

giving

$$\tilde{Z}_{m0} = \frac{Fe^{j\omega t}}{j\omega \frac{F}{kT} \tan kL \; e^{j\omega t}} = \frac{kT \cot kL}{j\omega} = -j\, r_L c \cot kL \,.$$

Recall that the mechanical impedance is the resistance plus the reactance: $\tilde{Z}_m = R_m + jX_m$

Resonance frequencies of any mechanical system are defined in general as those frequencies for which the input mechanical reactance, X_m, goes to zero:

$$\tilde{X}_{m0} = -j\, r_L c \cot kL = 0$$

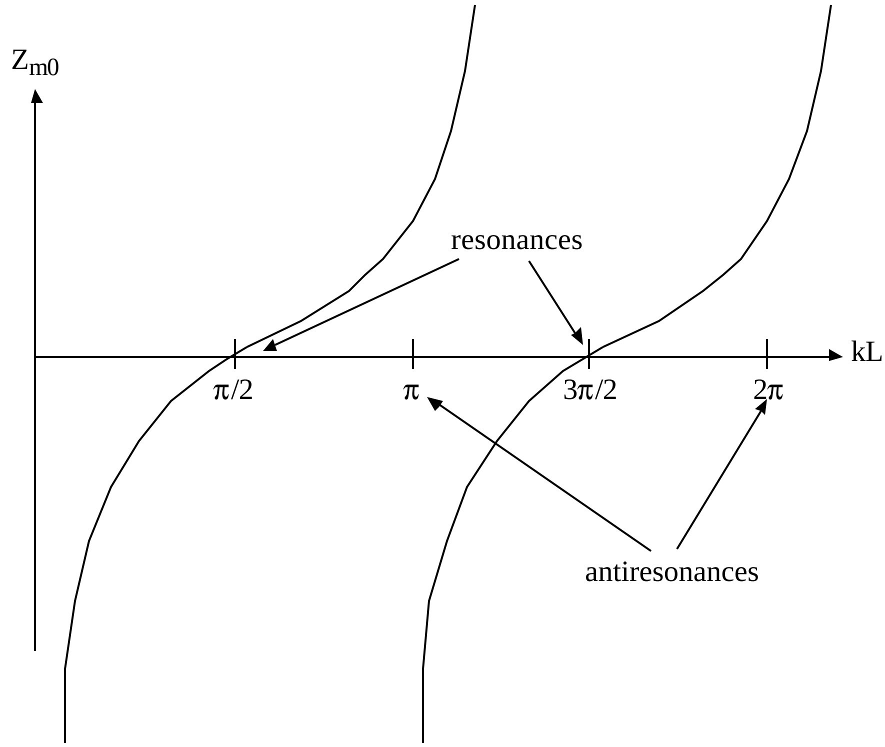
when

$$kL = \frac{2n-1}{2}\pi \,, \qquad n = 1, 2, 3, \ldots$$

or

$$f_r = \frac{(2n-1)c}{4L}.$$

At resonance frequencies the input mechanical impedance goes to zero.



At very low frequencies $\sin kL \rightarrow kL$ and $\cos kL \rightarrow 1$

$$\tilde{Z}_{m0} \cong -j r_L c \frac{1}{kL} = -j \frac{T}{wL}$$

**************************** Example 2.1 ****************************

Standing waves on a forced, fixed string

A forced, fixed string with a length of 1 m, a tension of 2 N, and a linear density of 0.02 kg/m (yielding a speed of 10 m/s) is driven at frequencies of 15 and 7.5 Hz. The wavelengths are 2/3 and 4/3 m and the propagation constants are $k = 3\pi$ and $3\pi/2$ for the two frequencies, respectively.

If the string is fixed at $x = L = 1$ m and is driven in the y direction by a force of amplitude F at $x = 0$, then the displacement in the y direction as a function of x and t is given by

$$y(x,t) = \frac{F}{kT} \frac{\sin[k(L-x)]}{\cos[k]} \cos[wt]$$

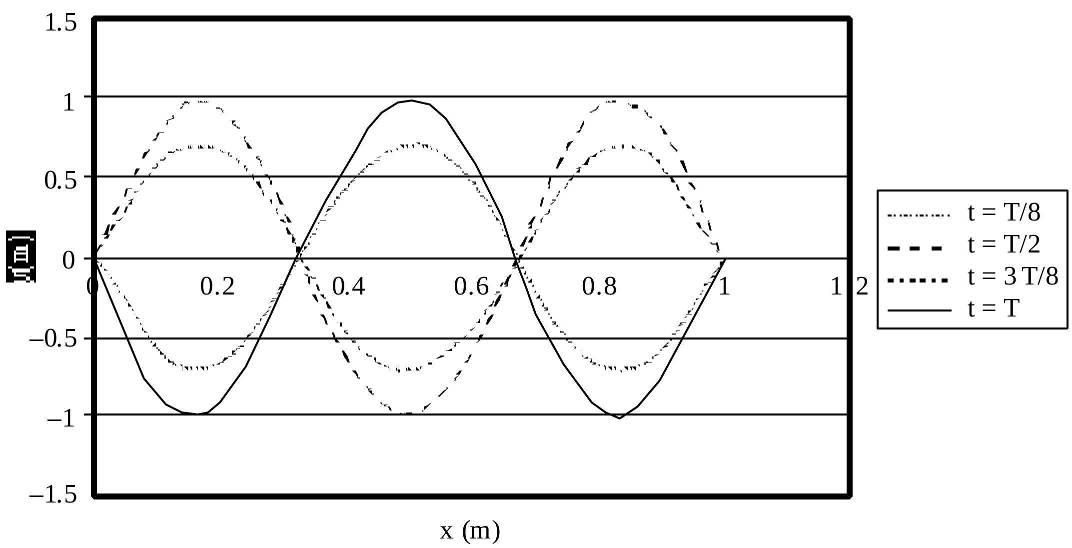
where k is the propagation constant, T is the tension, w is the angular frequency and t is time. Choosing values of F of 6π and 3π for the two frequencies, respectively, yields

$$y(x,t) = \frac{\sin[k(L-x)]}{\cos[k]} \cos[w\ t]$$

Examining the function at each frequency for values of $t = \tau/8$, $3\tau/8$, $\tau/2$ and τ, where τ is the period (denoted by T in the plots) gives the following results.

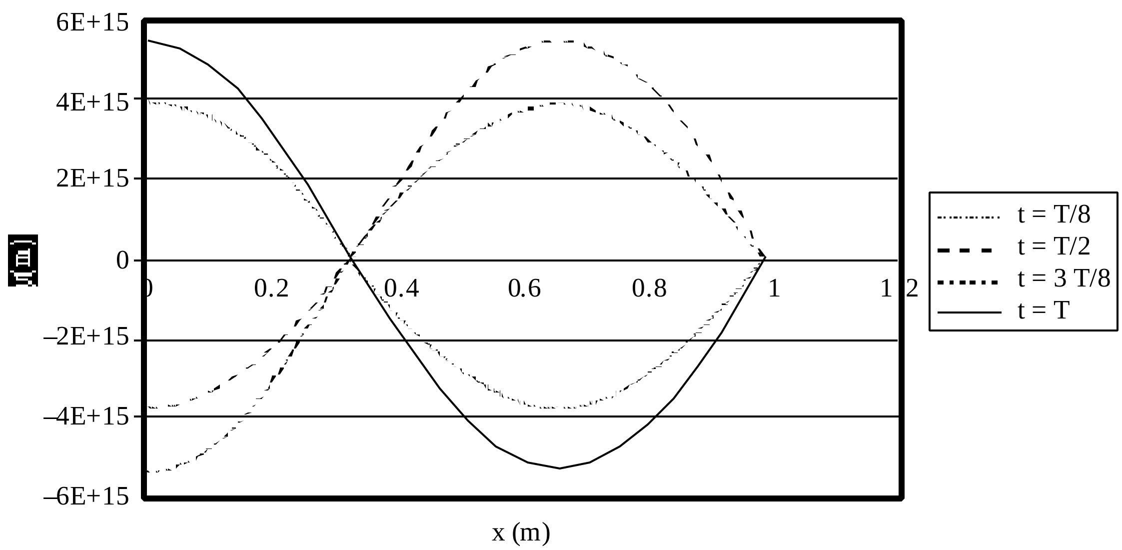
The results at 15 Hz (an antiresonance frequency) are plotted below. Note that at 15 Hz, $\cos[k] = -1$ and the amplitude will be at a minimum (this is an antiresonance frequency).

String Displacement at 15 Hz



The results at 7.5 Hz are plotted below. Note that 7.5 Hz is a resonance frequency, $\cos[k] = 0$, and the amplitude will be a maximum (theoretically it is infinite, but roundoff error gives the finite values plotted below).

String Displacement at 7.5 Hz



****

(2.10) Normal Modes of the Fixed, Fixed String

In this case we are looking at a string rigidly supported on both sides (x = 0 and x = L) so that the displacement at each end is y = 0. This is reminiscent of a guitar string.
First we assume a solution with both positive and negative going waves:

$$\tilde{y}(x,t) = \tilde{A}e^{j(wt-kx)} + \tilde{B}e^{j(wt+kx)}$$

Using our boundary conditions allows us to solve for the 2 unknowns.
At x = 0,

$$\tilde{y}(0,t) = \tilde{A}e^{jwt} + \tilde{B}e^{jwt} = 0$$

or

$$\tilde{B} = -\tilde{A}\,.$$

At x = L,

$$\tilde{y}(L,t) = \tilde{A}e^{j(wt-kL)} + \tilde{B}e^{j(wt+kL)} = 0$$

or

$$\tilde{A}e^{-jkL} + \tilde{B}e^{jkL} = 0\,.$$

Substituting in for $\tilde{B}$ gives

$$\tilde{A}e^{jkL} - \tilde{A}e^{-jkL} = 0$$

$$2j\tilde{A}\sin kL = 0$$

There is an obvious trivial solution of $\tilde{A} = 0$ where there is no motion, but we seek the nontrivial solution where

$$\sin kL = 0$$

or

$$kL = np \qquad \text{for } n = 1,2,3,\ldots$$

(Note: n = 0 is also a trivial solution since that would mean no motion of the string, also).
Thus, only discrete frequencies are solutions. Those frequencies are given by:

$$\frac{2p\, f_n L}{c} = np$$

$$f_n = n\frac{c}{2L} \quad or \quad L = \frac{nc}{2f_n} = n\frac{l_n}{2} \quad \text{multiples of } l/2$$

Back to our solution, for each n we have :

$$\tilde{y}_n(x,t) = \tilde{A}e^{j(w_n t - k_n x)} - \tilde{A}e^{j(w_n t + k_n x)}$$

$$\tilde{y}_n(x,t) = -2j\tilde{A}\sin(k_n x)e^{jw_n t} = \tilde{A}_n \sin(k_n x)e^{jw_n t}$$

So lets replace $\tilde{A}_n$ with $\tilde{A}_n = A_n - jB_n$ (where A_n and B_n are real values).

$$\tilde{y}_n(x,t) = (A_n - jB_n)\sin(k_n x)e^{jw_n t}$$

The real part of the solution is:

$$\mathrm{Re}\{\tilde{y}_n(x,t)\} = \mathrm{Re}\{(A_n - jB_n)\sin(k_n x)e^{jw_n t}\}$$

$$\mathrm{Re}\{\tilde{y}_n(x,t)\} = \mathrm{Re}\{(A_n - jB_n)\sin(k_n x)[\cos w_n t + j\sin jw_n t]\}$$

$$\mathrm{Re}\{\tilde{y}_n(x,t)\} = (A_n \cos w_n t + B_n \sin jw_n t)\sin(k_n x)$$

These functions are called eigenfunctions or normal modes and their solutions, A_n and B_n, are determined from the initial conditions, that is, $y(x,0)$ and $u(x,0)$.

There are also unique frequencies, called eigenfrequencies:

$$f_n = \frac{w_n}{2p} = n\frac{c}{2L} \quad , \quad n = 1,2,3,\ldots$$

For $n = 1$, the solution is called the fundamental mode and its equivalent frequency is called the fundamental frequency.

For $n = 2$, the second harmonic frequency is called the first overtone.

For $n = 2, 3, 4, \ldots$, the eigenfrequencies are called overtones.

The complete solution is the sum of all the individual modes of vibration:

$$\mathrm{Re}\{\tilde{y}(x,t)\} = \sum_{n=1}^{\infty}(A_n \cos w_n t + B_n \sin jw_n t)\sin(k_n x)$$

To determine the constants, we use the initial conditions:

$$\mathrm{Re}\{\tilde{y}(x,0)\} = \sum_{n=1}^{\infty} A_n \sin(k_n x)$$

and

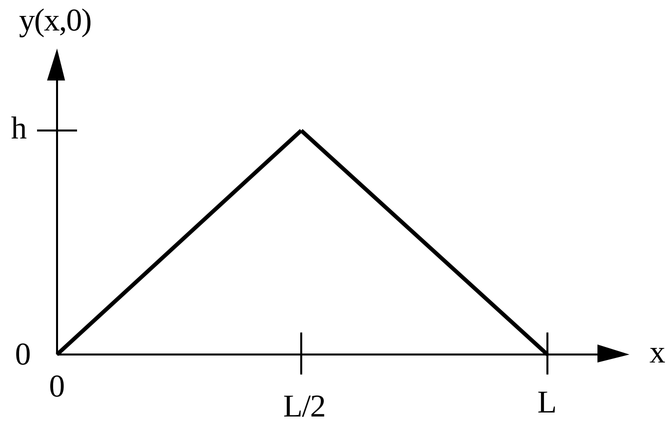
$$\mathrm{Re}\{u(x,0)\} = \sum_{n=1}^{\infty} w_n B_n \sin(k_n x).$$

The two constants can then be determined by applying the Fourier theorem:

$$A_n = \frac{2}{L}\int_0^L y(x,o)\sin(k_n x)\,dx$$

$$B_n = \frac{2}{w_n L}\int_0^L u(x,o)\sin(k_n x)\,dx$$

As an example, let's look at the plucked string. Suppose we have a string and the string is set into vibration by displacing the string a height, h, at its center ($x = L/2$). The displacement function at time $t = 0$ would look like:



The other initial condition is $u(x, 0) = 0$, which directly yields $B_n = 0$.

Insertion into the Fourier transform equation yields:

$$\left.\begin{aligned} A_n &= \frac{8h}{(np)^2}\sin\left(\frac{np}{2}\right), && n \text{ odd} \\ A_n &= 0, && n \text{ even} \end{aligned}\right\}$$

or more specifically

$$A_1 = \frac{8h}{p^2} \qquad \frac{A_3}{A_1} = -\frac{1}{9} \qquad \frac{A_5}{A_1} = \frac{1}{25} \quad \text{etc…}$$