Chapter 10 – Pipes, Resonators, and Filters

(10.1) Introduction

We will be discussing in this chapter the behavior of sound in rigid walled pipes, resonators and filters. The behavior of sound will be strongly dependent on the properties of the driver (source in the pipe), the length of the pipe, the cross-sectional area of the pipe, any obstructions or branches of the pipe and the way the pipe terminates. In the development in this chapter we will be focusing on wavelengths of sound that are sufficiently large so that the sound can be approximated as a collimated plane wave. This chapter applies to:

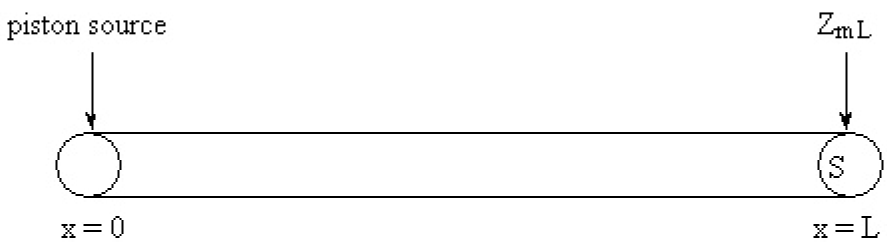
Musical instruments
- brasses
- woodwinds
- organ pipes

Ducts
- Ventilation systems

Mufflers and silencers

(10.2) Resonance in Pipes

We will start our discussion with simple pipes. We'll look at the resonance features of pipes, power radiated from open-ended pipes, standing wave patterns and the absorption of sound in pipes.



Now we've defined the specific acoustic impedance

Specific Acoustic Impedance at a point

$$Z_S = \frac{\text{acousticpressure}}{\text{particlevelocity}} = \frac{p}{u}$$

$$\text{Unit: } \frac{\text{Pa}}{\text{m / s}} = \frac{\text{N / m}^2}{\text{m / s}} = \frac{\text{N} \cdot \text{s}}{\text{m}^3} \quad \text{(MKS Rayl)}$$

In this section we will look at another couple of impedances

1. Acoustic Impedance on a surface
2. Mechanical Impedance (Radiation Impedance)

<u>Acoustic Impedance on a surface</u>

$$Z_A = \frac{\text{acousticpressure}}{\text{volumevelocity}} = \frac{p}{U_V}$$

$$= \frac{\text{acousticpressure}}{A(\text{particlevelocity})} = \frac{p}{Au}$$

$$\text{Unit: } \frac{\text{Pa}}{\text{m}^2(\text{m / s})} = \frac{\text{N / m}^2}{\text{m}^2(\text{m / s})} = \frac{\text{N}\cdot\text{s}}{\text{m}^5}$$

<u>Mechanical Impedance</u> (Radiation Impedance)

$$Z_m = \frac{\text{effectiveforce}}{\text{particlevelocity}}$$

$$\text{Unit: } \frac{\text{N}}{\text{m / s}} = \frac{\text{N}\cdot\text{s}}{\text{m}}$$

Let's consider the case where a (radius) << l $\Rightarrow$ plane waves are propagated (each point in the pipe is approximately at constant phase). Therefore a standing wave exists in the pipe so that

$$\tilde{p} = \tilde{A}e^{j(wt+k(L-x))} + \tilde{B}e^{j(wt-k(L-x))}$$

Let's apply the boundary conditions @ $x = 0$ and $x = L$. The continuity of <u>force</u> and <u>particle speed</u> must hold at the boundaries so that combined this yields continuity of mechanical impedance. From the above definition:

$$Z_m = \frac{F}{u}$$

where

$$F = pS$$

From Chapter 5 for a plane wave

$$u(x,t) = \pm\frac{p}{r_0 c} \quad \begin{array}{l} + \text{ pos. going} \\ - \text{ neg. going} \end{array}$$

so

$$Z_m = \frac{F}{u} = \pm\ r_0 cS$$

So, defining the impedance at each end:
<u>@ x = L</u>

$$\tilde{Z}_{mL} = r_0 cS\ \frac{\tilde{A}+\tilde{B}}{\tilde{A}-\tilde{B}}$$

@ x = 0

$$\tilde{Z}_{m0} = r_0 cS \; \frac{\tilde{A}e^{jkL} + \tilde{B}e^{-jkL}}{\tilde{A}e^{jkL} - \tilde{B}e^{-jkL}}$$

Combining these two relations allows us to eliminate $\tilde{A}$ and $\tilde{B}$ giving

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \frac{\dfrac{\tilde{Z}_{mL}}{r_0 cS} + j\tan(kL)}{1 + j\dfrac{\tilde{Z}_{mL}}{r_0 cS}\tan(kL)}$$

Of course we can represent the impedance in a complex form such that $\dfrac{\tilde{Z}_{mL}}{r_0 cS} = r + jx$ or

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \frac{r\left(\tan^2(kL)) + 1\right) - j\left[x\tan^2(kL) + \left(r^2 + x^2 - 1\right)\tan(kL) - x\right]}{(r+x)^2\tan^2(kL) - 2x\tan(kL) + 1}$$

At resonance and antiresonance the input reactance is zero so (chapter 3.7)

$$\frac{x\tan^2(kL) + \left(r^2 + x^2 - 1\right)\tan(kL) - x}{(r+x)^2\tan^2(kL) - 2x\tan(kL) + 1} = 0$$

resonance $\rightarrow$ small input impedance (resistance)
antiresonance $\rightarrow$ large input impedance (resistance)

So, Let's look at a few special Cases

Case 1: Rigid cap at $x = L$

As expected:

$$\tilde{Z}_{mL} = \infty \qquad \text{and} \qquad \frac{\tilde{Z}_{mL}}{r_0 cS} = \infty .$$

So,

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \left.\frac{\dfrac{\tilde{Z}_{mL}}{r_0 cS} + j\tan(kL)}{1 + j\dfrac{\tilde{Z}_{mL}}{r_0 cS}\tan(kL)}\right|_{\frac{\tilde{Z}_{mL}}{r_0 cS}\to\infty} = -j\cot(kL)$$

The reactance is not necessarily zero. However,

$$\cot(kL) = 0 = \frac{\cos(kL)}{\sin(kL)}$$

for $\quad k_n L = (2n-1)\frac{p}{2} \qquad$ for $\quad n = 1,2,3,\ldots$ (odd multiples of $\pi/2$)

and our resonance frequencies are given by

$$\boxed{f_n = \frac{(2n-1)}{4}\frac{c}{L} \quad or \quad L = \frac{(2n-1)}{4}\, l}$$

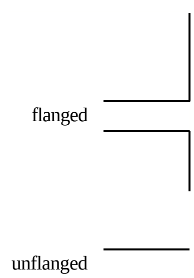
resonance @ $\; f = \frac{c}{4L}, \frac{3c}{4L}, 5\frac{c}{4L}\ldots$ i.e., at odd harmonics where $f_1 = \frac{c}{4L}$ is the fundamental frequency. The input mechanical impedance is zero at resonance. Also, we have a pressure antinode @ $x = L$ and a pressure node @ $x = 0$. Is it logical for the mechanical impedance to be zero at the driver?

Case 2: Open ended pipe

Is $\quad \tilde{Z}_{mL} = 0$?

No $\qquad \tilde{Z}_{mL} = \tilde{Z}_r \qquad$ radiation impedance. Some energy will be radiated into surrounding medium.

two cases (subcases)



<u>Flanged</u> (some musical instruments)

For <u>$l >> a$</u> or <u>ka small</u> and for the flange large relative to l then this can be approximated as a baffled piston source in the low frequency limit. So,

$$\frac{\tilde{Z}_{mL}}{r_0 cS} = \frac{\tilde{Z}_r}{r_0 cS} = \frac{\overbrace{\frac{1}{2} r_0 cS(ka)^2}^{R_r} + j \overbrace{r_0 cS \frac{8}{3p}(ka)}^{X_r}}{r_0 cS} = \frac{1}{2}(ka)^2 + j\frac{8}{3p} ka = r + jx$$

we obtained this from eq (7.5.11) for the baffled piston source.

Noting that $r << x$ and substituting into the expression for $\frac{\tilde{Z}_{m0}}{r_0 cS}$ we get

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \frac{\frac{\tilde{Z}_{mL}}{r_0 cS} + j\tan(kL)}{1 + j\frac{\tilde{Z}_{mL}}{r_0 cS}\tan(kL)} \cong \frac{jx + j\tan(kL)}{1 - x\tan(kL)}$$

Again, at resonance the reactance = 0 which means that

$$-x = \tan(kL)$$

yielding:

$$\tan\left(np - k_n L\right) = \frac{8}{3p} k_n a \underset{\substack{for \\ kasmall}}{\cong} \tan\left(\frac{8}{3p} k_n a\right)$$

which means that

$$np = k_n L + \frac{8}{3p} k_n a$$

giving us as our resonant frequencies:

$$f_n = \frac{n}{2}\frac{c}{L + \frac{8}{3p}a} = \frac{n}{2}\frac{c}{L_{eff}}$$

$$\text{or } \; L_{eff} = n\frac{l}{2}$$

for unflanged (I state without derivation) $\frac{\tilde{Z}_{mL}}{r_0 cS} = \frac{1}{4}(ka)^2 + j\,0.6\,ka$ giving

$$f_n = \frac{n}{2}\frac{c}{L + 0.6a} = \frac{n}{2}\frac{c}{L_{eff}}$$

***************************** Example 10.1 *****************************

Consider an open-ended pipe (air), with L = 1 m, a = 1 cm = 0.01 m

flanged	unflanged
$L_{eff} = L + \frac{8}{3p}a$	$L_{eff} = L + 0.6\,a$
$= 1 + (0.849)(.01)$	$= 1 + 0.6(.01)$
$= 1.0085$ m	$= 1.006$ m

$$f_n = \frac{n}{2}\frac{c}{L_{eff}}$$

f_n = n (170.05) Hz for flanged and f_n = n (170.5) Hz for unflanged

Flanged	Unflanged	
f_1 = 170.05	f_1 = 170.5	Hz
f_2 = 340.1	f_2 = 341.0	Hz
f_3 = 510.2	f_3 = 511.4	Hz

Check $$\frac{l_n}{a} = \frac{c}{f_n a} = \frac{343}{0.01}\frac{1}{f_n} = \frac{34{,}300}{f_n}$$

$\therefore\ l >> a$ for 1st 3 or more harmonics

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(10.3) Power Radiation from Open-Ended Pipes

Previously we found

$$\tilde{Z}_{mL} = r_0 cS\,\frac{\tilde{A}+\tilde{B}}{\tilde{A}-\tilde{B}}$$

rearranging

$$\frac{\tilde{B}}{\tilde{A}} = \frac{\dfrac{\tilde{Z}_{mL}}{r_0 cS}-1}{\dfrac{\tilde{Z}_{mL}}{r_0 cS}+1}$$

Note: this is just a pressure reflection coefficient.

So, the Power transmission coefficient is

$$T_\Pi = 1 - \left|\frac{B}{A}\right|^2$$

Let's examine for $l >> a$ $\quad (ka << 1)$

Flanged

$$\frac{\tilde{Z}_{mL}}{r_0 cS} = \frac{1}{2}(ka)^2 + j\,\frac{8}{3p}ka$$

giving

$$\frac{\tilde{B}}{\tilde{A}} = -\frac{1-\frac{1}{2}(ka)^2 - j\frac{8ka}{3p}}{1+\frac{1}{2}(ka)^2 + j\frac{8ka}{3p}}$$

since $ka << 1$ we can see that immediately $\frac{\tilde{B}}{\tilde{A}} \approx -1$ when we throw out all ka terms. If we keep the terms and substitute in we get for T_Π:

$$T_\Pi = 1 - \left| \frac{1-\frac{1}{2}(ka)^2 - j\frac{8ka}{3p}}{1+\frac{1}{2}(ka)^2 + j\frac{8ka}{3p}} \right|^2 = 1 - \frac{1-\frac{1}{2}(ka)^2 - j\frac{8ka}{3p}}{1+\frac{1}{2}(ka)^2 + j\frac{8ka}{3p}} \times \frac{1-\frac{1}{2}(ka)^2 + j\frac{8ka}{3p}}{1+\frac{1}{2}(ka)^2 - j\frac{8ka}{3p}}$$

$$= \frac{\left[1+\frac{1}{2}(ka)^2\right]^2 + \left(\frac{8ka}{3p}\right)^2}{\left[1+\frac{1}{2}(ka)^2\right]^2 + \left(\frac{8ka}{3p}\right)^2} - \frac{\left[1-\frac{1}{2}(ka)^2\right]^2 + \left(\frac{8ka}{3p}\right)^2}{\left[1+\frac{1}{2}(ka)^2\right]^2 + \left(\frac{8ka}{3p}\right)^2}$$

$$= \frac{2(ka)^2}{\left[1+\frac{1}{2}(ka)^2\right]^2 + \left(\frac{8ka}{3p}\right)^2}$$

$$\approx 2(ka)^2$$

$T_\Pi \approx 2(ka)^2$ is small since $ka << 1$ (also recall that $B/A \approx -1$). What does this mean about waves reflected from the open end of the pipe (flanged)? So, small amounts of energy will be radiated from the pipe when a is small. This coincides with what we saw earlier for small sources as inefficient radiators of acoustic energy.

Unflanged

The impedance for the unflanged is $\frac{\tilde{Z}_{mL}}{r_0 cS} = \frac{1}{4}(ka)^2 + j\,0.6\,ka$ for the open end. So

$$T_\Pi = 1 - \left| \frac{1-\frac{1}{4}(ka)^2 - j\,0.6ka}{1+\frac{1}{4}(ka)^2 + j\,0.6ka} \right|^2 = \frac{(ka)^2}{\left[1+\frac{1}{4}(ka)^2\right]^2 + \left[0.6\,ka\right]^2} \cong (ka)^2$$

This is $\frac{1}{2}$ radiation of flanged pipe.

Why?

Flanged end provides a better impedance match.

Later we will look at horns which provide an even better match (i.e. exponential horn).

There are some interesting behaviors near a resonance frequency. So let's look at the case where $w = w_n + \Delta w$ (the resonance frequency plus some small perturbation of the resonance). Recall that the input impedance is given by

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \frac{\frac{\tilde{Z}_{mL}}{r_0 cS} + j\tan(kL)}{1 + j\frac{\tilde{Z}_{mL}}{r_0 cS} + j\tan(kL)}$$

Looking again at our two cases (flanged and unflanged).

<u>Unflanged</u>

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \frac{\frac{1}{4}(ka)^2 + j\,0.6\,ka + j\tan(kL)}{1 - 0.6\,ka\tan(kL) + j\frac{1}{4}(ka)^2\tan(kL)}$$

at resonance the reactance is zero so

$$\frac{\tilde{Z}_{m0}}{r_0 cS} \cong \frac{1}{4}\left(k_n a\right)^2$$

near resonance

$$\frac{\tilde{Z}_{m0}}{r_0 cS} \approx \frac{1}{4}\left(k_n a\right)^2 + j\tan\Delta kL$$

but $\tan\Delta kL \approx \Delta kL = \frac{\Delta wL}{c}$ so

$$\frac{\tilde{Z}_{m0}}{r_0 cS} \approx \frac{1}{4}\left(k_n a\right)^2 + j\frac{\Delta wL}{c}$$

The reactance is no longer zero (as expected for off resonance behavior)

It is also important to define the quality factor of the radiator. The quality factor (or Q of the resonator) is defined as

$$Q = \frac{w_n}{w_u - w_l}$$

where w_u and w_l are the two angular frequencies, above and below resonance, respectively, at which the average power has dropped to one-half its resonance value. The Q factor is a measure of how the system responds to near resonance excitation. For the open ended unflanged pipe the half-power points (frequencies) are defined from

$$w_{u,1} = w_n \pm \frac{1}{4}\left(k_n a\right)^2 \frac{c}{L}.$$

This gives a Q of

$$Q_n = \frac{w_n}{w_u - w_1} = \frac{2}{np}\frac{L}{a}\frac{L+0.6a}{a}$$

it is important to note that the Q at the nth resonance varies with $1/n$. So what does this mean for larger resonances (larger values of Q)?

Power radiated at resonance is given by (recall 7.5.6 for acoustic sources)

$$\Pi = \frac{1}{2}R_{m0}U^2 = \frac{1}{2}R_{m0}\frac{F^2}{Z_{m0}^{\,2}}$$

$$\Pi_n = \frac{2}{(np)^2}\frac{F^2}{r_0 cS}\left(\frac{L+0.6a}{a}\right)^2$$

The radiated power is inversely proportional to n^2. So what happens at higher resonances? Why use R_{m0} and Z_{m0} instead of R_{mL} and Z_{mL}?

No loss in pipe $\Rightarrow$ power input to pipe is radiated.

**************************** **Example 10.2** ****************************

For an unflanged pipe let L = 1 m, a = 0.01 m and 0.1 m, then

$$f_n = \frac{n}{2}\frac{c}{L+0.6a}$$

$$Q_n = \frac{w_n}{w_u - w_l} = \frac{2}{np}\frac{L}{a}\frac{L+0.6a}{a}$$

$$\Delta f_n = \frac{f_n}{Q_n}$$

$$\Pi_n = \frac{2}{(2p)^2}\frac{F^2}{r_0 cS}\left(\frac{L+0.6a}{a}\right)^2$$

Parameter	a = 0.01 m	a = 0.1 m
f_1	170.5 Hz	161.8 Hz
Q_1	6404	67.48
Df_1	0.0266 Hz	2.4 Hz
$\Pi_1 r_0 cS / F^2$	512.7	5.69

Why is power greater for smaller radius when transmission is less for smaller radius?

The incident power is much greater because F was constant for this calculation, as opposed to U being constant!

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(10.4) Standing Wave Patterns

Similar to case examined previously for the reflection of waves from an interface between two fluids at normal incidence.

So let's choose $\tilde{A} = A$ and $\tilde{B} = Be^{j\phi}$ where A and B are real and positive. We can do this because we are concerned mainly with the ratio of $\frac{\tilde{B}}{\tilde{A}}$ in the impedance. So,

$$\frac{\tilde{Z}_{mL}}{\rho_0 cS} = \frac{1+\frac{A}{B}e^{j\phi}}{1-\frac{B}{A}e^{j\phi}}$$

We can then define the standing wave ratio in the pipe as

$$\text{SWR} = \frac{A+B}{A-B}$$

which is also defined as the pressure at the node divided by the pressure at the antinode. Likewise the magnitude of the pressure reflection coefficient in the pipe is

$$\frac{B}{A} = \frac{\text{SWR}-1}{\text{SWR}+1}$$

We can determine $\tilde{Z}_{mL}$ from (1) SWR and (2) location of 1st node (x_1)

$$\text{SWR} \Rightarrow \frac{B}{A}$$

And the phase angle can be determined from the distance of the first node from the end at $x = L$:

$$\phi = 2k\left(L - x_1\right) - \pi \qquad \text{(A good HW question or test question?)}$$

****************************** Example 10.3 ******************************

The standing wave ratio in some pipe with a termination giving an unknown impedance is measured to be SWR = 2 and the first node at the driven frequency is 3/8 of a wavelength from the end. Determine the impedance at $x = L$?

ANSWER:

$$L - x_1 = \frac{3\lambda}{8} \text{ so } \phi = 2k\left(L - x_1\right) - \pi = 2k\frac{3\lambda}{8} - \pi = \frac{4\pi}{\lambda}\frac{3\lambda}{8} - \pi = \frac{1}{2}\pi \,.$$

$$\frac{B}{A} = \frac{\text{SWR}-1}{\text{SWR}+1} = \frac{2-1}{2+1} = \frac{1}{3}.$$

Therefore,

$$\frac{\tilde{Z}_{mL}}{\rho_0 cS} = \frac{1+\frac{B}{A}e^{j\phi}}{1-\frac{B}{A}e^{j\phi}} = \frac{1+\frac{1}{3}e^{j\frac{\pi}{2}}}{1-\frac{1}{3}e^{j\frac{\pi}{2}}} = 0.8 + j0.6$$

In acoustics we can measure the impedance of a terminating material. We simple put a plug of some material we wish to measure the impedance of and determine the impedance from the position of the nodes and standing wave ratio.

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(10.5) Absorption of Sound in Pipes

Recall that for the lossless case of a pipe with a rigid cap we had

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \left.\frac{\dfrac{\tilde{Z}_{mL}}{r_0 cS} + j\tan(kL)}{1 + j\dfrac{\tilde{Z}_{mL}}{r_0 cS}\tan(kL)}\right|_{\substack{Z_{mL}\to\infty \\ \text{for rigid cap}}}$$

$$\Rightarrow \frac{1}{j\tan(kL)}$$

With loss we simply say that our wavenumber is complex so that $\tilde{k} = k - ja$. Substituting into the lossless solutions obtained previously for pipe yields:

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = \left.\frac{\dfrac{\tilde{Z}_{mL}}{r_0 cS} + j\tan(\tilde{k}L)}{1 + j\dfrac{\tilde{Z}_{mL}}{r_0 cS}\tan(\tilde{k}L)}\right|_{\substack{Z_{mL}\to\infty \\ \text{for rigid cap}}}$$

$$\Rightarrow \frac{1}{j\tan(\tilde{k}L)}$$

The input impedance is then given by

$$\tilde{Z}_{m0} = -j r_0 cS\cot(\tilde{k}L)$$

or rearranging and using some trig identities and cosh and sinh identities gives:

$$\frac{\tilde{Z}_{m0}}{r_0 cS} = -j\frac{1 + j\dfrac{a}{k}}{1 + \left(\dfrac{a}{k}\right)^2}\frac{\cos(kL)\sin(kL) + j\sinh(aL)\cosh(aL)}{\sin^2(kL)\cosh^2(aL) + \cos^2(kL)\sinh^2(aL)}$$

Now typically $\dfrac{a}{k} \ll 1$ and $aL \ll 1$ so

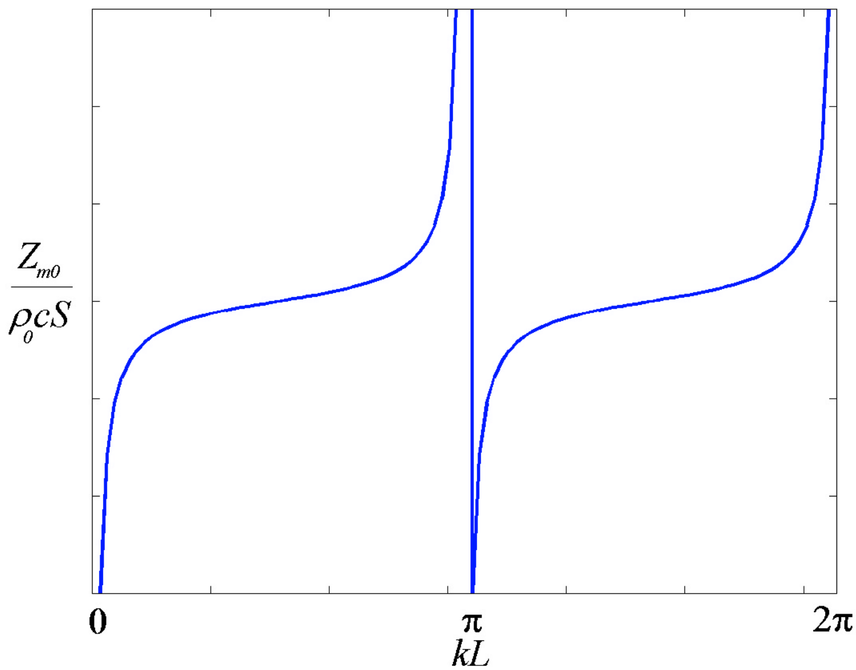
$$\frac{\tilde{Z}_{m0}}{r_0 cS} \approx \frac{aL - j\cos(kL)\sin(kL)}{\sin^2(kL) + (aL)^2\cos^2(kL)}$$

$$\underset{a=0}{\Rightarrow} -j\cot(kL)$$

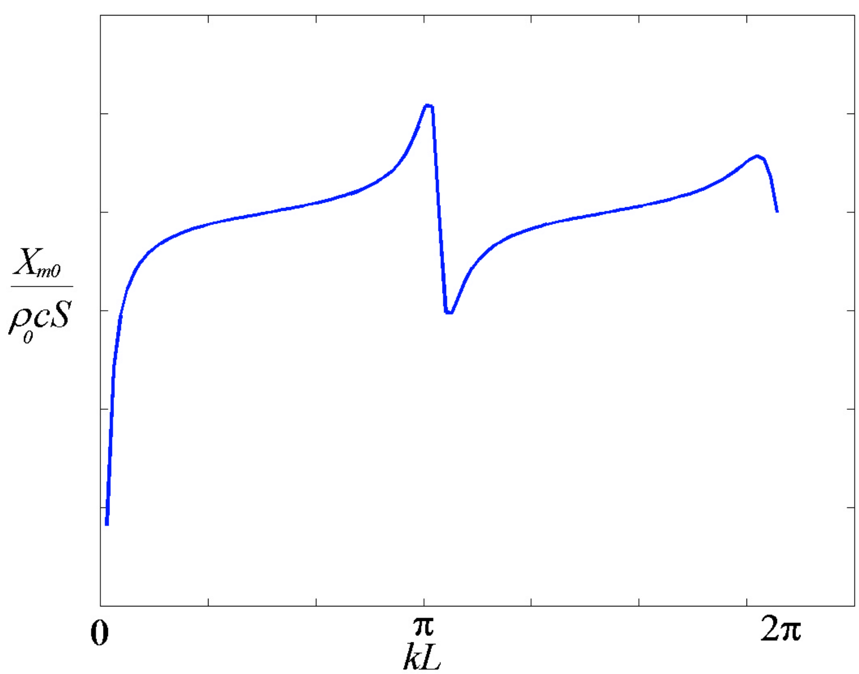
$\tilde{Z}_{m0} = R_{m0} + j\,X_{m0}$

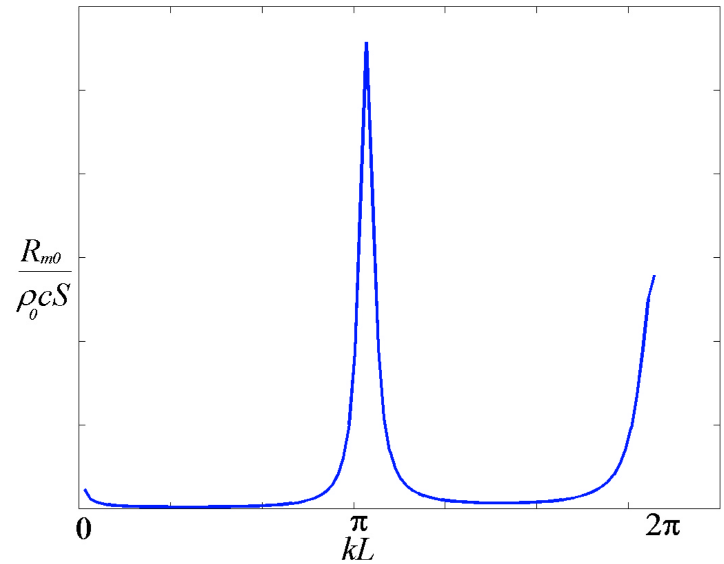
R_{m0} is no longer 0

Pipe with $\alpha = 0$



Pipe with $\alpha \neq 0$





The power dissipated by the pipe is given by

$$\Pi = \frac{1}{2}F^2\frac{R_{m0}}{Z_{m0}^2} = \frac{1}{2}\frac{F^2}{r_0 cS}a L\frac{\sin^2(kL)+\left(a L\right)^2\cos^2(kL)}{\left(a L\right)^2+\cos^2(kL)\sin^2(kL)}$$

At <u>Resonance</u> $\quad \cos(kL)=0 \qquad kL=(2n-1)\frac{p}{2}$ giving

$$\Pi_r = \frac{1}{2}\frac{F^2}{r_0 cS}\frac{1}{a L} \qquad \begin{pmatrix}\text{remember}\\ a L << 1\end{pmatrix}$$

At <u>Antiresonance</u> $\quad \sin(kL)=0 \qquad kL=np$ giving

$$\Pi_a = \frac{1}{2}\frac{F^2}{r_0 cS}a L$$

Near a resonance frequency we again define

$$w = w_n + \Delta w$$

and

$$kL = \frac{w_n+\Delta w}{c}L = (2n-1)\frac{p}{2}+\Delta w\frac{L}{c}$$

giving

$$\frac{\tilde{Z}_{m0}}{r_0 cS} \approx a L + j\Delta w\frac{L}{c}$$

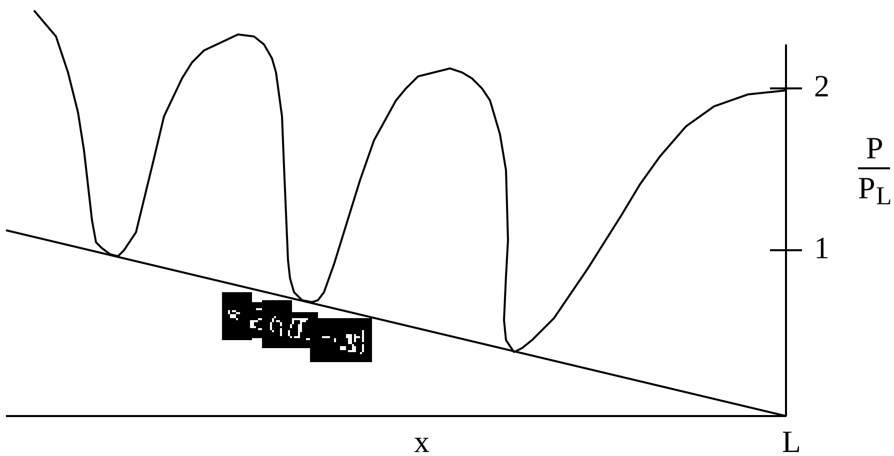
We have a half power point when

$$\Delta w \frac{L}{c} = a\, L \quad \text{or} \quad \Delta w = ca \;.$$

So our Q for the lossy pipe is given by

$$\boxed{Q = \frac{w_n}{2a\, c} = \frac{1}{2}\frac{1}{a/k_n}}$$

Standing wave in pipe (rigid cap)



where P_L is defined as the amplitude of reflected wave

Why is this the case? Let’s look at the total acoustic pressure given by

$$P = 2P_L \left\{ \cos^2\left[k(L-x)\right]\cosh^2\left[a\,(L-x)\right] + \sin^2\left[k(L-x)\right]\sinh^2\left[a\,(L-x)\right]\right\}^{1/2}$$

for a lossy pipe. We have Nodes at $k(L-x) = (2n-1)\,p/2 \qquad n = 1,2,3\ldots$
This gives a relative maximum amplitude of

$$\frac{P_{\min}}{P_L} = 2\sinh\left[a\,(L-x)\right] \approx 2a\,(L-x)$$

We also have Antinodes occurring at $k(L-x) = np \qquad n = 0,1,2\ldots$
This gives a relative minimum amplitude of

$$\frac{P_{max}}{P_L} = 2\cosh\left[\alpha(L - x)\right] » 2 + \left[\alpha(L - x)\right]^2$$

We can experimentally measure the magnitude and location of the minima (or maxima) and compute the value for a !!!

However, a for this case includes losses at the boundaries of the pipe in addition to the absorption in the fluid.