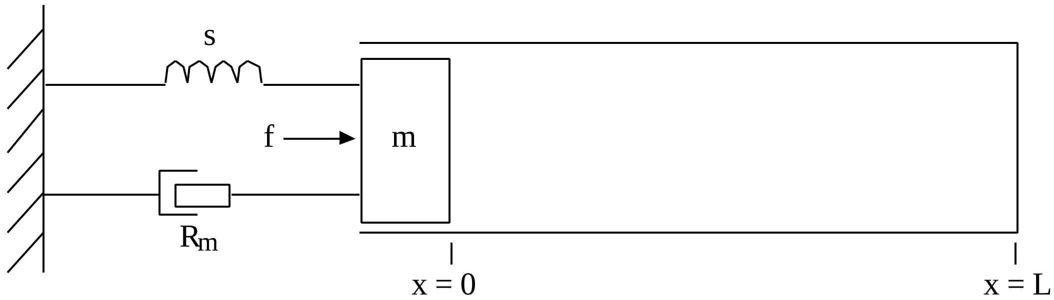
(10.6) Combined Driver-Pipe System



$f = Fe^{jwt}$

for combined system let's look at the differential equation of this diagram in terms of displacement of the mass in front of the driver. According to Newton's 2nd Law, $F = ma = m\dfrac{\partial^2 x}{\partial t^2}$ and

$$m\frac{d^2\tilde{x}}{dt^2} = -R_m\frac{d\tilde{x}}{dt} - s\tilde{x} - S\tilde{p}(0,t) + \tilde{f}$$

Converting things to the particle velocity

$$\tilde{f} = \tilde{u}(0,t)\left[R_m + j\left(wm - \frac{s}{w}\right) + \frac{S\tilde{p}(0,t)}{\tilde{u}(0,t)}\right]$$

Notice that the first two terms in the brackets are just the impedance of the driver $\tilde{Z}_{md} = R_m + j\left(wm - \dfrac{s}{w}\right)$ and the last term is the input mechanical impedance of the pipe. This means that the total impedance of the system

$$\tilde{Z}_m = \tilde{Z}_{md} + \tilde{Z}_{m0} = \frac{\tilde{f}}{\tilde{u}(0,t)}$$

For resonance of system (pipe and driver) occurs when the reactance goes to zero

$$\mathrm{Im}\left(\tilde{Z}_m\right) = \mathrm{Im}\left(\tilde{Z}_{md} + \tilde{Z}_{m0}\right) = 0$$

For rigid termination at $x = L$, the reactance at zero means that

$$\mathrm{Im}\left(\tilde{Z}_m\right) = 0 = wm - \frac{s}{w} - \frac{S r_0 c\cos(kL)\sin(kL)}{\sin^2(kL) + (aL)^2\cos^2(kL)}$$

Rearranging gives

$$\frac{\cos(kL)\sin(kL)}{\sin^2(kL) + (aL)^2\cos^2(kL)} = \frac{m}{\underbrace{S r_0 L}_{a}}kL - \underbrace{sL/S r_0 c^2}_{b}\frac{1}{kL}$$

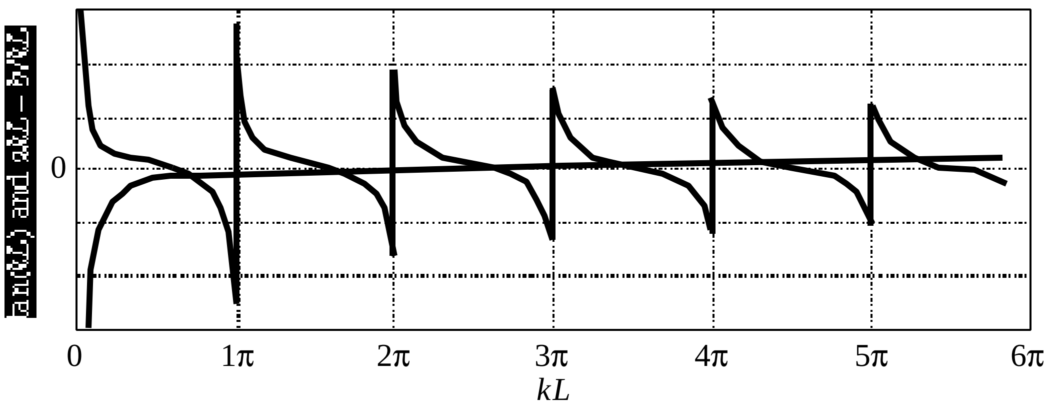
a: ratio of mass of piston to mass of fluid in tube

b: ratio of stiffness of the piston to stiffness of the fluid

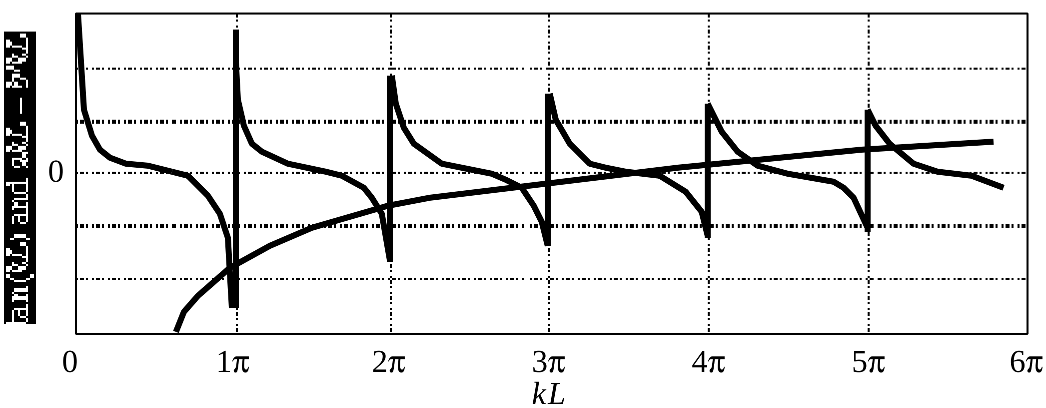
The above equation defines where the resonances occur. However, it is not easy to solve for the resonances, i.e. find the frequency values where the above equation holds. Typically, the resonant frequencies are found by solving graphically..

****************************** Example 10.4 ******************************

Let's compare the behavior between a pipe with driver light and flexible relative to surrounding fluid with a driver that is stiff and heavy relative to the surrounding fluid.
Using the graphical technique:



(a)



(b)

Figure 10.6.2 Graphical solution for the resonance frequencies of a rigidly terminated pipe of 1 m length and l cm radius driven by: (*a*) a light, flexible driver with a = 0.04 and b = 2.57; and (b) a heavy, stiff driver with a = 0.25 and b = 32.

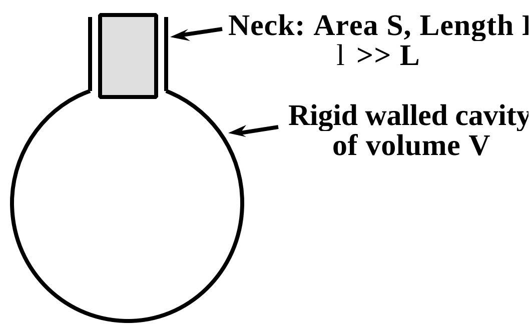
****

(10.7) The Long Wavelength Limit

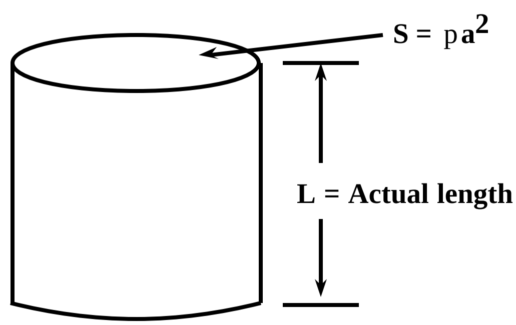
So far we have looked at l large compared to diameter of pipe but not necessarily long compared to the length of the pipe.

We can also have l large compared to all dimensions → behaves as <u>lumped acoustic element</u> with time dependent parameters but nearly the same throughout the element. This means the acoustic device behaves somewhat like a simple harmonic oscillator. Example: Helmholtz Resonator

(10.8) The Helmholtz Resonator



Looking more closely at the neck:



Relationship to simple harmonic oscillator:

For $l >> L$	<u>Mass</u>	⇒	fluid in neck
For $l >> V^{1/3}$	<u>Stiffness</u>	⇒	compressibility of fluid in resonator
For $l >> S^{1/2}$	<u>Resistance</u>	⇒	loss of energy associated with radiation from opening (can be viscous losses for a small diameter neck)

Mass

$$m = r_0 S L'$$

where

$$L' = L + 2(0.85a) = L + 1.7a$$ inner and outer ends flanged

$$L' = L + (0.85 + 0.6)a = L + 1.4a$$ inner end flanged outer unflanged

Note: if $L = 0$ then $L' = 1.6a$

Stiffness

$$\Delta V = -Sx$$

$$\text{stiffness} = s = \frac{f}{x} = \frac{pS}{-\Delta V/S} = \frac{pS^2}{-\Delta V}$$

$$\frac{-\Delta V}{V} = \frac{\Delta r}{r} \Rightarrow -\Delta V = V\frac{\Delta r}{r} = V\frac{p}{r_0 c^2}$$

$$s = \frac{pS^2}{V\,P/r_0 c^2} = \frac{r_0 c^2 S^2}{V}$$

Resistance

Radiation as from open-ended pipe for $l >> a$

$$R_r = r_0 c\frac{k^2 S^2}{2p} \qquad \text{(flanged)}$$

$$R_r = r_0 c\frac{k^2 S^2}{4p} \qquad \text{(unflanged)}$$

Add the driving force

$\vec{f} = S\,P\,e^{jwt}$ P – pressure amplitude for force driving inward

This gives the applicable differential equation is

$$m\frac{d^2x}{dt^2} + R_r\frac{dx}{dt} + sx = S\,P\,e^{jwt}$$

Substituting in our values for mass, spring stiffness and resistance yields (flanged)

$$\left(r_o SL'\right)\frac{d^2x}{dt^2} + \left(\frac{r_o c k^2 S^2}{2p}\right)\frac{dx}{dt} + \left(\frac{r_o c^2 S^2}{V}\right)x = SPe^{jwt}$$

Let's define a quantity of volume displacement where $X = \int\int \vec{x}\cdot d\vec{S} = x\,S$ (for normal surface movement, very reminiscent of a piston moving). The volume velocity is $U = \frac{\partial X}{\partial t} = S\frac{\partial x}{\partial t}$.

Substituting $X = xS \Rightarrow x = \frac{X}{S}$ into $\left(r_o SL'\right)\frac{d^2x}{dt^2} + \left(\frac{r_o ck^2 S^2}{2p}\right)\frac{dx}{dt} + \left(\frac{r_o c^2 S^2}{V}\right)x = SPe^{jwt}$ yields:

$$\left(r_o SL'\right)\frac{1}{S}\frac{d^2X}{dt^2} + \left(\frac{r_o ck^2 S^2}{2p}\right)\frac{1}{S}\frac{dX}{dt} + \left(\frac{r_o c^2 S^2}{V}\right)\frac{1}{S}X = SPe^{jwt}$$

$$\left(r_o SL'\right)\frac{d^2X}{dt^2} + \left(\frac{r_o ck^2 S^2}{2p}\right)\frac{dX}{dt} + \left(\frac{r_o c^2 S^2}{V}\right)X = S^2Pe^{jwt}$$

$$M\frac{d^2X}{dt^2} + R\frac{dX}{dt} + \frac{1}{C}X = S^2Pe^{jwt}$$

where

$$M = r_o SL', \quad R = \frac{r_o ck^2 S^2}{2p} \quad \text{and} \quad C = \frac{V}{r_o c^2 S^2}.$$

Assume $X = X_o e^{jwt}$ and $U = U_o e^{jwt}$, then $M\frac{d^2X}{dt^2} + R\frac{dX}{dt} + \frac{1}{C}X = S^2Pe^{jwt}$ gives:

$$-w^2 MX_o + jwRX_o + \frac{1}{C}X_o = S^2P$$

$$\left(\frac{1}{C} - w^2M + jwR\right)X_o = S^2P$$

Therefore,

$$Z = \frac{Pe^{jwt}}{U} = \frac{Pe^{jwt}}{jwX_o e^{jwt}} = \frac{P}{jwX_o} = \frac{\frac{1}{S^2}\left(\frac{1}{C} - w^2M + jwR\right)X_o}{jwX_o}$$

$$= \frac{R}{S^2} + jw\frac{M}{S^2} - j\frac{1}{wS^2C} = \frac{R}{S^2} + j\left(w\frac{M}{S^2} - \frac{1}{wS^2C}\right)$$

$$= R' + j\left(wM' - \frac{1}{wC'}\right)$$

Z is the Acoustic Impedance of a Helmholtz Resonator ($\mathrm{N \cdot s / m^5}$) where

$$R' = \frac{R}{S^2} = \frac{r_o ck^2}{2p}, \qquad M' = \frac{M}{S^2} = \frac{r_o L'}{S} \qquad \text{and} \qquad C' = S^2C = \frac{V}{r_o c^2}$$

Note that Z is the acoustic impedance on a surface

$$R' = \frac{r_o ck^2}{2p}\ \left(\frac{\mathrm{N \cdot s}}{\mathrm{m^5}}\right), \quad wM' = \frac{wr_o L'}{S}\ \left(\frac{\mathrm{N \cdot s}}{\mathrm{m^5}}\right) \quad \text{and} \quad \frac{1}{wC'} = \frac{r_o c^2}{wV}\ \left(\frac{\mathrm{N \cdot s}}{\mathrm{m^5}}\right)$$

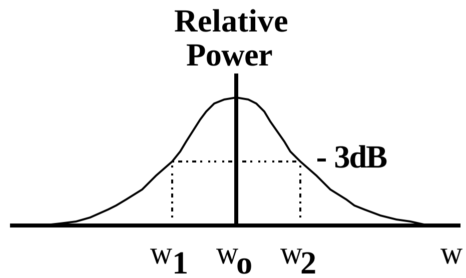
so,

$$Z = R' + j\left(wM' - \frac{1}{wC'}\right) = \frac{r_o ck^2}{2p} + j\left(\frac{wr_o L'}{S} - \frac{r_o c^2}{wV}\right)$$

At resonance; $\text{Im}\{Z\} = 0$ (reactance is zero) and

$$\frac{w_o r_o L'}{S} - \frac{r_o c^2}{w_o V} = 0 \quad \Rightarrow \quad w_o = \sqrt{\frac{c^2 S}{L'V}} = c\sqrt{\frac{S}{L'V}} \quad \text{(Eq. 10.8.8)}$$

At w_o, cavity shape is not important; volume is most important, that is, depends upon motion of gas at plug. Overtones ($2w_o$, $3w_o$,...) depend upon cavity shape.
Graphically (see Sec 1.10):



So we can define the quality factor for the Helmholtz resonator:

$$Q = \frac{w_o}{w_2 - w_1} = \frac{w_o}{\Delta w} \quad \text{or} \quad Q = \frac{f_o}{\Delta f} \quad \text{(Definition, Eq. 1.10.3)}$$

For Helmholtz Resonator, $Q = \frac{w_o M'}{R'} = 2p\sqrt{V\left(\frac{L'}{S}\right)^3}$ (flanged)

$$= 4p\sqrt{V\left(\frac{L'}{S}\right)^3} \quad \text{(unflanged)}$$

***************************** Example 10.5 *****************************

(A) Derive Eq. (10.8.12) in Kinsler et al., i.e., $Q = 2p\sqrt{V\left(\frac{L'}{S}\right)^3}$, for a flanged source, from the facts that (1) $m = r_o SL'$, (2) $w_o = c\sqrt{\frac{S}{L'V}}$, (3) $R_r = r_o c\frac{k^2 S^2}{2p}$, and (4) $Q = \frac{w_o m}{R_r}$.
(B) How does this derivation change if the source is unflanged?

ANSWER:

(A) Starting with (4), $Q = \frac{w_o m}{R_r} = \dfrac{c\sqrt{\frac{S}{L'V}}\left(r_o SL'\right)}{r_o c\frac{k^2 S^2}{2p}} = 2p\sqrt{V\left(\frac{L'}{S}\right)^3}$

(B) In the case of the unflanged source, we have the relation $R_r = \rho_o c \dfrac{k^2 S^2}{4\pi}$, instead of $R_r = \rho_o c \dfrac{k^2 S^2}{2\pi}$. Keeping track of the 4π, we have.... $Q = 4\pi\sqrt{V\left(\dfrac{L'}{S}\right)^3}$. That is, the quality factor of the resonance frequency goes as the *inverse* of the radiation impedance.

****

<u>Pressure amplification at resonance</u>

$$\frac{P_c}{P} = \frac{\text{pressure in cavity}}{\text{pressure of incident sound wave}} \qquad (P \text{ is the driving pressure})$$

$$P_c = \frac{\rho_0 c^2 S}{V} x$$

$$\frac{dx}{dt} = \frac{F}{\tilde{Z}_m} = \frac{PS}{R_r}$$

$$\therefore x = \frac{PS}{\omega_0 R_r}$$

$$\frac{P_c}{P} = \frac{\rho_0 c^2 S}{V}\frac{PS}{\omega_0}\frac{1}{\rho_0 c \dfrac{k^2 S^2}{2\pi}}$$

$$= 2\pi\sqrt{V\left(\frac{L'}{S}\right)^3} = Q$$

$$\boxed{\frac{P_c}{P} = Q}$$

the pressure gain is equal to the quality factor Q

*************************** **Example 10.6** ***************************

If the diameter of a spherical Helmholtz resonator is 20 cm = 0.2 m, then the volume is

$$V = \frac{4}{3}\pi r^3 = 0.00419\, m^3$$

$$L = 1\,\text{cm} = 0.01\,\text{m}$$
$$a = 2\ \text{cm} = 0.02\ \text{m}$$
$$S = \pi a^2 = 0.001257\ \text{m}^2$$

unflanged $\Rightarrow$ $L' = L + 1.4\,a = 0.01 + 1.4\,(0.02) = 0.038$ m

$$m = r_0 S L' = 1.21(0.001257)(0.038) = 5.78\times10^{-5}\,\text{kg}$$

$$s = r_0 c^2 \frac{S^2}{V} = 1.21(343)^2 \frac{(0.001257)^2}{0.00419} = 53.68\,\frac{\text{N•s}}{\text{m}}$$

$$w_0 = c\sqrt{\frac{S}{L'V}} = 199\ \frac{\text{krad}}{\text{s}}$$

$$f_0 = 31.7\ \text{kHz}$$

$$Q = \text{gain} = 2p\sqrt{V\left(\frac{L'}{S}\right)^3} = 67.6 \text{ at resonance}$$

****

Loudspeakers in a closed cabinet can be modeled as a Helmholtz resonator by taking into account the additional stiffness and resistance of the speaker.

****************************** Example 10.7 ******************************

Coca-Cola®, in 1995, copyrighted a new design of 20 fl.oz. (591 mL) Coke bottles (from the “spider” ads). The neck of these bottles is approximately 3 cm in length, and the diameter of the opening is approximately 2 cm. If you blow into an empty bottle, what would be the approximate resonant frequency?

ANSWER:

Think of the Coke bottle as a Helmholtz resonator. Using the formula $w_o = c\sqrt{\frac{S}{L'V}}$, the problem is trivial except for conversion factors. One mL is one cubic centimeter. (The value for c is taken from Appendix A10, for air at 20°C).

$$w_o = c\sqrt{\frac{S}{L'V}} = c\sqrt{\frac{p\,\frac{d^2}{4}}{L'V}} = 343\text{m/s}\sqrt{\frac{\frac{p}{4}\left(2\times10^{-2}\text{ m}\right)^2}{\left(3\times10^{-2}\text{ m}\right)\left(591\text{mL}\right)\left(\frac{1\text{cm}^3}{1\text{mL}}\right)}} = 1444 \text{ r/s}$$

Thus the resonant frequency that is heard is 230 Hz.

****

(10.9) Acoustic Impedance

<u>Summary of impedances</u>

characteristic impedance $\boxed{r_0 c}$ $\dfrac{\text{Pa}\cdot\text{s}}{\text{m}}$

specific acoustic impedance $\boxed{z = \dfrac{p}{u}}$ $\dfrac{\text{Pa}\cdot\text{s}}{\text{m}}$

mechanical impedance
(radiation impedance) $\boxed{Z_m = \dfrac{f}{u}} = \dfrac{pS}{u} = z\,S$ (for plane waves) Pasm or

mechanical ohm

acoustic impedance at a surface $\boxed{Z = \dfrac{p}{U}} = \dfrac{z}{S}$ $\dfrac{\text{Pa•s}}{\text{m}^3}$ or acoustic ohm

Lumped Acoustic Impedance

$$Z = \frac{p}{U} = R + j\left(wM - \frac{1}{wC}\right)$$ acoustic impedance

$$R = \frac{R_m}{S^2} - \text{acoustic resistance}$$

$$M = \frac{m}{S^2} - \text{acoustic inertance}$$

$$C = \frac{S^2}{s} - \text{acoustic compliance}$$

Helmholtz resonator is an example (lumped impedance). There are analogies to simple harmonic oscillator and circuits.

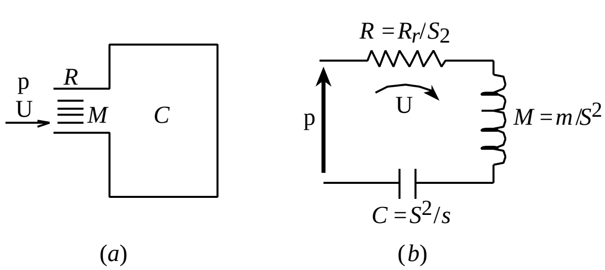
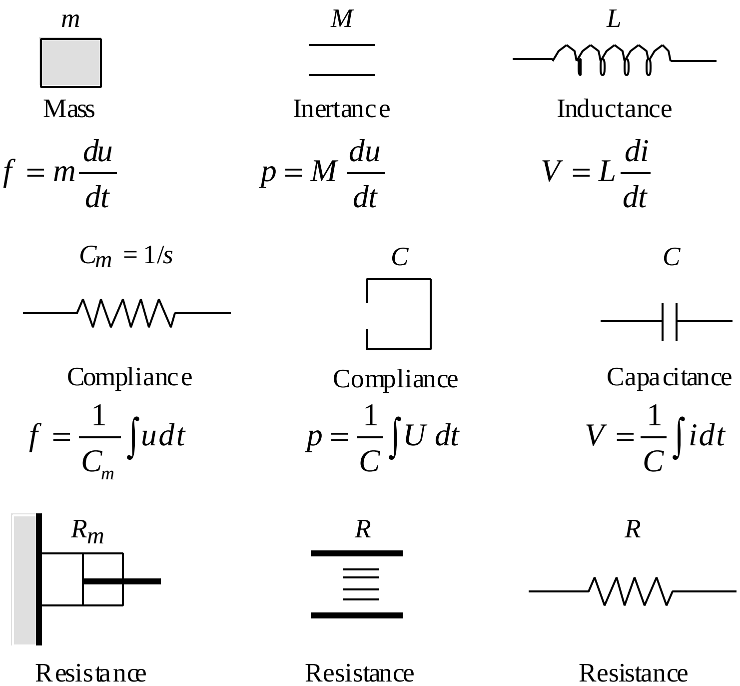


Figure 10.9.1 Schematic representation of a Helmholtz resonator. (*a*) Acoustic analog with inertance *M*, resistance *R*, and compliance *C*. The oscillator is driven by an incident pressure **p** and the air in the neck moves with volume velocity **U**. (*b*) Electrical analog with inductance *M*, resistance *R*, and capacitance *C*. This series circuit is driven by voltage **p** and carries current **U**.



MECHANICAL	ACOUSTICAL	ELECTRICAL
Mass: $f = m\frac{du}{dt}$	Inertance: $p = M\frac{du}{dt}$	Inductance: $V = L\frac{di}{dt}$
Compliance: $f = \frac{1}{C_m}\int u\,dt$	Compliance: $p = \frac{1}{C}\int U\,dt$	Capacitance: $V = \frac{1}{C}\int i\,dt$
Resistance: $f = R_m u$	Resistance: $p = R\,U$	Resistance: $V = R_i$

Acoustic, electrical and mechanical analogs.

Dimensions (at least one) not small compared to l

Acoustics (Ex. pipe)	Electrical (Ex. transmission line)
$Z = \frac{p}{U} = \frac{p}{uS} = \frac{r_0 c}{S} = \sqrt{\frac{M_1}{C_1}}$	$Z_0 = \sqrt{\frac{L}{C}}$
for plane waves	characteristic impedance
$M_1 = \frac{m_1}{S^2} = \frac{r_0}{S}$	L – inductance/unit length
inertance/unit length	
$C_1 = \frac{S}{r_0 c^2}$	C – capacitance/unit length
compliance/unit length	