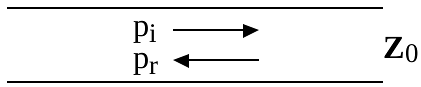
(10.10) Reflection and Transmission of Waves in a Pipe

General case: Acoustic impedance discontinuity at $x = 0$



For a pipe,

$$Z = \frac{r_0 c}{S} \quad \text{(for a traveling wave)}$$

and

$$p_i = A e^{j(wt - kx)}$$

$$p_r = B e^{j(wt + kx)}$$

for the standing wave case

$$\tilde{Z} = \frac{p_i + p_r}{U_i + U_r} = \frac{r_0 c}{S} \frac{\tilde{A} e^{-jkx} + \tilde{B} e^{jkx}}{\tilde{A} e^{-jkx} - \tilde{B} e^{jkx}}$$

$$\tilde{Z} = Z_0 = \frac{r_0 c}{S} \frac{\tilde{A} + \tilde{B}}{\tilde{A} - \tilde{B}} \quad \text{@ interface (x=0)}$$

$$\frac{\tilde{B}}{\tilde{A}} = \frac{Z_0 - \frac{r_0 c}{S}}{Z_0 + \frac{r_0 c}{S}}$$

For $\tilde{Z}_0 = R_0 + jX_0$ the power reflection and transmission coefficients are

$$R_p = \left| \frac{B}{A} \right|^2 = \frac{\left(R_0 - \frac{r_0 c}{S} \right)^2 + X_0^2}{\left(R_0 + \frac{r_0 c}{S} \right)^2 + X_0^2}$$

$$T_p = \frac{4 R_0 \frac{r_0 c}{S}}{\left(R_0 + \frac{r_0 c}{S} \right)^2 + X_0^2}$$

Change in acoustic impedance can be associated with a change in fluid or a change in cross sectional area of the pipe without a change of medium (this can also come from a branch or a side port in a pipe).

Case 1: Simple change in cross-sectional area

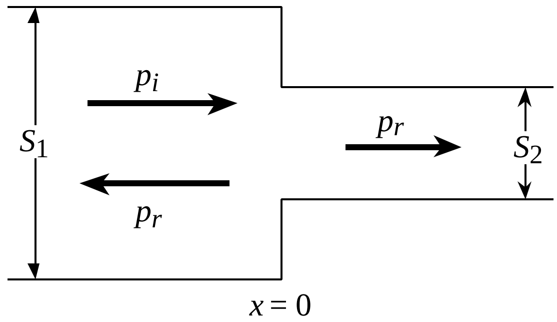


Figure 10.10.1 Transmission and reflection of a plane wave in the vicinity of a junction between two pipes where the cross-sectional area changes from S_1 to S_2.

$$R_\Pi = \frac{(S_1 - S_2)^2}{(S_1 + S_2)^2}$$

$$T_\Pi = \frac{4S_1 S_2}{(S_1 + S_2)^2}$$

Case 2: Branch in pipe

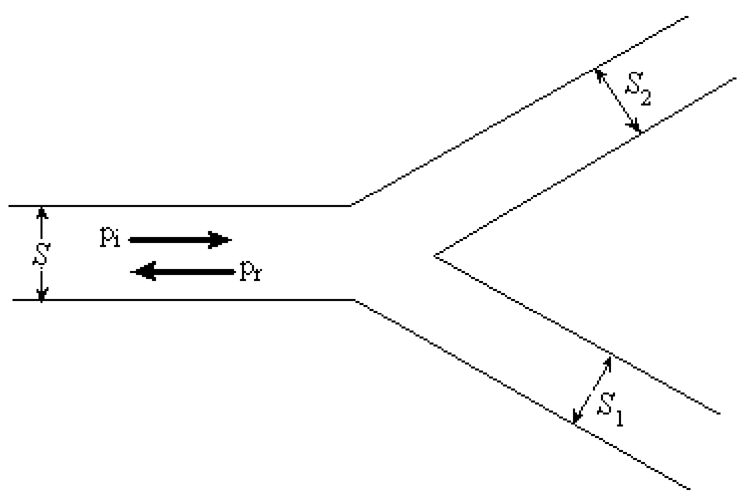


Figure 10.10.2 Conditions in the vicinity of a branch. The branches have cross-sectional areas S_1 and S_2 and input acoustic impedances Z_1 and Z_2.

$$p_i = P_i e^{jwt} \qquad p_r = P_r e^{jwt}$$

$$p_1 = Z_1 U_1 e^{jwt} \qquad p_2 = Z_2 U_2 e^{jwt}$$

The boundary conditions at $x = 0$ (location of branch)

$p_i + p_r = p_1 = p_2 \quad \Leftrightarrow$ cont. of pressure

$U_i + U_r = U_1 + U_2 \quad \Leftrightarrow$ cont. of volume velocity

Application of these boundary conditions give

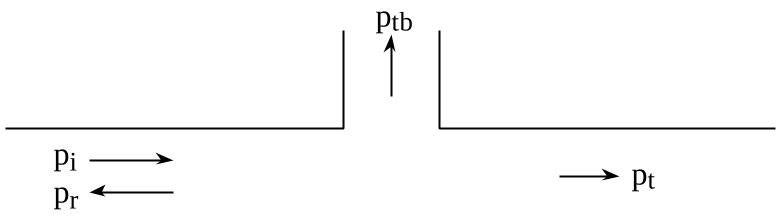
$$\frac{1}{Z_0} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

admittances add to give admittance $@ x = 0$

$$Z_0 = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

If two impedances in parallel

Case 3: Side branch



$$Z_b = R_b + j X_b$$

$$R_\Pi = \frac{\left(\frac{r_0 c}{2S}\right)^2}{\left(\frac{r_0 c}{2S} + R_b\right)^2 + X_b^2}$$

$$T_\Pi = \frac{R_b^2 + X_b^2}{\left(\frac{r_0 c}{2S} + R_b\right)^2 + X_b^2}$$

$$T_{\Pi b} = 1 - R_\Pi - T_\Pi$$

$$= \frac{\frac{r_0 c}{S} R_b}{\left(\frac{r_0 c}{2S} + R_b\right)^2 + X_b^2}$$

Special Cases

$R_b = X_b = 0$

$T_\Pi = 0$

$R_\Pi = 1$

$T_{\Pi b} = 0$

R_b finite but not 0	Some power trans. into branch	
$R_b = X_b = \infty$	$T_\Pi = 1$ $R_\Pi = 0$ $T_{\Pi b} = 0$	As though branch is not there.

$R_b = 0 \qquad X_b \neq 0$

No power into branch.
(Ex. branch is pipe with rigid termination)
X_b value does affect power trans. and reflected.

(10.11) Acoustic Filters

(a) Low-Pass Filters
Insert length L of pipe with different, S_1, cross sectional area.

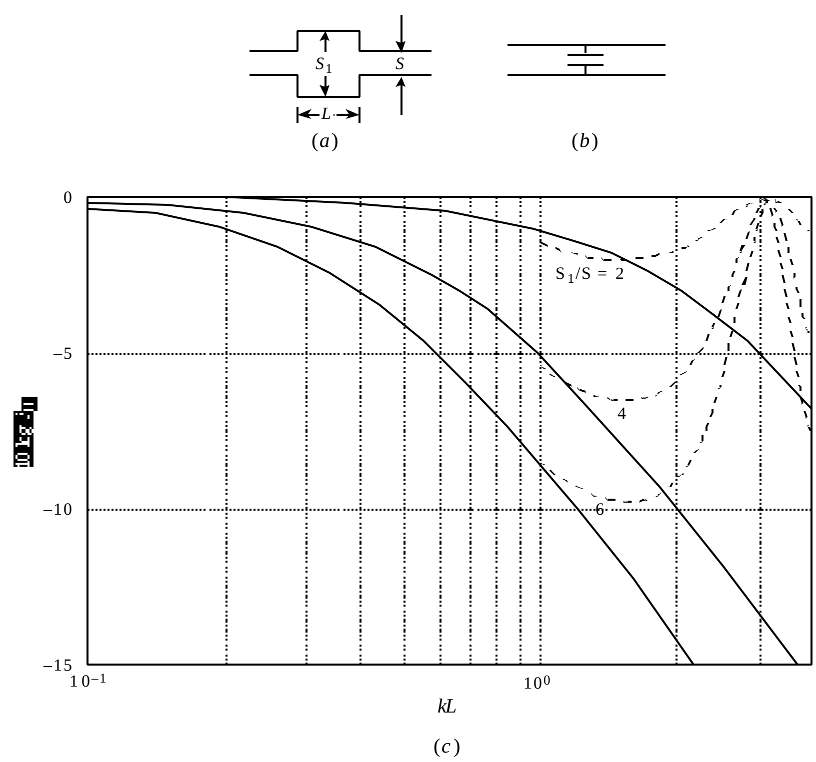


Figure 10.11.1 A simple low-pass acoustic filter consists of an enlarged section of cross-sectional area S_1 and length L in a pipe of cross-sectional area S.
(*a*) Schematic. (*b*) Analogous electric filter. (*c*) Attenuation for several values of S_1/S. Solid lines are from (10.11.2) for $kL << 1$. Dashed lines are from Problem 10.11.6 for $kL >> 1$.

At low frequency kL << 1. The extra length of pipe acts like a side branch with

compliance $C = \frac{\mathrm{V}}{r_0 \mathrm{c}^2}$ and $V = (S_1 - S)L$ giving an acoustic impedance for the chamber of

$$Z_b \approx 0 - j\frac{r_0 c^2}{w(S_1 - S)L}$$

$$T_\Pi \approx \frac{1}{1 + \left(\frac{S_1 - S}{2S} kL\right)^2}$$

True ONLY for kL << 1

Actually, it acts similar to the transmission through a layer examined earlier in chapter 6

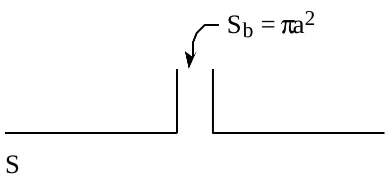
$$T_\Pi = \frac{4}{4 + \left(\frac{S_1}{S} - \frac{S}{S_1}\right)^2 \sin^2(kL)}$$

for small kL and for the middle layer being analogous to characteristic impedance of z_2.

Like the layer transmission case, at higher frequency, the pipe will have frequencies for complete transmission $\left(L = m\frac{l}{2}\right)$ and others where there is a minimum.

Used in mufflers, gun silencers, sound-absorbing plenum chambers in ventilation systems.

(b) High-Pass Filters



Short unflanged branch of radius a and length L. $L¢ = L + 1.4\,a$

$$Z_b = \frac{r_0 c k^2}{4p} + jw\left(\frac{r_0 L'}{p a^2}\right) = R_b + jX_b$$

giving

$$T_\Pi = \frac{1}{1 + \left(\frac{p a^2}{2SL'k}\right)^2}$$

Most power is reflected at low frequencies. This implies that low frequencies are not transmitted down the pipe and radiated out (like a wind instrument).

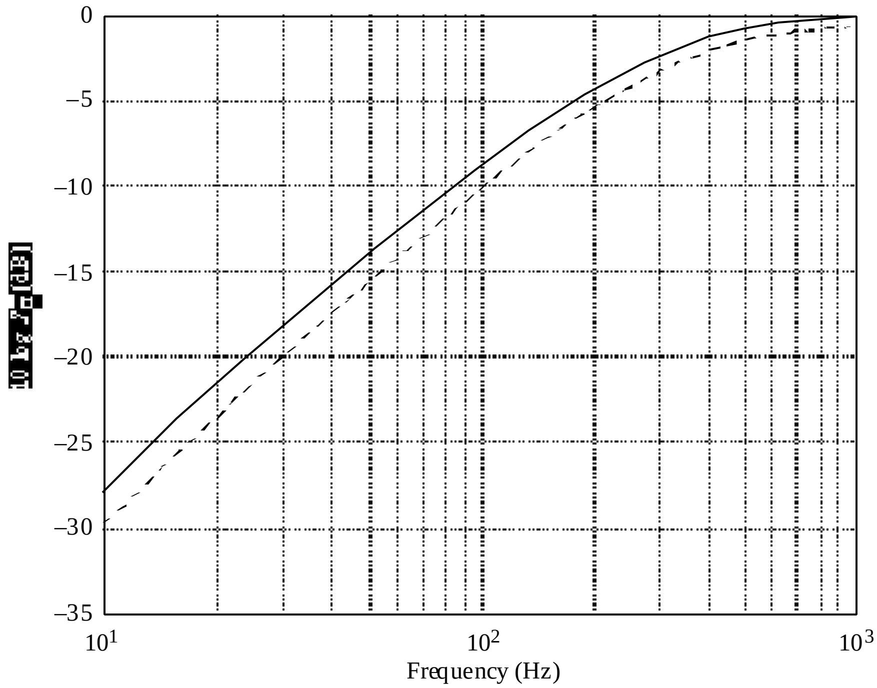
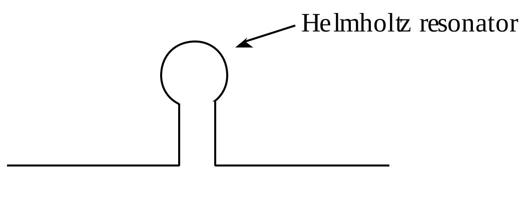


Figure 10.11.3 Attenuation for a high-pass filter in a pipe of cross-sectional area S = 28 cm^2. The side branch has a 1.55 cm radius. The solid curve is for L = 0.6 cm and the dashed curve for L = 0.

(c) Band-Stop Filters



$$Z_b = 0 + j r_0 \left(\frac{wL'}{S_b} - \frac{c^2}{wV} \right)$$

$$T_\Pi = \frac{1}{1 + \left(\dfrac{c/2S}{\dfrac{wL'}{S_b} - \dfrac{c^2}{wV}} \right)^2}$$

$$T_\Pi = 0 \text{ for } w = c\sqrt{\frac{S_b}{L'V}} =$$ resonance freq. of Helmholtz resonator

All energy is reflected $\rightarrow$ none is lost in resonator.

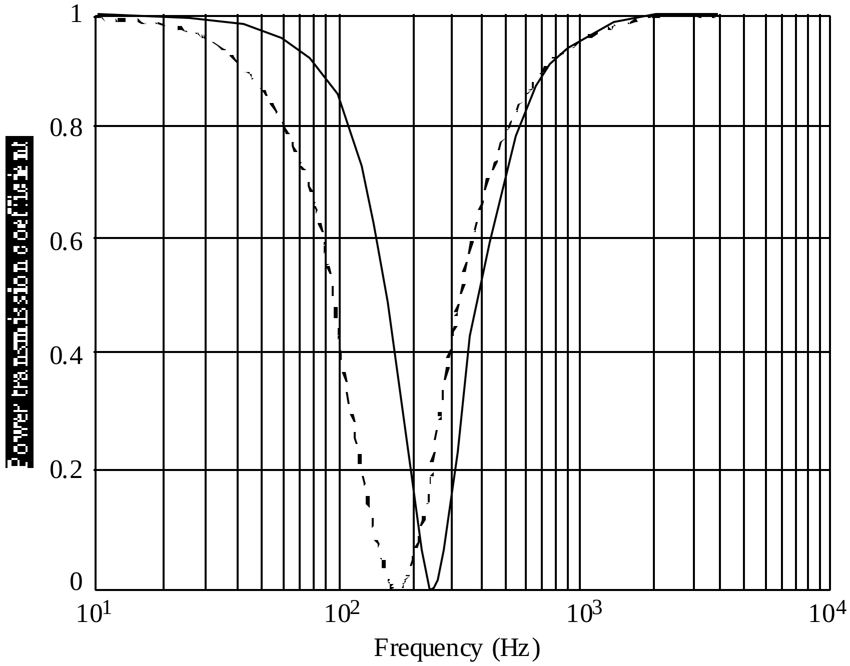


Figure 10.11.4 The power transmission coefficient for a band-stop filter consisting of a Helmholtz resonator. The resonator has a neck of length 0.6 cm and radius 1.55 cm. The pipe has a cross-sectional area 28 cm^2. The solid line is for a resonator volume of 1120 cm^3. The dashed line is for a resonator volume of 2240 cm^3.