(5.7) Harmonic Plane Waves

The classical wave equation is of the form:

$$\frac{\partial^2 \tilde{p}(\vec{r},t)}{\partial t^2} = c^2 \nabla^2 \tilde{p}(\vec{r},t)$$

where $\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$ is the position vector.
If we assume that the solution is of the form:

$$\tilde{p}(\vec{r},t) = \tilde{g}(\vec{r})\tilde{h}(t)$$

where

$$h(t) = e^{jwt}$$

then

$$\nabla^2 \tilde{g}(\vec{r}) + k^2 \tilde{g}(\vec{r}) = 0. \qquad \text{(Helmholtz Equation)}$$

The Helmholtz equation is a convenient form to solve propagation factors (wavenumber) in different coordinate systems. The use of rectangular coordinates is useful in describing <u>plane waves</u>. A plane wave is an acoustic wave in which the acoustic variables have constant amplitude and phase on any plane perpendicular to the direction of propagation. The Helmholtz equation in rectangular coordinates gives:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(Laplacian operator in rectangular)}$$

$$\frac{\partial^2 \tilde{g}}{\partial x^2} + \frac{\partial^2 \tilde{g}}{\partial y^2} + \frac{\partial^2 \tilde{g}}{\partial z^2} + k^2 \tilde{g} = 0.$$

Let's try the solution:

$$\tilde{g}(\vec{r}) = \tilde{X}(x)\tilde{Y}(y)\tilde{Z}(z)$$

then

$$\frac{\partial^2 \tilde{X}(x)}{\partial x^2}\tilde{Y}(y)\tilde{Z}(z) + \frac{\partial^2 \tilde{Y}(y)}{\partial y^2}\tilde{X}(x)\tilde{Z}(z) + \frac{\partial^2 \tilde{Z}(z)}{\partial z^2}\tilde{X}(x)\tilde{Y}(y) + k^2\tilde{X}(x)\tilde{Y}(y)\tilde{Z}(z) = 0$$

dividing by $\tilde{X}(x)\tilde{Y}(y)\tilde{Z}(z)$ yields

$$\frac{1}{\tilde{X}(x)}\frac{\partial^2 \tilde{X}(x)}{\partial x^2} + \frac{1}{\tilde{Y}(y)}\frac{\partial^2 \tilde{Y}(y)}{\partial y^2} + \frac{1}{\tilde{Z}(z)}\frac{\partial^2 \tilde{Z}(z)}{\partial z^2} = -k^2.$$

In order for the equation above to hold for all x, y and z each term must be equal to a constant, so

$$\frac{1}{\tilde{X}(x)}\frac{\partial^2 \tilde{X}(x)}{\partial x^2} = -k_x^2, \qquad \frac{1}{\tilde{Y}(y)}\frac{\partial^2 \tilde{Y}(y)}{\partial y^2} = -k_y^2, \qquad \frac{1}{\tilde{Z}(z)}\frac{\partial^2 \tilde{Z}(z)}{\partial z^2} = -k_z^2$$

and

$$k^2 = k_x^2 + k_y^2 + k_z^2 \qquad \text{and} \qquad \vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \ \text{(the wave vector)}$$

This represents the 3D acoustic wavenumber for rectangular coordinates.
Each of the three terms above can be solved and have the solution

$$\tilde{X}(x) = e^{-jk_x x}$$

for a simple plane wave propagating in the + direction.
This gives a total solution for $\tilde{g}(\vec{r})$ of:

$$\tilde{g}(\vec{r}) = Ae^{-jk_x x}e^{-jk_y y}e^{-jk_z z}$$

$$\tilde{g}(\vec{r}) = Ae^{-j\vec{k}\cdot\vec{r}}$$

The pressure is then:

$$\tilde{p}(\vec{r},t) = Ae^{jwt}e^{-j\vec{k}\vec{r}} = Ae^{j(wt-\vec{k}\cdot\vec{r})}. \qquad \text{(+ going wave).}$$

Let's take as an example a plane wave propagating in both + and – directions and constrained to move only in the x-direction (a 1D plane wave solution). The solution to this will be:

$$\tilde{p}(x,t) = Ae^{j(wt-kx)} + Be^{j(wt+kx)}$$

or

$$\tilde{p}(x,t) = p_+ + p_-$$

Note: Since we are talking about a real physical acoustic wave, we may represent the solution using the complex exponential form but we are really interested in the real part of the solution.

So, we have the constants *A* and *B*, which in general are complex but for this solution we are assuming them to be real. The acoustic pressure that would be measured is the real part:

$$p(x,t) = A\cos(wt-kx) + B\sin(wt+kx).$$

To obtain the particle velocity, $\vec{u}$, we use the force equation (Euler's equation) $r_0\frac{\partial\vec{u}}{\partial t} = -\nabla p$:

$$\frac{\partial\vec{u}}{\partial t} = -\frac{1}{r_0}\left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right]p$$

But since we are talking about a plane wave propagating only in the x-direction then:

$$\frac{\partial\vec{u}}{\partial t} = -\frac{1}{r_0}\frac{\partial p}{\partial x}\hat{x}$$

$$\frac{\partial\vec{u}}{\partial t} = -\frac{1}{r_0}\left[(-jk)Ae^{j(wt-kx)} + (jk)Be^{j(wt+kx)}\right]\hat{x}$$

where we have again used the exponential form due to the ease of manipulation and operations. What we have derived is the particle acceleration of a plane harmonic wave

$$\vec{a} = \frac{\partial\vec{u}}{\partial t} = \left[\frac{jkA}{r_0}e^{j(wt-kx)} - \frac{jkB}{r_0}e^{j(wt+kx)}\right]\hat{x}$$

To get $\vec{u} = u_x\hat{x}$ we must integrate wrt time. If differentiating our solution wrt time mean multiplying by jw then integrating wrt time mean dividing our solution by jw.

$$u_x = \frac{jkA}{jwr_0}e^{j(wt-kx)} - \frac{jkB}{jwr_0}e^{j(wt+kx)}$$

$$= \frac{A}{r_0c}e^{j(wt-kx)} - \frac{B}{r_0c}e^{j(wt+kx)}$$

where we used $c = w/k$.

We can then relate this to pressure:

$$u_x = \frac{p_+}{r_0 c} - \frac{p_-}{r_0 c}$$

and

$$a_x = \frac{jkp_+}{r_0} - \frac{jkp_-}{r_0}.$$

So, for a positive going wave,

u_x is in phase with p

a_x leads p by 90°

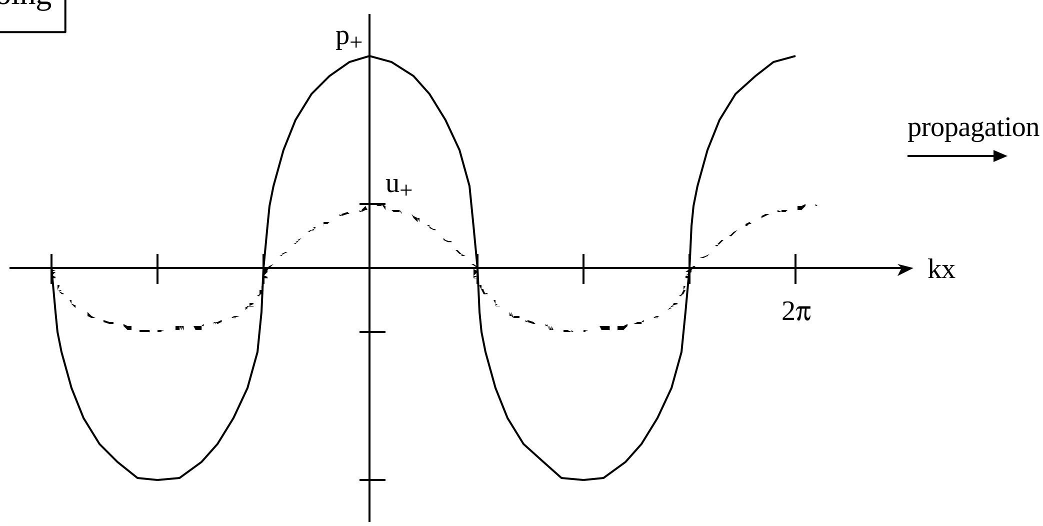
For the negative going wave,

u_x is 180° out of phase with p

a_x lags p by 90°

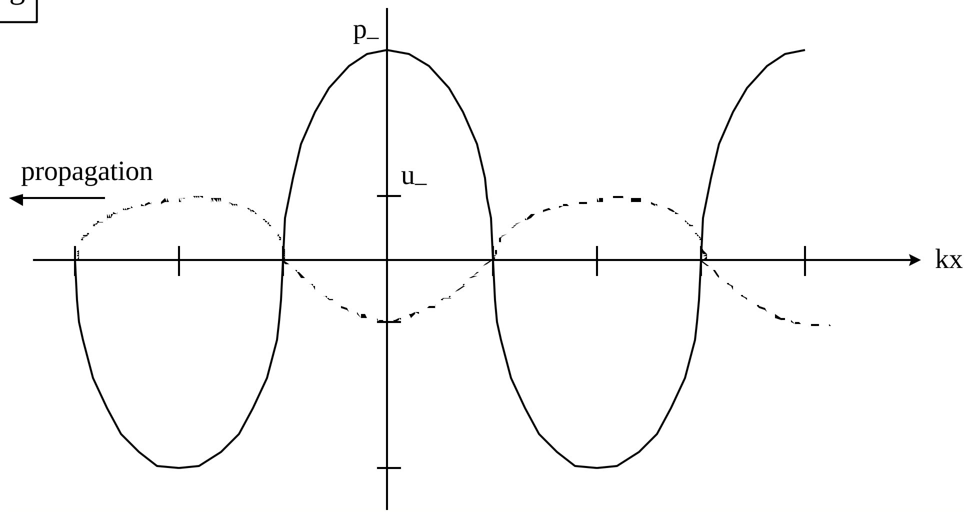
This is illustrated by the following figures:

Positive going



At peak in compression u is in direction of propagation.

Negative going



Note that at the peak in compression *u* is in direction of propagation in both cases. (Negative amplitude with negative going wave).

Other wave parameters:

$$x_x = \int u_x dt = \frac{1}{jw} u_x = \frac{A}{jwr_0 c} e^{j(wt-kx)} - \frac{B}{jwr_0 c} e^{j(wt+kx)}$$

$$\boxed{x_x = \frac{-j}{wr_0 c} p^+ + \frac{j}{wr_0 c} p^-}$$

Also $\vec{u} = Ñ\phi$ so can get ϕ, the velocity potential, and can get s

$$\boxed{\begin{aligned} \Phi &= -\frac{1}{jwr_0} p^+ - \frac{1}{jwr_0} p^- \\ s &= \frac{p^+}{r_0 c^2} + \frac{p^-}{r_0 c^2} \end{aligned}}$$

<u>Review</u>

$j \Rightarrow e^{j90°} \Rightarrow$ +90° phase shift

$j^2 \Rightarrow -1 \Rightarrow$ 180° phase shift

$j3 = -j \Rightarrow e^{-j90°} \Rightarrow$ –90° phase shift

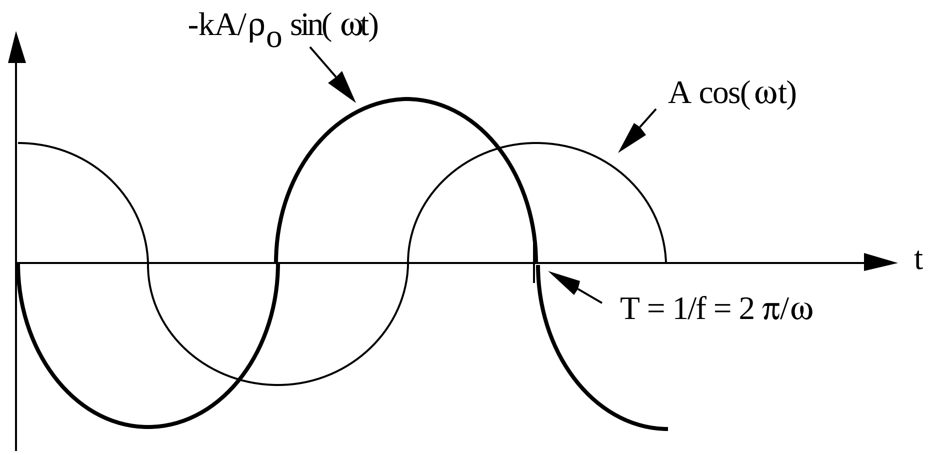
Plots

take $\mathrm{Re}\left\{p^{+}\right\} = A\cos(wt - kx) = \mathrm{Re}\left\{Ae^{j(wt-kx)}\right\}$

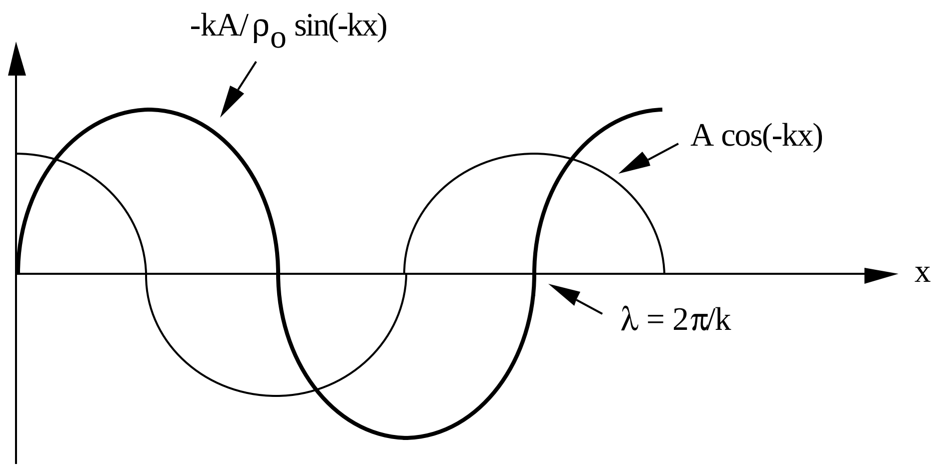
then $\mathrm{Re}\left\{a_x^{+}\right\} = \mathrm{Re}\left\{\dfrac{jk}{r_0}p^{+}\right\} = \mathrm{Re}\left\{\dfrac{jkA}{r_0}e^{j(wt-kx)}\right\}$

$$= -\frac{kA}{r_0}\sin(wt - kx)$$

versus t for kx = 0



versus x for ωt = 0



Note that for +x going wave the maximum in acceleration in direction of propagation leads the pressure maximum by T/4 or $\frac{\lambda}{4}$.

The relation will be different for positive and negative going parameters that are vectors.

Summary

$$p = Ae^{j(wt-kx)} + Be^{j(wt-kx)} = p^+ + p^-$$

$$\vec{u}^{\pm} = \pm\frac{p^{\pm}}{r_0 c}\hat{x} \qquad s^{\pm} = \frac{p^{\pm}}{r_0 c^2}$$

$$\vec{a}^{\pm} = \pm j\frac{kp^{\pm}}{r_0}\hat{x} \qquad \Phi^{\pm} = -\frac{p^{\pm}}{jwr_0}$$

$$\vec{x}^{\pm} = \mp j\frac{p^{\pm}}{wr_0 c}\hat{x}$$

Back to the 3D acoustic wave:

$$\tilde{p}(\vec{r},t) = \tilde{A}e^{j\left(wt-\vec{k}\cdot\vec{r}\right)}$$

Looking at the real part:

$$p(\vec{r},t) = \mathrm{Re}\left\{\tilde{p}(\vec{r},t)\right\} = \mathrm{Re}\left\{\left|\tilde{A}\right|e^{jf}e^{j\left(wt-\vec{k}\cdot\vec{r}\right)}\right\}$$

$$= \mathrm{Re}\left\{Ae^{j\left(wt-\vec{k}\cdot\vec{r}+f\right)}\right\}$$

$$= A\cos\left(wt-\vec{k}\cdot\vec{r}+f\right)$$

The phase of the wave is the argument of the cos term. At a given time, *t*, all points having the same phase, Ω, obey:

$$wt-\vec{k}\cdot\vec{r}+f = \Omega .$$

So

$$\vec{k}\cdot\vec{r} = wt + f - \Omega = const$$

defines an equation of a plane perpendicular to $\vec{k}$. This also means that maximum and minimum displacements move as a plane perpendicular to $\vec{k}$. If we take the derivative of the above relation with respect to time, *t*,

$$\frac{\partial}{\partial t}\left\{\vec{k}\cdot\vec{r} = wt + f - \Omega\right\}$$

$$\vec{k}\cdot\frac{\partial\vec{r}}{\partial t} = w .$$

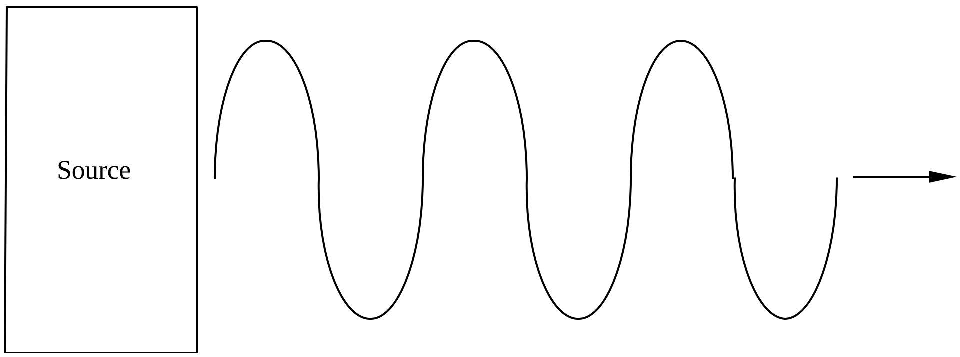
Substituting $\vec{k} = k\hat{k}$ and rearranging gives:

$$\hat{k}\cdot\frac{\partial\vec{r}}{\partial t} = \frac{w}{k} = c$$

which tells us the plane wave advances with a speed c along the $\hat{k}$ component.

(5.8) Energy Density

Energy is generated by the source and transported by the traveling acoustic wave in the direction of propagation.



The energy in the wave may, at any time and location, have two contributions or forms

(1) Potential energy – fluid is compressed, storing energy

(2) Kinetic energy – particle is moving

Examine kinetic energy first

Consider small volume (particle) at mass $r_0 V_0$ moving with particle velocity $\vec{u}$

$$E_{kinetic} = E_k = \frac{1}{2} mass\ (speed)^2 = \frac{1}{2} r_0 V_0 u^2$$

Potential energy

Potential energy is equal to the work done on a volume, which is stored as potential energy.

$$E_{potential} = E_p = -\int_{V_0}^{V} p\,dV$$

Negative sign
For positive p the volume is decreased ⇒ positive potential energy

Need relation between p and V.

$$rV = r_0 V_0 \ \text{– conservation of mass}$$

We showed earlier that $r_0 = r\left(1 + \frac{\partial x_x}{\partial x}\right)$ so that

$$V=\frac{\rho_0 V_0}{\rho}=\frac{\cancel{\rho}\left(1+\frac{\partial \xi_x}{\partial x}\right)V_0}{\cancel{\rho}}$$

and from the Equation of State $p=\mathrm{B}s=\rho_0 c^2 s=-\rho_0 c^2\frac{\partial \xi_x}{\partial x}$ so that

$$\therefore V=V_0\left(1-\frac{p}{\rho_0 c^2}\right)$$

Taking the derivative yields

$$dV=-\frac{V_0}{\rho_0 c^2}dp$$

so that our Potential Energy term is given by:

$$E_p=-\int_{V_0}^{V} p\,dV=-\int_0^p p\left(-\frac{V_0}{\rho_0 c^2}\right)dp=\frac{1}{2}p^2\frac{V_0}{\rho_0 c^2}$$

Total Acoustic Energy

$$E=E_k+E_p=\frac{1}{2}\rho_0 V_0 u^2+\frac{1}{2}\frac{V_0 p^2}{\rho_0 c^2}$$

Instantaneous energy density is defined as the energy per unit volume

$$e_{instantaneous}=e_i=\frac{1}{2}\rho_0\left(u^2+\frac{p^2}{\rho_0^2 c^2}\right)$$

Energy density is the time average of the instantaneous energy density over an acoustic cycle

$$e=\langle e_i\rangle_t=\frac{1}{T}\int_o^T e_i\,dt$$ where T is one period of a harmonic wave.

Consider a positive going plane wave

$$p=P\cos(\omega t-kx) \text{ and } u_x=\frac{P}{\rho_0 c}\cos(\omega t-kx)$$

then the instantaneous energy density is given by

$$\begin{aligned} e_i&=\frac{1}{2}\rho_0\left(u^2+\frac{p^2}{\rho_0^2 c^2}\right)\\ &=\frac{1}{2}\rho_0\left[\left(\frac{P}{\rho_0 c}\right)^2\cos^2(\omega t-kx)+\frac{P^2}{\rho_0^2 c^2}\cos^2(\omega t-kx)\right]. \end{aligned}$$

Integrating over the acoustic period gives

$$e=\frac{1}{T}\int_0^T \rho_0\left(\frac{P}{\rho_0 c}\right)^2\cos^2(\omega t-kx)\,dt \quad \text{where } T=\frac{1}{f}=\frac{2\pi}{\omega}$$

$$\boxed{e=\frac{1}{2}\frac{P^2}{\rho_0 c^2}}\ \frac{\mathrm{J}}{\mathrm{m}^3}$$

Recall that $P = r_0 c\, U$

$$\therefore e = \frac{1}{2}\frac{PU}{c} = \frac{1}{2} r_0 U^2 .$$

(5.9) Acoustic Intensity

Intensity is a concept generally used in connection with progressive (traveling) plane waves in a fluid. Intensity is a <u>vector</u>. It is a measure of power flowing at normal incidence to the specified unit area.

<u>Instantaneous Intensity</u> is defined as: $i = p\dfrac{\partial x}{\partial t} = pu$.

<u>Average Intensity</u>: $I = \langle pu \rangle_t = \dfrac{1}{T}\displaystyle\int_0^T pudt$

From

$$x(x,t) = x_{o+}\cos(wt-kx) + x_{o-}\cos(wt+kx)$$

$$u(x,t) = -U_{o+}\sin(wt-kx) - U_{o-}\sin(wt+kx)$$

and since for a positive going wave, u_x is in phase with p and for the negative going wave, u_x is 180° out of phase with p so,

$$p(x,t) = -P_{o+}\sin(wt-kx) + P_{o-}\sin(wt+kx).$$

Therefore,

$$p(x,t)u(x,t) = P_{o+}U_{o+}\sin^2(wt-kx) + P_{o+}U_{o-}\sin(wt-kx)\sin(wt+kx)$$
$$-P_{o-}U_{o+}\sin(wt+kx)\sin(wt-kx) - P_{o-}U_{o-}\sin^2(wt+kx)$$

and the Average Intensity gives:

$$I = \frac{1}{T}\int_0^T pudt = \frac{1}{T}\int_0^T \{P_{o+}U_{o+}\sin^2(wt-kx) + P_{o+}U_{o-}\sin(wt-kx)\sin(wt+kx)$$
$$-P_{o-}U_{o+}\sin(wt+kx)\sin(wt-kx) - P_{o-}U_{o-}sin^2(wt+kx)\}\,dt$$

Using the trigonometric relations:

$$\sin^2(wt-kx) = \frac{1}{2}\{1-\cos(2wt-2kx)\}$$

$$\sin^2(wt+kx) = \frac{1}{2}\{1-\cos(2wt+2kx)\}$$

$$\sin(wt-kx)\sin(wt+kx) = \frac{1}{2}\{\cos(2kx)-\cos(2wt)\}$$

then

$$I = \frac{w}{2p}\int_0^{2p/w} \frac{1}{2}\{P_{o+}U_{o+}\{1-\cos(2wt-2kx)\} + P_{o+}U_{o-}\{\cos(2kx)-\cos(2wt)\}$$
$$-P_{o-}U_{o+}\{\cos(2kx)-\cos(2wt)\} - P_{o-}U_{o-}\{1-\cos(2wt+2kx)\}\}\,dt$$

A little rearranging yields:

$$I = \frac{w}{2p}\int_0^{2p/w} \frac{1}{2}\Big\{P_{o+}U_{o+}\{1-\cos(2wt-2kx)\} - P_{o-}U_{o-}\{1-\cos(2wt+2kx)\}$$
$$+\left(P_{o+}U_{o-} - P_{o-}U_{o+}\right)\{\cos(2kx)-\cos(2wt)\}\Big\}\,dt$$

When you do the integration to yield the following expression (why?):

$$I = \frac{1}{2}\left\{\left(P_{o+}U_{o+} - P_{o-}U_{o-}\right) + \left(P_{o+}U_{o-} - P_{o-}U_{o+}\right)\cos(2kx)\right\}$$

From (Note: plane waves)

$$z_+ = \frac{p_+}{u_+} = \frac{-P_{o+}}{-U_{o+}} = r_o c_o\,,\ P_{o+} = r_o c_o U_{o+}$$
$$z_- = \frac{p_-}{u_-} = \frac{P_{o-}}{-U_{o-}} = -r_o c_o\,,\ P_{o-} = r_o c_o U_{o-}$$

so that,

$$I = \frac{1}{2}\left\{\left(r_o c_o U_{o+}U_{o+} - r_o c_o U_{o-}U_{o-}\right) + \left(r_o c_o U_{o+}U_{o-} - r_o c_o U_{o-}U_{o+}\right)\cos(2kx)\right\}$$
$$I = \frac{r_o c_o}{2}\left\{\left(U_{o+}^2 - U_{o-}^2\right) + \left(U_{o+}U_{o-} - U_{o-}U_{o+}\right)\cos(2kx)\right\}$$
$$I = \frac{r_o c_o}{2}\left\{U_{o+}^2 - U_{o-}^2\right\} = \frac{1}{2 r_o c_o}\left\{P_{o+}^2 - P_{o-}^2\right\}$$

Let's discuss traveling vs. standing wave

$$e = \frac{r_o}{2}\left\{U_{o+}^2 + U_{o-}^2\right\} \qquad \text{(Energy density)}$$
$$I = \frac{r_o c_o}{2}\left\{U_{o+}^2 - U_{o-}^2\right\} \qquad \text{(PLANE WAVE!)}$$

For a progressive, plane wave in the +x direction

$$e = \frac{r_o}{2}U_{o+}^2$$
$$I = \frac{r_o c_o}{2}U_{o+}^2$$

***************************** Example 5.4 *****************************

Consider fresh water at 20°C in which the time-average acoustic intensity is $1\ \mathrm{W/cm^2}$. Calculate the peak values x_o, U_o, A_o and p_o at the three frequencies of 1, 10 and 100 MHz.

ANSWER:

For fresh water at 20°C, two important quantities are $c_0 = 1481\frac{\mathrm{m}}{\mathrm{s}}$ and the density $r_0 = 998\frac{\mathrm{kg}}{\mathrm{m}^3}$ (Appendix A10, a very important part of the book!!!)

From the energy density equation:

$$I = \frac{r_o c_o}{2} U_{o+}^2 \qquad \text{or} \qquad U_{o+} = \sqrt{\frac{2I}{r_o c_o}}$$

This gives us the value for the velocity:

	1 MHz	10 MHz	100 MHz
$U_o\,(\text{cm/s})$	11.6	11.6	11.6

and it is not frequency dependent. Likewise from the relation $P_{o+} = r_o c_o U_{o+}$ we have for the pressure:

$p_o\,(\text{atm})$	1.72	1.72	1.72

The displacement and acceleration we get from integrating the velocity and taking the derivative of the velocity, respectively. This makes the values frequency dependent because we merely divide by (displacement) or multiply by jw. So, $x_o = \frac{U_o}{w} = \frac{U_o}{2p\,f}$ and $A_o = wU_o = 2p\,fU_o$. giving

$x_o\,(\dot{\text{A}})$	185	18.5	1.85
$A_o\,(\mu\text{m/s}^2)$	0.731	7.31	73.1

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*************************** Example 5.5 ***************************

A pulse of frequency 1000 Hz and intensity level of 10^{-12} W/m^2 and is traveling in air at 20°C in the +x direction. (a) What is the particle displacement amplitude? (b) Give a numerical formula for the condensation, *s*, as a function of x?

ANSWER:

(a) $\mathrm{I} = \frac{|\mathrm{p_o}|^2}{2\rho_o c_o} = \frac{(r_o c_o w x_o)^2}{2 r_o c_o} = \frac{r_o c_o w^2 x_o^2}{2} = 10^{-12}\ \mathrm{W/m^2}$

$x_o = \sqrt{\frac{2I}{r_o c_o w^2}} = \sqrt{\frac{2(10^{-12}\ \mathrm{W/m^2})}{(1.21\,\mathrm{kg/m^3})(343\ \mathrm{m/s})(2000\pi/\mathrm{s})^2}}$ = 11 pm (peak)

(b) $s = \frac{r - r_o}{r_o} = -\frac{\partial x}{\partial x} = -\frac{\partial}{\partial x}\left[x_o \cos(wt - kx)\right] = kx_o \sin(wt - kx)$, so

$\mathrm{s} = (6.44 \times 10^{-11})\sin(2000\pi\mathrm{t} - 5.831\pi\mathrm{x})$ (peak)

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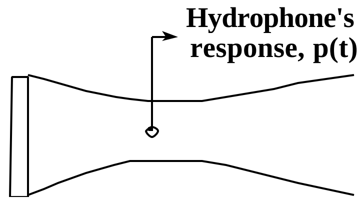
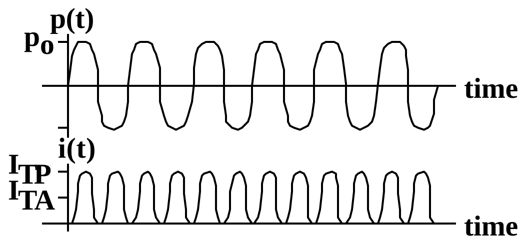
When attempting to quantify (*via* a measurement process usually) intensity, both time (temporal) and space (spatial) aspects must be considered.

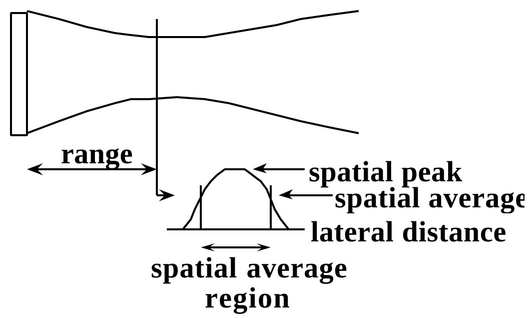


For continuous wave ultrasound at one location in space:



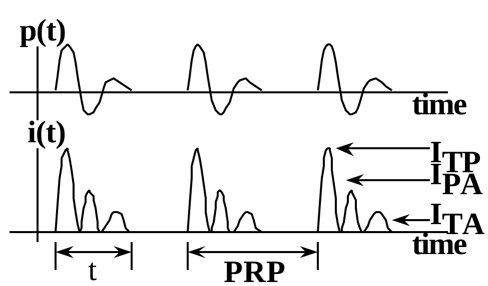
where $I_{TP} = \dfrac{p_o^2}{\rho_o c_o}$ and $I_{TA} = \dfrac{p_o^2}{2\rho_o c_o}$.

In some instances you measure over a spatial range (i.e. in the focus of a focused source):



I_{SPTP}: Spatial peak temporal peak intensity
I_{SPTA}: Spatial peak temporal average intensity
I_{SATP}: Spatial average temporal peak intensity
I_{SATA}: Spatial average temporal average intensity

Sometimes "spatial peak" refers to a global peak. For pulsed wave ultrasound, and at one location in space:



I_{SPTP}: Spatial peak temporal peak intensity
I_{SPPA}: Spatial peak pulse average intensity
I_{SPTA}: Spatial peak temporal average intensity
I_{SATP}: Spatial average temporal peak intensity
I_{SAPA}: Spatial average pulse average intensity
I_{SATA}: Spatial average temporal average intensity