(5.10) Specific Acoustic Impedance

Specific acoustic impedance

$$\tilde{z} = \frac{\tilde{p}}{\tilde{u}} \left(\frac{Pa \bullet s}{m} \right) \; or \; (ray1)$$

For plane waves

$$z = \pm \; r_0 c \;\; \text{for} \;\; \pm \text{going waves}$$

The product $r_0 c$ is the characteristic impedance. It is specified by the properties of the material. The characteristic impedance is analogous to the $\sqrt{m/e}$ in dielectric properties. So for example, a sound wave incident on some interface of two different fluids will have a reflected and transmitted wave dependent on the characteristic impedance of the two fluids at the boundary. Similarly, if you have an EM wave incident on some dielectric interface, you will have a reflected and transmitted wave based on the dielectric properties of the two media.

In general, **z** is complex (i.e. for spherical waves and lossy materials)

$$\tilde{z} = r + j\; x$$

where r is the specific acoustic resistance and x is the specific acoustic reactance.

**************************** Example 5.6 ****************************

Characteristic impedance = specific acoustic impedance for plane wave at 20°C and 1 atm.

Air $\qquad r = r_0 c = 1.21 \left(\frac{\text{kg}}{\text{m}^3} \right) 343 \left(\frac{\text{m}}{\text{s}} \right) = 415 \text{ ray1}$

Water $\qquad r = r_0 c = 998 \left(\frac{\text{kg}}{\text{m}^3} \right) 1481 \left(\frac{\text{m}}{\text{s}} \right) = 1.48 \times 10^6 \text{ ray1} = 1.48 \text{ Mrayl}$

This is a very large characteristic impedance difference.
The true significance of this will be shown when we talk about reflection and transmission at a boundary between two propagating media (Chapter 6)

****

(5.11) Spherical Waves

We have been talking about plane wave, now we talk about spherical waves. Spherical waves are useful in describing waves from a small source and for determining the field from an arbitrarily shaped source (like a piston source).

In this case we begin again with the 3D Helmholtz equation:

$$\nabla^2 \tilde{g}(\vec{r}) + k^2 \tilde{g}(\vec{r}) = 0$$

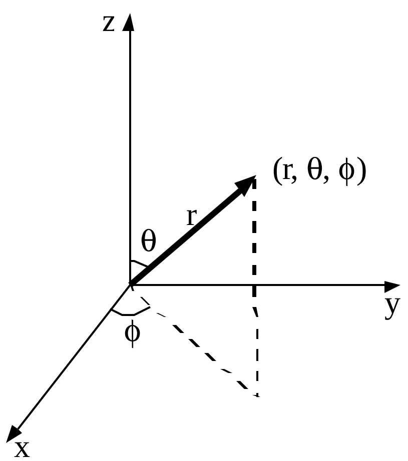
where

$$\tilde{p}(\vec{r},t) = \tilde{g}(\vec{r})\tilde{h}(t) \qquad \text{and} \qquad h(t) = e^{jwt}$$

This time we used the Laplacian operator for Spherical coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin q}\frac{\partial}{\partial q}\left(\sin q \frac{\partial}{\partial q}\right) + \frac{1}{r^2 \sin q}\frac{\partial^2}{\partial f^2} \qquad \text{(Appendix A7)}$$

where the coordinate system relations between Cartesian and Spherical are given by:



x = r sin θ cosφ

y = r sin θsin φ

x = r cos θ

If the waves have spherical symmetry (which we can assume for small sources if we are looking at the pressure field far enough away from the source), then the pressure p is a function of the radial distance and time but not the angular coordinates (angular symmetry). Hence we simplify the Laplacian as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

This gives for the Helmholtz equation:

$$\frac{\partial^2 \tilde{g}(\vec{r})}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\tilde{g}(\vec{r}) + k^2 \tilde{g}(\vec{r}) = 0.$$

Now, as an aside we note that:

$$\frac{\partial}{\partial r}(r\tilde{g}) = \tilde{g} + r\frac{\partial \tilde{g}}{\partial r}$$

so

$$\frac{\partial^2}{\partial r^2}(r\tilde{g}) = \frac{\partial}{\partial r}\left[\tilde{g} + r\frac{\partial \tilde{g}}{\partial r}\right] = \frac{\partial \tilde{g}}{\partial r} + \frac{\partial \tilde{g}}{\partial r} + r\frac{\partial^2 \tilde{g}}{\partial r^2} = 2\frac{\partial \tilde{g}}{\partial r} + r\frac{\partial^2 \tilde{g}}{\partial r^2}.$$

or

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\tilde{g}) = \frac{\partial^2 \tilde{g}}{\partial r^2} + \frac{2}{r}\frac{\partial \tilde{g}}{\partial r}.$$

This gives for our Helmholtz equation:

$$\frac{\partial^2\left[r\tilde{g}(\vec{r})\right]}{\partial r^2} + k^2\left[r\tilde{g}(\vec{r})\right] = 0.$$

If we let $f(r) = r\tilde{g}(r)$ then we have

$$\frac{\partial^2 f(r)}{\partial r^2} + k^2 f(r) = 0$$

which has the simple and well-known solution:

$$f(r) = r\tilde{g}(r) = Ae^{-jkr} + Be^{jkr}$$

so that

$$\tilde{g}(r) = \frac{A}{r}e^{-jkr} + \frac{B}{r}e^{jkr}$$

and

$$\tilde{p}(r,t) = \tilde{h}(t)\tilde{g}(r) = \frac{A}{r}e^{j(wt-kr)} + \frac{B}{r}e^{j(wt+kr)}.$$

The first term represents an outgoing (positive r going) wave that is diverging. Physically what does this mean? Example

The second term is just the opposite, it is an inward going wave (negative r going) that is converging at some point. Physically what does this mean? Example.

Both waves have spherical fronts. (The diverging is more common).

Next lets look at the particle velocity for a spherically diverging wave. We use the diverging wave since it is the most common seen.

To get $\vec{u}$ we'll use the velocity potential $p = -r_0\frac{\partial\Phi}{\partial t}$ and $\vec{u} = \nabla\Phi$

For diverging wave then

$$\Phi = \int -\frac{p}{r_0}dt = -\frac{A}{r_0 rjw}e^{j(wt-kr)} = \frac{jA}{r_0 wr}e^{j(wt-kr)}$$

$$\vec{u} = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{r} = \left[-\frac{(jk)jA}{r_0 wr} - \frac{jA}{r_0 wr^2}\right]e^{j(wt-kr)}\hat{r}$$

$$= \left[\frac{k}{r_0 w} - \frac{j}{r_0 wr}\right]\frac{A}{r}e^{j(wt-kr)}\hat{r}$$

$$\vec{u} = \left[1 - \frac{j}{kr}\right]\frac{1}{r_0 c}\frac{A}{r}e^{j(wt-kr)}$$

$$\therefore\ z = \frac{p}{u} = \rho_0 c \frac{kr}{kr - j} = \rho_0 c \frac{(kr)^2 + jkr}{(kr)^2 + 1}$$

$$= \rho_0 ckr \frac{kr + j}{(kr)^2 + 1}$$

Notice that in this case we have a real and imaginary part to z:

$$z = \rho_0 c \left[\frac{(kr)^2}{1 + (kr)^2} + j \frac{kr}{1 + (kr)^2} \right] = r_s + jx_s$$

where the specific acoustic resistance is $r_s = \rho_0 c \frac{(kr)^2}{1 + (kr)^2}$ and the specific acoustic reactance is $x_s = \rho_0 c \frac{kr}{1 + (kr)^2}$.

Putting it in polar form yields:

$$z = \rho_0 c \frac{kr}{\sqrt{1 + (kr)^2}} e^{j \tan^{-1}\left(\frac{1}{kr}\right)}$$

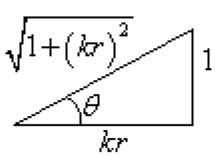
Notes: $kr = \frac{2\pi r}{\lambda}$ $\therefore$ relates r and λ

For very large kr (kr >> 1)

$$z \cong \rho_0 c \left[1 + j \frac{1}{kr} \right] \quad \rightarrow$$ approaches $\rho_0 c$ for large *kr*, as for plane wave.

At large *kr*, the spherical wave looks like a plane wave.

Looking at it another way, if $\theta = \tan^{-1} \frac{1}{kr}$



then

$$\cos\theta = \frac{kr}{\sqrt{1 + (kr)^2}}$$

and

$$z = \rho_0 c \cos\theta \, e^{j\theta}$$

As $kr \rightarrow \infty$, $\theta \rightarrow 0$ and $\cos\theta \rightarrow 1$, then $z \rightarrow \rho_0 c$ as for plane waves.

The real pressure is given by:

$$p = \frac{A}{r}\cos(wt - kr)$$

The particle velocity is determined by:

$$\tilde{u} = \frac{\tilde{p}}{\tilde{z}} = \frac{A}{r\tilde{z}} e^{j(wt-kr)} = \frac{A}{r\, r_0 c \cos q} e^{j(wt-kr-q)}$$

The real part gives for the particle velocity:

$$\tilde{u} = \frac{A}{r\, r_0 c \cos q}\cos(wt - kr - q)$$

Recall, that $\cos q$ is a function of r so that particle velocity does <u>not</u> vary as $1/r$.

Let's look now at the intensity of a spherical wave:

$$I = \langle pu \rangle_t = \frac{1}{T}\int_0^T \frac{A}{r}\cos(wt - kr)\frac{A}{r}\frac{1}{r_0 \cos q}\cos(wt - kr - q)dt$$

Integrating yields:

$$I = \frac{1}{2}\left(\frac{A}{r}\right)^2 \frac{1}{r_0 c \cos q}\cos q$$

or rearranging a little more gives:

$$I = \frac{1}{2}\frac{(A/r)^2}{r_0 c} = \frac{1}{2}\frac{|P|^2}{r_0 c}$$ the same as for plane waves (not true for U)

Also, it is important to note that the intensity decreases as $\frac{1}{r^2}$

Now, let's look at the energy density in a spherical wave:

For a plane wave recall that the instantaneous KE density is $E_{ki} = \frac{r_0}{2}u^2$ (see Eq 5.8.1) and the instantaneous PE density is $E_{pi} = \frac{1}{2 r_0 c_0^2} p^2$ (see Eq 5.8.5).

The time average KE density is determined by $\langle E_{ki} \rangle = \frac{1}{T}\int_0^T E_{ki}dt = \frac{r_o}{4}|u_o|^2$ and similarly the average PE density is given by $\langle E_{pi} \rangle = \frac{1}{4 r_0 c_0^2}|p_0|^2$

Things will be slightly different for a Spherical Wave:
For a spherical wave the particle velocity is

$$\tilde{u} = \frac{\tilde{A}}{r_0 r (jw)}\left(\frac{1}{r} + jk\right)e^{j(wt-kr)} = \tilde{u}_0 e^{j(wt-kr)}$$

where $\tilde{u}_0 = \frac{\tilde{A}}{r_0 r (jw)}\left(\frac{1}{r} + jk\right)$.

Using this to get the average KE density for a spherical wave is then:

$$\langle E_{ki}\rangle = \frac{r_0}{4}|u_0|^2 = \frac{r_0}{4}\left|\frac{\tilde{A}}{r_0 r(jw)}\left(\frac{1}{r}+jk\right)\right|^2$$

$$= \frac{r_0}{4}\left\{\frac{\tilde{A}}{r_0 r(jw)}\left(\frac{1}{r}+jk\right)\right\}\cdot\left\{\frac{\tilde{A}}{r_0 r(jw)}\left(\frac{1}{r}+jk\right)\right\}^*$$

$$= \frac{r_0}{4}\left\{\frac{|\tilde{A}|^2}{r_0^2 r^2 w^2 (j)(-j)}\right\}\cdot\left(\frac{1}{r}+jk\right)\left(\frac{1}{r}-jk\right) = \frac{r_0|\tilde{A}|^2}{4r_0^2 r^2 w^2}\left\{\frac{1}{r^2}+k^2\right\}$$

$$= \frac{k^2|\tilde{A}|^2}{4r_0 r^2 (kc_0)^2}\left\{1+\frac{1}{(kr)^2}\right\}$$

$$\langle E_{ki}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}\left\{1+\frac{1}{(kr)^2}\right\}$$

The pressure for a spherical wave is simply

$$\tilde{p}(r,t) = \frac{\tilde{A}}{r}e^{j(wt-kr)} = \tilde{p}_0 e^{j(wt-kr)}$$

where $\tilde{p}_0 = \frac{\tilde{A}}{r}$.

Using this to get the average PE density for a spherical wave:

$$\langle E_i\rangle = \frac{1}{4r_0 c_0^2}|p_0|^2 = \frac{1}{4r_0 c_0^2}\left|\frac{\tilde{A}}{r}\right|^2 = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}$$

Time out ——————————

Observe what happens when $kr >> 1$

$$\langle E_{ki}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}\left\{1+\frac{1}{(kr)^2}\right\} \underset{kr>>1}{\longrightarrow} \langle E_{ki}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}$$

$$\langle E_{pi}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2} \underset{kr>>1}{\longrightarrow} \langle E_{pi}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}$$

Therefore,

$$\langle E_{ki}\rangle = \langle E_{pi}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}$$

So that the average energy density (total):

$$\langle E\rangle = \langle E_{ki}\rangle + \langle E_{pi}\rangle = \frac{|\tilde{A}|^2}{2r^2 r_o c_o^2} = \frac{p_0^2}{2r_0 c_0^2}$$

where p_0 is the peak amplitude acoustic pressure at r which is the SAME AS FOR PLANE WAVE when kr >> 1. Again this makes sense for large r because the spherical wave front appears as a plane wave front (relatively speaking).
Time in ——————————

Average energy density for a <u>spherical wave</u>, in general, is $\langle E\rangle = \langle E_{ki}\rangle + \langle E_{pi}\rangle$ where

$$\langle E_{ki}\rangle = \frac{|A|^2}{4r^2 r_0 c_0^2}\left\{1+\frac{1}{(kr)^2}\right\} \quad \text{and} \quad \langle E_{pi}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}$$

so that

$$\langle E\rangle = \langle E_{ki}\rangle + \langle E_{pi}\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}\left\{1+\frac{1}{(kr)^2}\right\} + \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}$$

$$\langle E\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}\left\{2+\frac{1}{(kr)^2}\right\}.$$

Again, observe what happens when kr >> 1

$$\langle E\rangle = \frac{|\tilde{A}|^2}{4r^2 r_0 c_0^2}\left\{2+\frac{1}{(kr)^2}\right\} \underset{kr>>1}{\rightarrow} \langle E\rangle = \frac{|\tilde{A}|^2}{2r^2 r_0 c_0^2}$$

The total acoustic power is defined as the average rate at which energy flows through a closed spherical surface of radius r surrounding a source of symmetric spherical waves:

$$\mathrm{W_A} = \int_{Area} I = 4p\, r^2 I = 4p\, r^2 \frac{(A/r)^2}{2 r_0 c} = \frac{2p\, A^2}{r_0 c}$$

This is constant, independent of r, since there is no loss within the medium and the energy must be conserved.

Comparison to other fields

	acoustic plane wave	electromagnetic plane wave	transmission line
phase velocity	$c = \sqrt{\frac{B}{\rho_0}} = \frac{1}{\sqrt{\rho_0 \beta_s}}$	$\frac{1}{\sqrt{\mu\varepsilon}}$	$\frac{1}{\sqrt{LC}}$
impedance	$\frac{p}{u} = \rho_0 c = \sqrt{\frac{\rho_0}{\beta_s}}$	$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$	$\frac{V}{I} = \sqrt{\frac{L}{C}}$

(5.12) Decibel Scales

In acoustics, it is customary to describe sound pressures and intensities in a decibel (logarithmic) scale. There are two reasons for using logarithmic scale:

(1) Large range of intensities (range of hearing 10^{-12} – 10 W/m^2). By using a logarithmic scale, the large range of numbers used to describe the intensities is compressed.

(2) Human ear determines relative loudness of two sounds by the ratio of intensities.

In practice the following two scales are used to describe sound levels:

Intensity Level $$IL = 10\log\left(\frac{I}{I_{ref}}\right)$$

Sound Pressure Level $$SPL = 20\log\left(\frac{P_e}{P_{ref}}\right)$$

where P_e is the effective or rms pressure of the sound. I_{ref}, P_{ref} represent reference levels for sound. So, the *IL* or *SPL* represent the sound level above some reference sound level. The reference depends on the medium of propagation. Generally for air:

$$I_{ref} = 10^{-12}\ \mathrm{W/m^2} \Rightarrow P_{ref_{calculated}} \cong 20.4\ \mu\mathrm{Pa}\ \text{(Note: microPascals)}$$

However, we use P_{ref} = 20 μPa = 20 x 10^{-6} Pa which makes them nearly the same.

These are for traveling plane waves.

Note: $$IL = 10\log\left(\frac{I}{I_{ref}}\right) = 10\log\frac{P_e^2 / r_0 c}{P_{ref}^2 / r_0 c} = 20\ \log\frac{P_e}{P_{ref}}$$

*************************** Example 5.7 ***************************

Take P = 1 Pa (in air) then $P_e = \frac{1}{\sqrt{2}} = 0.7071\mathrm{Pa}$

and

$$SPL = 20\ \log\frac{0.7071}{2\times10^{-5}} = 90.97\ \mathrm{dB}$$

$$\text{Note: } I = \frac{P_e^2}{r_0 c} = \frac{0.7071}{1.21(343)} = 1.704\times10^{-3}\,\frac{\text{W}}{\text{m}^2}$$

$$IL = 10\ \log\frac{1.704\times10^{-3}}{10^{-12}} = 92.31\text{dB}$$

****

Threshold of audibility → 10 dB
Threshold of feeling → 120 dB
Threshold of pain → 140 dB

<u>For water</u>

Three different references used (see Table 5.12.1).

$$10^5\ \mu\text{Pa} \quad Þ \quad 6.76\ ´\ 10^{-9}\ \text{W/m}^2$$

$$20\ \mu\text{Pa} \quad Þ \quad 2.70\ ´\ 10^{-16}\ \text{W/m}^2$$

$$1\ \mu\text{Pa} \quad Þ \quad 6.76\ ´\ 10^{-19}\ \text{W/m}^2$$

Must specify reference, otherwise you can confuse people as to what level you are trying to show. The first reference corresponds to 1 μbar. The second corresponds to the standard used in air. The third is what is used as a standard today for underwater acoustics.

*************************** Example 5.8 ****************************

Consider a 40 cm diameter acoustic beam in water of uniform intensity. The total acoustic power W_A = 100 W and frequency = 24 kHz. In water we can determine the wavelength from the speed of sound in water

$$l = \frac{c}{f} = \frac{1481\ \text{m/s}}{24{,}000\ \text{Hz}} = 0.0617\ \text{m}$$

<u>Intensity</u>

Since I is uniform [recall $W_A = (I)$ (Area)] then we can determine the intensity over the beam cross-section

$$I = \frac{W_A}{\text{area}} = \frac{100}{p(0.20)^2} = 796\ \text{W/m}^2$$

<u>Sound pressure amplitude</u>

$$I = \frac{P^2}{2 r_0 c}$$

so

$$P = \sqrt{2\rho_0 c I} = \sqrt{2\left(1.48\times10^{6}\right)796}$$
$$= 48{,}533 \text{ Pa} \cong 0.479\text{atm}$$

Particle velocity amplitude

$$U = \frac{P}{\rho c} = \frac{48{,}500}{1.48\times10^{6}} = 0.0328 \text{ m/s}$$

Particle displacement amplitude

$$\xi = \frac{U}{\omega} = \frac{0.0328}{2\pi(24{,}000)} = 2.17\times10^{-7} \text{ m} = 0.217\ \mu\text{m}$$

Condensation amplitude

$$s = \frac{P}{\rho_0 c^2} = \frac{4.85\times10^{4}}{(998)(1481)^{2}} = 2.22\times10^{-5}$$

Effective or rms pressure amplitude

$$P_{eff} = P_{rms} = \sqrt{\frac{1}{T}\int_0^T p^2(t)} = \frac{P}{\sqrt{2}} = \frac{48{,}500}{\sqrt{2}}$$
$$= 34{,}300 \text{ Pa}_{\text{rms}} = 3.43\times10^{4} \text{ Pa}_{\text{rms}}$$

Sound pressure level

$$SPL = 20 \log\left(\frac{3.43\times10^{4}}{2\times10^{-5}}\right) = 184.69 \text{ dB}_{\text{re}20\mu\text{Pa}}$$

****

**************************** Example 5.9 ****************************

Consider a SPL = 70 dB at 1 kHz in air at STP. Calculate p_o, ξ_o, U_o, A_o, E and I.

ANSWER:

(a) $SPL = 70 = 20\log_{10}\frac{p_o}{p_{ref}}$,

so $\log_{10}\frac{p_o}{p_{ref}} = \frac{70}{20} = 3.5 \rightarrow \frac{p_o}{p_{ref}} = 10^{3.5} \rightarrow p_o = 10^{3.5} p_{ref}$.

Amplitude acoustic pressure is p_o = 0.063 Pa (rms) or 0.089 Pa (peak)

(b) particle displacement amplitude ξ_o = 33.2 nm (peak)

(c) particle velocity amplitude $U_o = 209\ \mu\text{m/s}$ (peak)

(d) particle acceleration amplitude $A_o = 1.31\text{m/s}^2$ (peak)

(e) time-average energy density $E = \dfrac{p_0^2}{2 r_0 c^2} = 28.1\text{nJ/m}^3$

(f) time-average acoustic intensity $I = \dfrac{p_o^2}{2 r_o c_o} = 9.33\,\mu\text{W/m}^2$

As a check for a plane progressive wave, $E = \dfrac{I}{c_o} = \dfrac{9.33\,\mu\text{W/m}^2}{331.6\text{m/s}} = 28.1\text{nJ/m}^3$

****

**************************** Example 5.10 ****************************

Amplitudes associated with a plane wave in water @ 20°C

(r = 998 kg/m^3 and c = 1481 m/s)

Frequency (kHz)	I (W/m^2)	P (kPa)	P_{rms} (kPa$_{rms}$)	ξ_0 (μm)	U (m/s)	A (m/s^2)	s	SPL (dB re 20 μPa)
24	796	48.5403	34.3232	0.2175	0.0328	4945.7415	2.2E-05	184.69
1	1,000	54.4059	38.4708	5.85066	0.03676	230.97429	2.5E-05	185.68
1,000	1,000	54.4059	38.4708	0.00585	0.03676	230974.29	2.5E-05	185.68
1	1	1.72047	1.21655	0.18501	0.00116	7.3040484	7.9E-07	155.68
1,000	1	1.72047	1.21655	0.00019	0.00116	7304.0484	7.9E-07	155.68

Amplitudes associated with a plane wave in air @ 20°C

(r = 1.21 kg/m^3 and c = 343 m/s)

Frequency (kHz)	I (W/m^2)	P (kPa)	P_{rms} (kPa$_{rms}$)	ξ_0 (μm)	U (m/s)	A (m/s^2)	s	SPL (dB re 20 μPa)
1	1,000	0.91104	0.6442	349.391	2.19529	13793.372	0.0064	150.16
1,000	1,000	0.91104	0.6442	0.34939	2.19529	13793372	0.0064	150.16
1	1	0.02881	0.02037	11.0487	0.06942	436.18472	0.0002	120.16
1,000	1	0.02881	0.02037	0.01105	0.06942	436184.72	0.0002	120.16
1	0.001	0.00091	0.00064	0.34939	0.0022	13.793372	6.4E-06	90.16
1,000	0.001	0.00091	0.00064	0.00035	0.0022	13793.372	6.4E-06	90.16

where

I = Intensity

P = Acoustic pressure amplitude

P_{rms} = RMS acoustic pressure amplitude

ξ_0 = Particle displacement amplitude

U = Particle velocity amplitude

A = Particle acceleration amplitude

s = Condensation amplitude

SPL = Sound pressure level

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