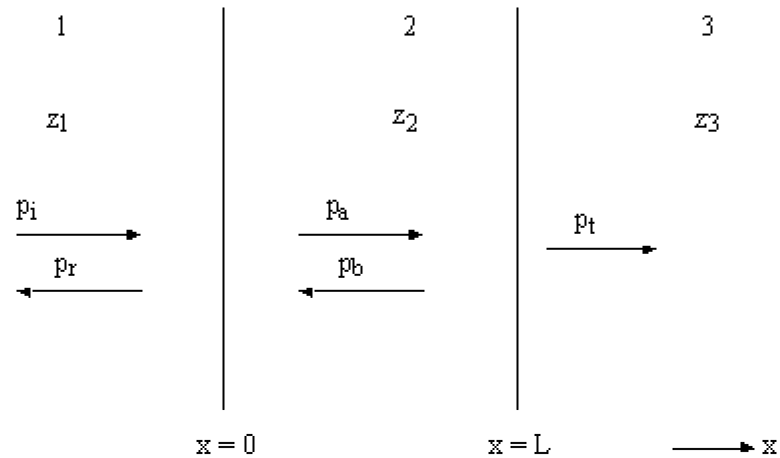
(6.3) Transmission Through a Layer: Normal Incidence

Applicable to:

(1) Effect of walls, barriers, etc.
(2) Separation layers $\rightarrow$ acoustic windows
(3) Matching layers



For pulses short compared to $2L/c_2$ this looks like two separate interfaces. For continuous wave or much longer pulses the multiple reflections set up a steady-state reflection from and transmission through layer 2.

Incident wave – $p_i = P_i e^{j(wt - k_1 x)}$

Reflected wave – $p_r = P_r e^{j(wt + k_1 x)}$

Positive and Neg. going waves in Medium 2 $\begin{cases} p_a = A e^{j(wt - k_2 x)} \\ p_b = B e^{j(wt + k_2 x)} \end{cases}$

Transmitted wave – $p_t = P_t e^{j(wt - k_3 x)}$

Apply two B.C. (continuity of pressure and particle velocity) at both interfaces.

B.C. #1 $\left.(p_i + p_r)\right|_{x=0} = \left.(p_a + p_b)\right|_{x=0}$

$$\boxed{P_i + P_r = A + B}$$

B.C. #2 $\left.(u_i + u_r)\right|_{x=0} = \left.(u_a + u_b)\right|_{x=0}$

$$\frac{P_i - P_r}{z_1} = \frac{A - B}{z_2}$$

<u>B.C. #3</u> $\left.(p_a + p_b)\right|_{x=L} = \left.p_t\right|_{x=L}$

$$A e^{-jk_2L} + B e^{jk_2L} = P_t e^{-jk_3L}$$

<u>B.C. #4</u> $\left.(u_a + u_b)\right|_{x=L} = \left.u_t\right|_{x=L}$

$$\frac{A e^{-jk_2L} - B e^{jk_2L}}{z_2} = \frac{P_t}{z_3} e^{-jk_3L}$$

From four equations we can get ratios to P_i again, as for single interface.

$$R = \frac{P_r}{P_i} = \frac{\left(1 - \frac{z_1}{z_3}\right)\cos k_2 L + j\left(\frac{z_2}{z_3} - \frac{z_1}{z_2}\right)\sin k_2 L}{\left(1 + \frac{z_1}{z_3}\right)\cos k_2 L + j\left(\frac{z_2}{z_3} + \frac{z_1}{z_2}\right)\sin k_2 L}$$

$R_I = R_p = \frac{I_r}{I_i} = |R|^2$ [Note: R is in general complex.]

$R_I + T_I = 1$ Conservation of energy

$$T_I = \frac{4}{2 + \left(\frac{z_3}{z_1} + \frac{z_1}{z_3}\right)\cos^2 k_2 L + \left(\frac{z_2^2}{z_1 z_3} + \frac{z_1 z_3}{z_2^2}\right)\sin^2 k_2 L}$$

Let’s look at Special cases of interest

Case 1: $z_1 = z_3$

For this case

$$T_I = \frac{1}{1 + \frac{1}{4}\left(\frac{z_2}{z_1} - \frac{z_1}{z_2}\right)^2 \sin^2 k_2 L}$$

If $z_2 >> z_1$

$$T_I \approx \frac{1}{1 + \frac{1}{4}\left(\frac{z_2}{z_1}\right)^2 \sin^2 k_2 L}$$

For $\sin k_2 L \approx 0$

(1) For $k_2 L << 1$ $\quad T_I \approx 1$

or (2) For $k_2 L = 0, p, 2p, \ldots np$

or $L = n\frac{l}{2}$ for $n = 0, 1, 2, \ldots$

Then $T_I = 1$ and medium 2 acts as an acoustic window.

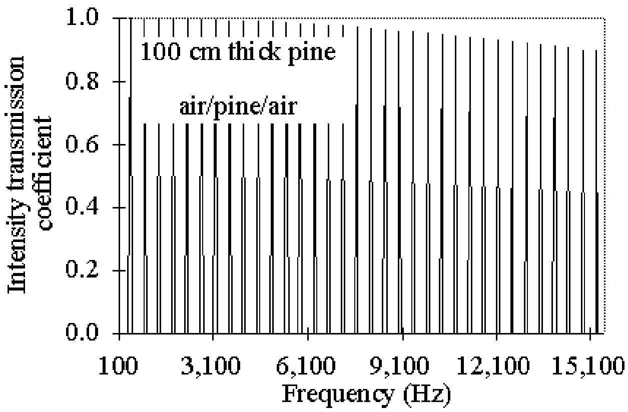
************************ Example 6.4 **************************

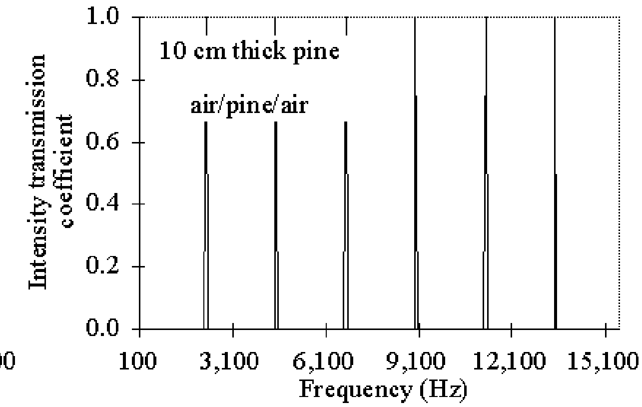
Consider what happens when we have a slab of pine of different thicknesses surrounded by air.

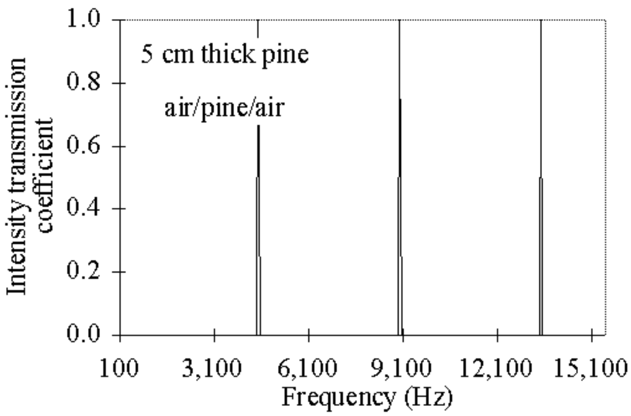
For air, $z_1 = z_3 = z_{air} = 415\text{rayls}$,
For pine, $z_2 = z_{pine} = 1.575\text{Mrayls}$ and $c_2 = c_{pine} = 3500\text{m/s}$.
In this case we note that $z_2 \gg z_1$ so we use

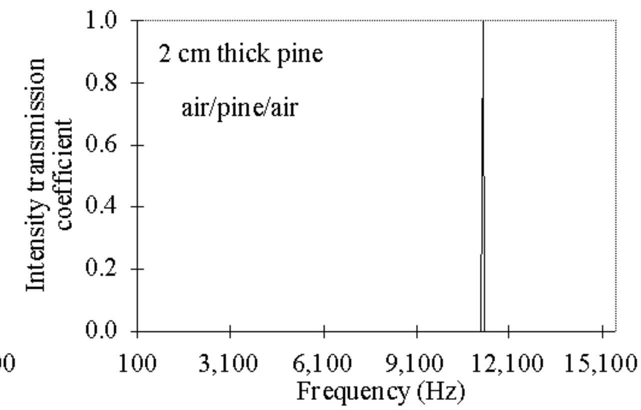
$$T_I \approx \frac{1}{1 + \frac{1}{4}\left(\frac{z_2}{z_1}\right)^2 \sin^2 k_2 L} = \frac{1}{1 + 3.6e^6 \sin^2 k_2 L}.$$

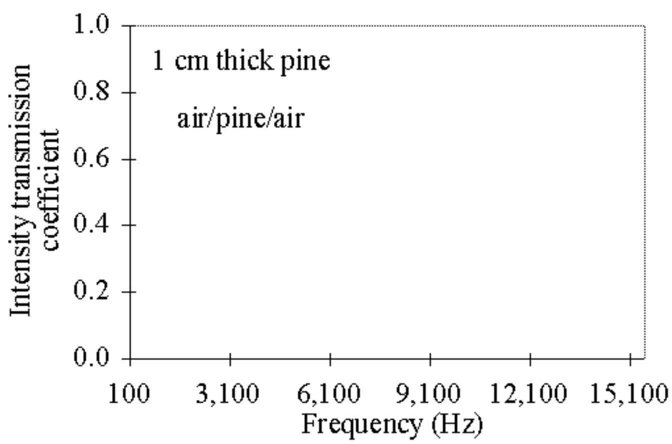
You can see from the solution that the Intensity Transmission Coefficient is going to be essentially zero unless $\sin k_2 L \approx 0$. Let's look at the plots of the Intensity Transmission Coefficient versus frequency for different pine of variable thickness.











It is interesting to note that when $k_2L = 0, \pi, 2\pi, \ldots\ n\pi,$ the transmission coefficient acts as if there isn't even a substance in between media 1 and 3.

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Case 2: $z_1 \neq z_3$

Take $k_2L << 1$ again or $L = n\dfrac{\lambda}{2}$

Then

$$T_I = \frac{4z_1z_3}{\left(z_1+z_3\right)^2}$$

In effect, a thin membrane may act as a barrier between two fluids or gases and not have any effect at particular frequencies and will appear to be just two media separated by one interface.

Case 3: Matching layer

If $k_2L = (2n-1)\dfrac{p}{2}$ $\quad for\ n = 1,\ 2,\ \ldots$ then $\qquad L = (2n-1)\dfrac{l_2}{4}$ odd multiples of a quarter wavelength. This means that $\cos k_2L = 0$ and $\sin k_2L = 1$ and

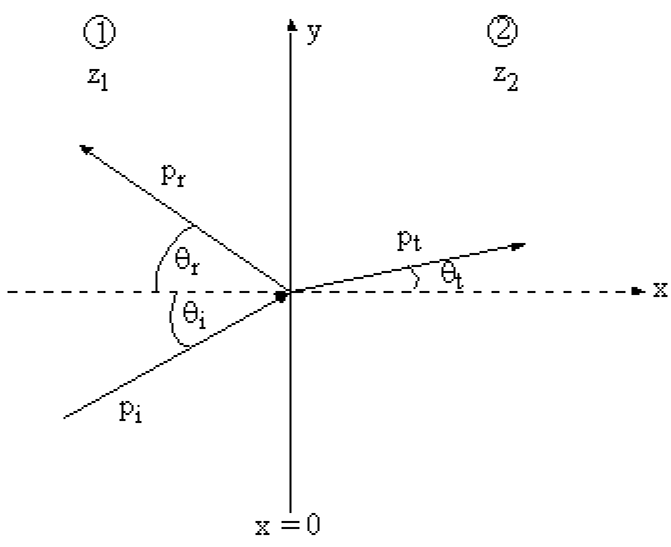
$$T_I = \frac{4z_1z_3}{\left(z_2 + \frac{z_1z_3}{z_2}\right)^2}$$

Further, if $z_2 = \sqrt{z_1z_3}$ then

$$T_I = \frac{4z_2^2}{\left(z_2 + \frac{z_2^2}{z_2}\right)^2} = \frac{4z_2^2}{\left(2z_2\right)^2} = 1.$$

So, this means that if you have an acoustic window that is an odd multiple of a quarter wavelength and $z_2 = \sqrt{z_1z_3}$ you will have 100% acoustic transmission. Of course this means that you only get the 100% transmission for selective frequency bandwidths.

6.4 Transmission from One Fluid to Another: Oblique Incidence



Incident wave: $p_i = P_i\, e^{j(\omega t - \vec{k}_1 \bullet \vec{r})}$

$$= P_i\, e^{j[\omega t - k_1(\cos\theta_i \hat{x} + \sin\theta_i \hat{y})\bullet(x\hat{x} + y\hat{y} + z\hat{z})]}$$

$$= P_i\, e^{j(\omega t - k_1 \cos\theta_i x - k_1 \sin\theta_i y)}$$

Reflected wave: $p_r = P_r\, e^{j(\omega t + k_1 \cos\theta_r x - k_1 \sin\theta_r y)}$

Transmitted wave: $p_t = P_t\, e^{j(\omega t - k_2 \cos\theta_t x - k_2 \sin\theta_t y)}$

where θ_t can be complex

So, Let's again apply our boundary conditions

B.C. #1 Continuity of pressure at x = 0

$$P_i\, e^{-jk_1 y \sin\theta_i} + P_r\, e^{-jk_1 y \sin\theta_r} = P_t\, e^{-jk_2 y \sin\theta_t}$$

This must be true for any value of y (independent of y). Thus the coefficients of y must be equivalent:

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \qquad \underline{\text{Snell's law}}$$

Snell's law gives us a couple of important pieces of information: the reflected and transmitted angles

$$k_1 \sin\theta_r = k_1 \sin\theta_i \;\Downarrow\; \boxed{\theta_r = \theta_i} \qquad\qquad k_2 \sin\theta_t = k_1 \sin\theta_i \;\Downarrow\; \boxed{\frac{\sin\theta_t}{c_2} = \frac{\sin\theta_i}{c_1}}$$

Since $P_i\, e^{-jk_1 y \sin\theta_i} + P_r\, e^{-jk_1 y \sin\theta_r} = P_t\, e^{-jk_2 y \sin\theta_t}$ must be true for all y then

$$\boxed{\begin{array}{l} P_i + P_r = P_t \\ \text{or} \quad 1 + R = T \end{array}} \qquad \text{(I)}$$

B.C. #2 Continuity of <u>normal component</u> of particle velocity at x = 0

$$u_i \cos\theta_i + u_r \cos\theta_r = u_t \cos\theta_t$$

$$\frac{P_i}{z_1}\cos\theta_i - \frac{P_r}{z_1}\cos\theta_r = \frac{P_t}{z_2}\cos\theta_t$$

We can view this as similar to form for normal incidence if we consider

$$z_i = \frac{z_1}{\cos\theta_i}, \quad z_r = -\frac{z_1}{\cos\theta_r}, \text{ and } \quad z_t = \frac{z_2}{\cos\theta_t}$$

and remembering that $\theta_r = \theta_i$ then we have

$$\boxed{1 - R = \frac{z_1}{z_2}\frac{\cos\theta_t}{\cos\theta_i}T} \qquad \text{(II)}$$

Combining (I) and (II) gives:

$$1 - R = \frac{z_1}{z_2}\frac{\cos\theta_t}{\cos\theta_i}(1 + R)$$

Solve for *R*, called the <u>Rayleigh Reflection Coefficient</u> gives:

$$R = \frac{1 - \dfrac{z_1 \cos\theta_t}{z_2 \cos\theta_i}}{1 + \dfrac{z_1 \cos\theta_t}{z_2 \cos\theta_i}} = \frac{z_2 \cos\theta_i - z_1 \cos\theta_t}{z_2 \cos\theta_i + z_1 \cos\theta_t} = \frac{z_2/z_1 - \dfrac{\cos\theta_t}{\cos\theta_i}}{z_2/z_1 + \dfrac{\cos\theta_t}{\cos\theta_i}}$$

Using $T = 1 + R$ as before gives:

$$T = 1 + R = \frac{\dfrac{2z_2}{\cos\theta_t}}{\left(\dfrac{z_2}{\cos\theta_t} + \dfrac{z_1}{\cos\theta_i}\right)}.$$

Likewise we can define the Intensity Reflection and Transmission Coefficients for oblique incidence:

$$R_I = R^2$$

$$T_I = \frac{z_1}{z_2}|T|^2 = \frac{4\dfrac{z_1 z_2}{\cos^2\theta_t}}{\left(\dfrac{z_2}{\cos\theta_t} + \dfrac{z_1}{\cos\theta_i}\right)^2}$$

and we can define the Power Reflection and Transmission Coefficients for oblique incidence:

$$R_p = R_I$$

but in general for oblique incidence $T_\pi \neq T_I$ so:

$$T_p = \frac{\cos\theta_t}{\cos\theta_i} T_I = \frac{4\frac{z_1 z_2}{\cos\theta_t \cos\theta_i}}{\left(\frac{z_2}{\cos\theta_t} + \frac{z_1}{\cos\theta_i}\right)^2} = \frac{4\frac{z_2 \cos\theta_t}{z_1 \cos\theta_i}}{\left(\frac{z_2}{z_1} + \frac{\cos\theta_t}{\cos\theta_i}\right)^2}$$

As a sanity check, what happens to each of the reflection and transmission coefficients when $\theta_i \to 0$, that is, back to normal incidence:

$$R = \frac{z_2\cos\theta_i - z_1\cos\theta_t}{z_2\cos\theta_i + z_1\cos\theta_t} \to \frac{z_2 - z_1}{z_2 + z_1}$$

$$T = \frac{2z_2\cos\theta_i}{z_2\cos\theta_i + z_1\cos\theta_t} \to \frac{2z_2}{z_1 + z_2}$$

$$R_I = \left(\frac{z_2\cos\theta_i - z_1\cos\theta_t}{z_2\cos\theta_i + z_1\cos\theta_t}\right)^2 \to \left(\frac{z_2 - z_1}{z_2 + z_1}\right)^2$$

$$T_I = \frac{4z_1 z_2\cos^2\theta_i}{(z_2\cos\theta_i + z_1\cos\theta_t)^2} \to \frac{4z_1 z_2}{(z_1 + z_2)^2}$$

$$R_p = \left(\frac{z_2\cos\theta_i - z_1\cos\theta_t}{z_2\cos\theta_i + z_1\cos\theta_t}\right)^2 \to \left(\frac{z_2 - z_1}{z_2 + z_1}\right)^2$$

$$T_p = \frac{4z_1 z_2\cos\theta_i\cos\theta_t}{(z_2\cos\theta_i + z_1\cos\theta_t)^2} \to \frac{4z_1 z_2}{(z_1 + z_2)^2}$$

so we recover the equation for normal incidence back.

Let's consider 5 cases for oblique incidence:

Case 1: $\frac{z_2}{z_1} \to \infty$

$$R = \frac{z_2\cos\theta_i - z_1\cos\theta_t}{z_2\cos\theta_i + z_1\cos\theta_t} = \frac{\left(z_2/z_1\right)\cos\theta_i - \cos\theta_t}{\left(z_2/z_1\right)\cos\theta_i + \cos\theta_t} \to 1$$

$$T = \frac{2z_2\text{Cos}\theta_i}{z_2\text{Cos}\theta_i + z_1\text{Cos}\theta_t} = \frac{2\left(z_2/z_1\right)\text{Cos}\theta_i}{\left(z_2/z_1\right)\text{Cos}\theta_i + \text{Cos}\theta_t} \to 2$$

$$R_I = \left(\frac{z_2\cos\theta_i - z_1\cos\theta_t}{z_2\cos\theta_i + z_1\cos\theta_t}\right)^2 = \left(\frac{\left(z_2/z_1\right)\cos\theta_i - \cos\theta_t}{\left(z_2/z_1\right)\cos\theta_i + \cos\theta_t}\right)^2 \to 1$$

$$T_I = \frac{4z_1 z_2 \cos^2 q_i}{\left(z_2 \cos q_i + z_1 \cos q_t\right)^2} = \frac{4\left(z_2 / z_1\right)\cos^2 q_i}{\left(\left(z_2 / z_1\right)\cos q_i + \cos q_t\right)^2} \to 0$$

Case 2: $\frac{z_2}{z_1} \to 0$

$$R = \frac{z_2 \cos q_i - z_1 \cos q_t}{z_2 \cos q_i + z_1 \cos q_t} = \frac{\left(z_2 / z_1\right)\cos q_i - \cos q_t}{\left(z_2 / z_1\right)\cos q_i + \cos q_t} \to -1$$

$$T = \frac{2z_2 \cos q_i}{z_2 \cos q_i + z_1 \cos q_t} = \frac{2\left(z_2 / z_1\right)\cos q_i}{\left(z_2 / z_1\right)\cos q_i + \cos q_t} \to 0$$

$$R_I = \left(\frac{z_2 \cos q_i - z_1 \cos q_t}{z_2 \cos q_i + z_1 \cos q_t}\right)^2 = \left(\frac{\left(z_2 / z_1\right)\cos q_i - \cos q_t}{\left(z_2 / z_1\right)\cos q_i + \cos q_t}\right)^2 \to 1$$

$$T_I = \frac{4z_1 z_2 \cos^2 q_i}{\left(z_2 \cos q_i + z_1 \cos q_t\right)^2} = \frac{4\left(z_2 / z_1\right)\cos^2 q_i}{\left(\left(z_2 / z_1\right)\cos q_i + \cos q_t\right)^2} \to 0$$

Case 3: $z_2 = z_1$

$$R = \frac{z_2 \cos q_i - z_1 \cos q_t}{z_2 \cos q_i + z_1 \cos q_t} = \frac{cosq_i - cosq_t}{cosq_i + cosq_t}$$

$$T = \frac{2z_2 \cos q_i}{z_2 \cos q_i + z_1 \cos q_t} = \frac{2\cos q_i}{\cos q_i + \cos q_t}$$

$$R_I = \left(\frac{z_2 \cos q_i - z_1 \cos q_t}{z_2 \cos q_i + z_1 \cos q_t}\right)^2 = \left(\frac{\cos q_i - \cos q_t}{\cos q_i + \cos q_t}\right)^2$$

$$T_I = \frac{4z_1 z_2 \cos^2 q_i}{\left(z_2 \cos q_i + z_1 \cos q_t\right)^2} = \frac{4\cos^2 q_i}{\left(\cos q_i + \cos q_t\right)^2}$$

For $z_2 = z_1$, that is, $r_2 c_2 = r_1 c_1$, in order for

$$R_I = R_p = \left(\frac{\cos q_i - \cos q_t}{\cos q_i + \cos q_t}\right)^2 = 0$$ (which you would expect with no real interface)

$$T_I = \frac{4\cos^2 q_i}{\left(\cos q_i + \cos q_t\right)^2} = 1$$

the propagation speeds in both media must also be the <u>same</u>, that is, $c_2 = c_1$, which, from Snell's Law, yields $q_i = q_t$.

Case 4: $q_t = 90^o$ when $q_i < 90^o$ (Critical Angle)

Important is the fact that at the critical angle $\cos q_t = \cos 90^\circ = 0$ so

$$R = \frac{z_2 \cos q_i - z_1 \cos q_t}{z_2 \cos q_i + z_1 \cos q_t} = 1$$

$$T = \frac{2 z_2 \cos q_i}{z_2 \cos q_i + z_1 \cos q_t} = 2$$

$$R_I = \left(\frac{z_2 \cos q_i - z_1 \cos q_t}{z_2 \cos q_i + z_1 \cos q_t} \right)^2 = 1$$

$$T_I = \frac{4 z_1 z_2 \cos^2 q_i}{\left(z_2 \cos q_i + z_1 \cos q_t\right)^2} = 4\left(z_1 / z_2 \right)$$

From Snell's Law,

$$\frac{c_1}{\sin q_i} = \frac{c_2}{\sin q_t} = \frac{c_2}{\sin 90^o} = c_2$$

so we define the critical angle from

$$\sin q_i = \sin q_c = \frac{c_1}{c_2}$$

Therefore, the critical angle q_c is

$$q_c = \sin^{-1}\left(\frac{c_1}{c_2} \right) \text{ where } c_1 < c_2$$

Consider

$$c_1 = 1130\text{m/s} \text{ and } c_2 = 1600\text{m/s}$$

$$q_c = \sin^{-1}\left(\frac{c_1}{c_2} \right) = \sin^{-1}\left(\frac{1130\text{m/s}}{1600\text{m/s}} \right) = 44.9^o \text{ is the critical angle.}$$

Now, what if $q_i = 50^o$

$$\sin q_t = \frac{c_2}{c_1} \sin q_i = \frac{1600\text{m/s}}{1130\text{m/s}} \sin 50^o = 1.085$$

Let's assume that $q_t = q_{RE} + j q_{IM}$ (q_t is complex).
Then

$$\sin\theta_t = \sin(\theta_{RE} + j\theta_{IM}) = \sin(\theta_{RE})\cos(j\theta_{IM}) + \cos(\theta_{RE})\sin(j\theta_{IM})$$
$$= \sin(\theta_{RE})\cosh(\theta_{IM}) + j\cos(\theta_{RE})\sinh(\theta_{IM}) = 1.085$$

Comparing real to real and imaginary to imaginary then:

$$\sin(\theta_{RE})\cosh(\theta_{IM}) = 1.085$$
$$\cos(\theta_{RE})\sinh(\theta_{IM}) = 0.$$

So

$$\cos(\theta_{RE})\sinh(\theta_{IM}) = 0 \text{ when } \theta_{RE} = \frac{\pi}{2},\ldots \text{ or } \theta_{IM} = 0$$

When $\theta_{RE} = \frac{\pi}{2}$ and $\theta_{IM} \neq 0$,

$$\sin\left(\frac{\pi}{2}\right)\cosh(\theta_{IM}) = \cosh(\theta_{IM}) = 1.085$$
$$\theta_{IM} = \cosh^{-1}(1.085) \approx 0.41$$
$$\theta_t = \theta_{RE} + j\theta_{IM} = 1.57 + j0.41 \quad \text{(rad)}$$

Case 5: Angle of intromission

Taking the Power Transmission Coefficient and rearranging (multiply by $(z_1\cos\theta_i)^{-2}$) gives:

$$T_p = \frac{4z_1z_2\cos\theta_i\cos\theta_t}{(z_2\cos\theta_i + z_1\cos\theta_t)^2} \times \frac{\dfrac{1}{(z_1\cos\theta_i)^2}}{\dfrac{1}{(z_1\cos\theta_i)^2}} = \frac{4\left(\dfrac{z_2}{z_1}\right)\left(\dfrac{\cos\theta_t}{\cos\theta_i}\right)}{\left(\dfrac{z_2}{z_1} + \dfrac{\cos\theta_t}{\cos\theta_i}\right)^2}$$

Note that when $\frac{z_2}{z_1} = \frac{\cos\theta_t}{\cos\theta_i}$, $T_p = 1$; therefore, $R_p = 0$.

Combining the condition $\frac{z_2}{z_1} = \frac{\cos\theta_t}{\cos\theta_i}$ with Snell's law $\frac{c_1}{\sin\theta_i} = \frac{c_2}{\sin\theta_t}$ yields for the angle of incidence for which there is no reflected power and, therefore, complete transmission of power (EXERCISE for the student in homework: Solve for Eq. 6.4.18):

$$\sin\theta_I = \sqrt{\frac{\left(\dfrac{z_2}{z_1}\right)^2 - 1}{\left(\dfrac{z_2}{z_1}\right)^2 - \left(\dfrac{c_2}{c_1}\right)^2}} = \sqrt{\frac{1 - \left(\dfrac{z_1}{z_2}\right)^2}{1 - \left(\dfrac{\rho_1}{\rho_2}\right)^2}} \quad \text{(Eq 6.4.18)}$$

where θ_I is called the <u>angle of intromission</u>

Following are some examples (corrected from versions in the text) of the pressure reflection coefficient for various ratios of propagation speed and characteristic impedance.

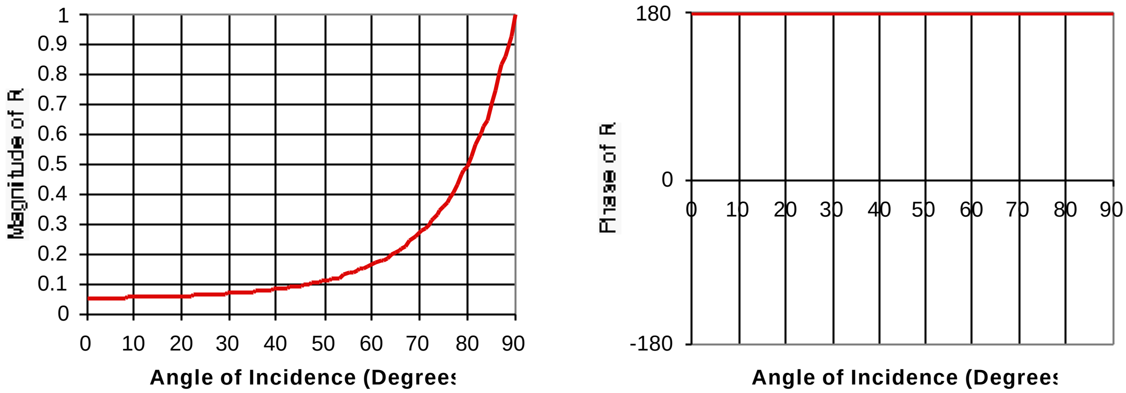


Figure 6.11 Magnitude and phase of the pressure reflection coefficient with $c_2/c_1 = 0.9$ and $z_2/z_1 = 0.9$.

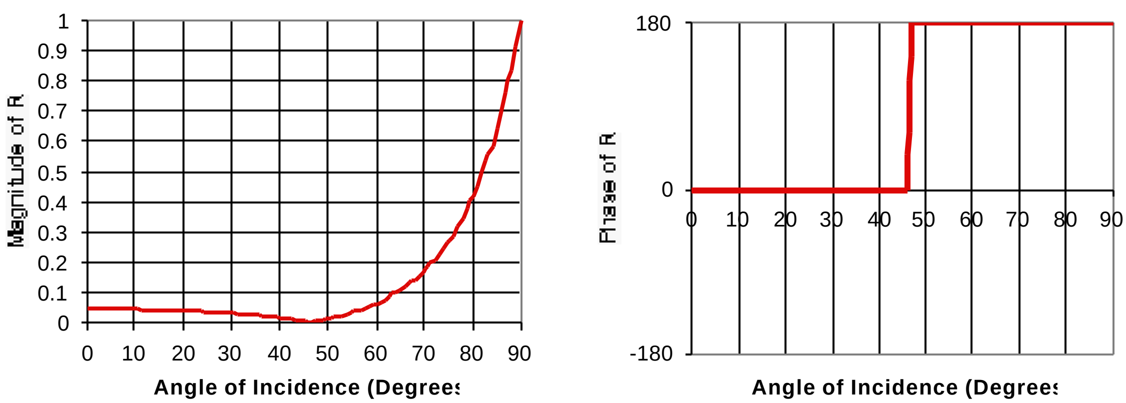


Figure 6.12 Magnitude and phase of the pressure reflection coefficient with $c_2/c_1 = 0.9$ and $z_2/z_1 = 1.1$. Note the angle of intromission at 46.4°.

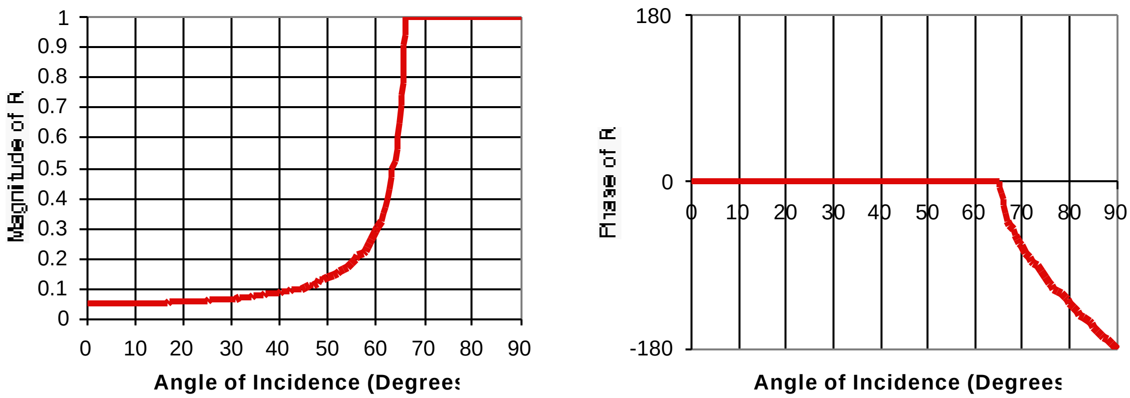


Figure 6.13 Magnitude and phase of the pressure reflection coefficient with $c_2/c_1 = 1.1$ and $z_2/z_1 = 1.1$. Note the critical angle at 65.4°.

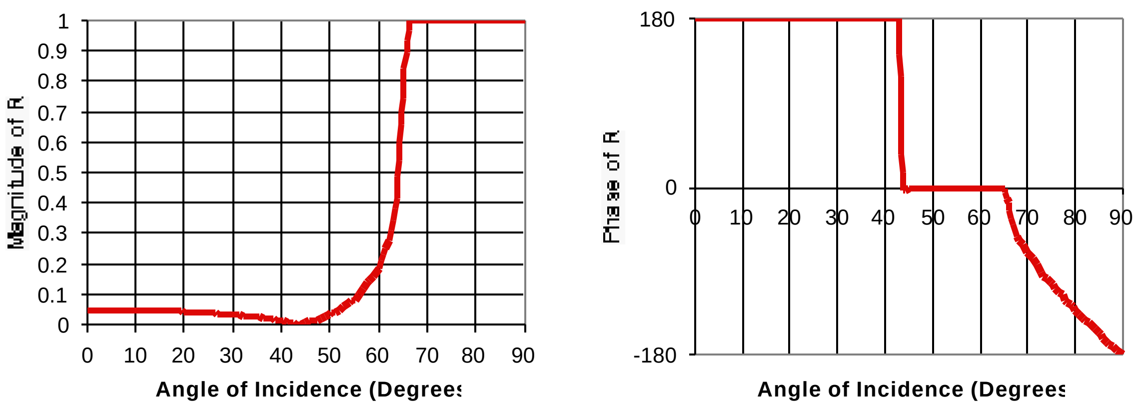


Figure 6.14 Magnitude and phase of the pressure reflection coefficient with $c_2/c_1 = 1.1$ and $z_2/z_1 = 0.9$. Note the angle of intromission at 43.2° and critical angle at 65.4°.

************************** Example 6.5 ***************************

Confirming that the angle of intromission is 46.4° in Fig. 6.12.

cos(46.4°) = 0.6896

$$\sin\theta_t = \frac{c_2}{c_1}\sin\theta_i = 0.9\ \sin(46.4^\circ) = 0.65175$$

$$\theta_t = 40.67^\circ$$
$$\cos\theta_t = 0.758$$

$$\frac{\cos\theta_i}{\cos\theta_t} = \frac{0.758}{0.6896} \cong 1.1$$

$$R = \frac{\frac{z_2}{\cos\theta_t} - \frac{z_1}{\cos\theta_i}}{\frac{z_2}{\cos\theta_t} + \frac{z_1}{\cos\theta_i}} = \frac{\frac{z_2}{z_1} - \frac{\cos\theta_t}{\cos\theta_i}}{\frac{z_2}{z_1} + \frac{\cos\theta_t}{\cos\theta_i}}$$

but $z_2/z_1 = 1.1$ so

$$R = 0$$

and we have complete transmission!

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