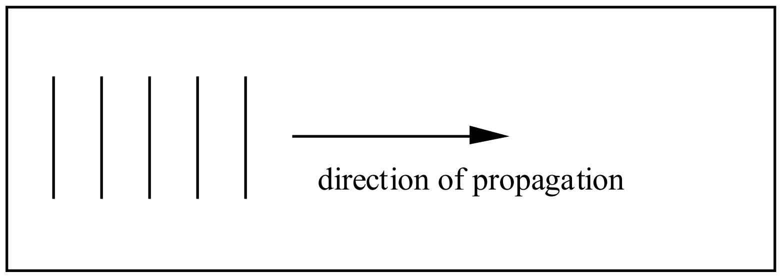
(Appendix 1) Solids

We will divide our discussion of waves in solids into two different conditions: (1) bulk propagation, (2) bar propagation. Each of these will be considered in order below.

(1) Propagation of bulk waves in a solid. Bulk waves have the material extending laterally beyond the limits of the acoustic beam (as seen in the figure below), or the beam is many wavelengths wide.



There are two different types of bulk waves. One is a longitudinal or compressional wave and the other is a transverse or shear wave. The longitudinal wave is the same type of wave that propagates in fluids. The particle motion is in the direction of propagation and there are regions of compression and regions of rarefraction associated with the wave as for fluids, although the expression for the wave propagation speed is different as seen below. The shear wave on the other hand has particle motion transverse to the direction of propagation and a propagation speed that is dependent on the shear modulus of elasticity, *G*. These waves have shear stress associated with them but no regions of compression of the solid.

The propagation speeds of these two waves are as follows:

Longitudinal wave speed,

$$c_L = \sqrt{\frac{B + \frac{4}{3} G}{\rho_o}} ,$$

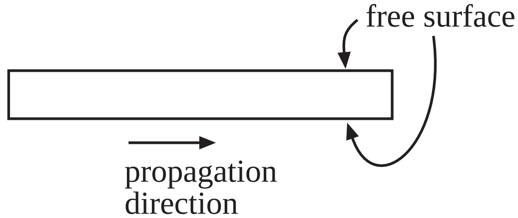
where *B* is the bulk modulus of elasticity that has been used previously and, as noted above, *G* is the shear modulus of elasticity. What are the units of *B* and *G*?
Note that *G* is zero in fluids so that we get the previous expression for speed of propagation in fluids.

Shear wave speed,

$$c_S = \sqrt{\frac{G}{\rho_o}} ,$$

where all parameters have been defined previously. Note that the shear wave speed is always less than the longitudinal wave speed.

(2) Propagation of bar waves in a solid. Bar waves are waves that are propagating in a solid with lateral dimensions smaller than the lateral beam width (see figure below) and with free boundary conditions. Thus the pressure is zero at the lateral boundary of the material.



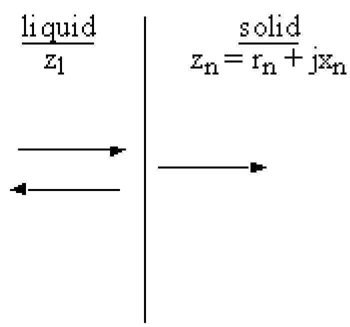
The speed of propagation for this wave is

$$c_{bar} = \sqrt{\frac{Y}{\rho_o}},$$

where Y is Young's modulus of elasticity. The Young's modulus relates the stress in a material to its strain. Even though this wave is a compressional wave its speed is less than the speed of a longitudinal bulk wave because the boundary conditions are different. In the case of the bar, the material can move laterally as well as along the direction of propagation which changes the effective elasticity to Young's modulus.

Reflection at a solid - normal incidence

Lossy solid: (ex. acoustic tile or ocean bottom)
normal specific acoustic impedance is continuous at the boundary.



$$z_n = \frac{p}{\vec{u} \bullet \hat{n}} = r_n + jx_n$$

Continuity of z_n @ $x = 0$

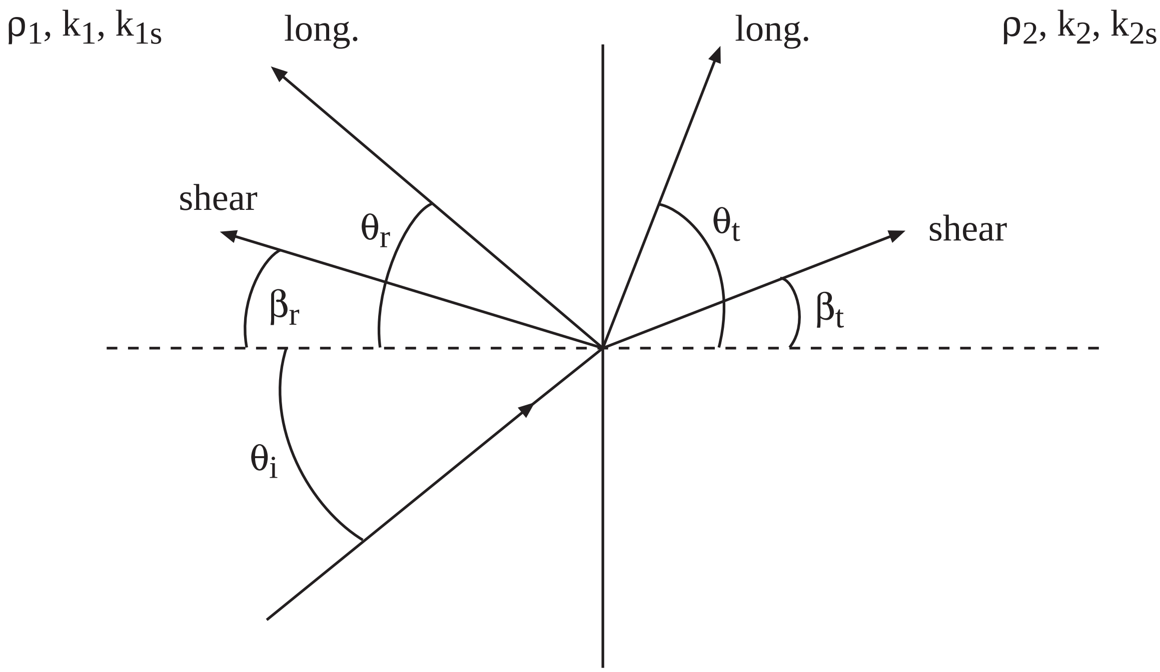
$$\tilde{R} = \frac{(r_n - z_1) + jx_n}{(r_n + z_1) + jx_n}$$

Note: $\tilde{R}$ is complex.

$$R_\pi = R_I = \frac{(r_n - z_1)^2 + x_n^2}{(r_n + z_1)^2 + x_n^2}$$

$$T_\pi = T_I = \frac{4 z_1 r_n}{(r_n + z_1)^2 + x_n^2}$$

Reflection at a solid-solid interface – oblique incidence



4 Boundary Conditions
- Continuity of normal force (stress)
- Continuity of tangential force (stress)
- Continuity of normal $\vec{u}$
- Continuity of tangential $\vec{u}$

Snell's law applies and we get the relations derived previously plus the following 2 additional equations.

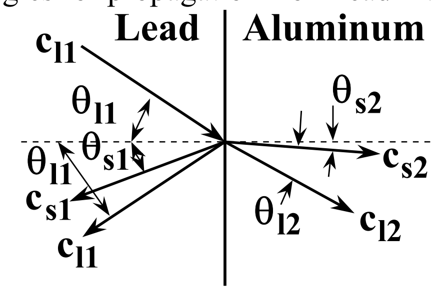
$$k_{2s} \sin \beta_t = k_1 \sin \theta_i$$
$$k_{1s} \sin \beta_r = k_1 \sin \theta_i$$

Critical angles (2 possible)

If $c_{L2} > c_{L1}$, then there is a critical angle for the transmitted longitudinal wave
If $c_{s2} > c_{L1}$, then there is a critical angle for the transmitted shear wave

**************************** Example 6.6 ****************************

Critical angles for propagation from lead into aluminum.



Lead	Aluminum
$\rho_0 = 11,300$	$\rho_0 = 2,700$
$c_{L1} = 2,050$	$c_{L2} = 6,300$
$c_{s1} = 698$	$c_{s2} = 2,980$

If $\theta_i = 10^\circ$ $\qquad$ $\theta_r = 10^\circ$

$$\beta_r = \sin^{-1}\left(\frac{c_{s1}}{c_{L1}}\sin\theta_i\right) = 3.4^\circ$$

$$\theta_t = \sin^{-1}\left(\frac{c_{L2}}{c_{L1}}\sin\theta_i\right) = 32^\circ$$

$$\beta_t = \sin^{-1}\left(\frac{c_{s2}}{c_{L1}}\sin\theta_i\right) = 14.6^\circ$$

Critical angles

$$\theta_{i_{L\ critical}} = \sin^{-1}\left(\frac{c_{L1}}{c_{L2}}\right) = 19^\circ$$

$$\theta_{i_{s\ critical}} = \sin^{-1}\left(\frac{c_{L1}}{c_{s2}}\right) = 43.5^\circ$$

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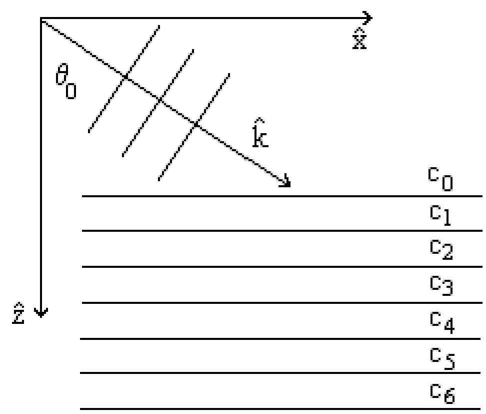
(Appendix 2) Thermoclines



Suppose you have a plane wave obliquely incident on an interface between two fluid media whose only appreciable difference is in the speed of sound (see above figure). According to Snell's Law

$$\frac{\sin\theta_0}{c_0} = \frac{\sin\theta_1}{c_1}.$$

The transmitted wave will propagate through the first layer at an angle θ_1. After the transmitted wave propagates the length of the first layer it is incident on another layer so that

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$

Comparing with the first relation we see that

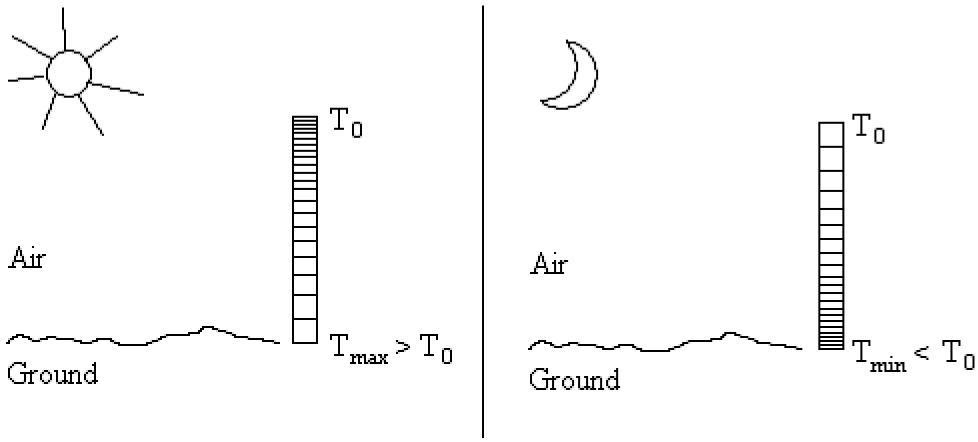
$$\frac{\sin\theta_0}{c_0} = \frac{\sin\theta_2}{c_2}$$

so that in general

$$\frac{\sin\theta_0}{c_0} = \frac{\sin\theta_n}{c_n}$$

where *n* denotes a layer in general.

One way that this sort of process can occur is when you have thermoclines. A thermocline is defined as a temperature gradient in some medium. For example, when the sun heats up the ground in the morning, the ground readily absorbs the heat while the air above the ground heats up less rapidly. This sets up a temperature gradient from the hot ground to the ambient temperature of the surrounding air because the temperature must be continuous at the ground. The inverse happens at night when the ground quickly looses its heat and the air above it has a less gradual heat loss. This sets up a thermocline inverted to that from the heat of the day. This is illustrated by the following figure.



From the perfect gas in an adiabatic process we saw that

$$c = \sqrt{\rho r T_k}\ .$$

If the temperature is a function of the level above the ground then

$$T = T(z) \qquad \rightarrow \qquad c = c(z).$$

*************************** Example 6.7 ***************************

Consider what happens when you have a plane wave propagating towards the ground during the day at an incidence angle of θ_0 where there is a slight thermocline starting at 1 meter above the ground and increasing in temperature until the wave bounces from the ground. Suppose the thermocline can be described by

$$T(z) = T_0 + m_T z$$

where T_g is the temperature of the ground at z = 1 m and the temperature at 1 m above the ground (z = 0) is T_0, m_T is the slope of the temperature gradient (assumed to be linear). Using our temperature dependent function for speed

$$c(z) = \sqrt{\rho r T(z)} \qquad \text{and} \qquad c_0 = \sqrt{\rho r T_0}$$

assuming the thermocline is not large relative to the ambient temperature (in Kelvin) and that the density remains fairly constant then

$$\frac{c(z) = \sqrt{\rho r T(z)}}{c_0 = \sqrt{\rho r T_0}}$$

or

$$c(z) = c_0\sqrt{\frac{T(z)}{T_0}} = c_0\sqrt{\frac{T_0 + m_T z}{T_0}} = c_0(1 + Bz)^{1/2}$$

where $B=\frac{m_T}{T_0}$. For a small thermocline, *B* is small and therefore

$$c(z)\simeq c_0\left(1+\frac{1}{2}Bz\right).$$

If we think of the thermocline as just a bunch of small layers then from Snell's law

$$\frac{\sin\theta_0}{c_0}=\frac{\sin\theta_z}{c(z)}.$$

Solving for $\sin\theta_z$ gives

$$\sin\theta_z=\frac{c(z)}{c_0}\sin\theta_0=\left(1+\frac{1}{2}Bz\right)\sin\theta_0.$$

Taking the derivative yields

$$\cos\theta_z d\theta_z=\frac{1}{2}B\sin\theta_0 dz.$$

Using the trigonometric identity that

$$\cos\theta=\sqrt{1-\sin^2\theta}$$

then

$$\cos\theta_z=\sqrt{1-\sin^2\theta_z}=\sqrt{1-\left(1+\frac{1}{2}Bz\right)^2\sin^2\theta_0}\simeq\sqrt{1-(1+Bz)\sin^2\theta_0}$$
$$=\sqrt{1-\sin^2\theta_0-Bz\sin^2\theta_0}=\sqrt{\cos^2\theta_0-Bz\sin^2\theta_0}=\cos^2\theta_0\sqrt{1_0-Bz\tan^2\theta_0}$$

giving

$$\frac{d\theta_z}{dz}=\frac{1}{2}\frac{B\sin\theta_0}{\cos\theta_0\left(1-Bz\tan^2\theta_0\right)^{1/2}}=-\frac{1}{2}B\tan\theta_0\left(1-Bz\tan^2\theta_0\right)^{-1/2}$$

approximating again for small *B* gives

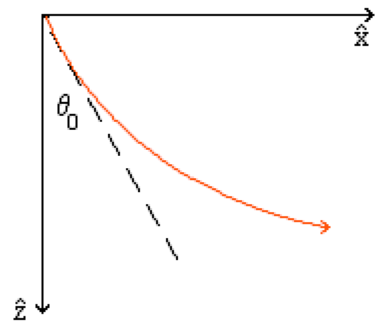
$$\frac{d\theta_z}{dz}=\frac{1}{2}B\tan\theta_0\left(1+\frac{1}{2}Bz\tan^2\theta_0\right).$$

Since *B* is small we keep only the linear terms in *B* and

$$\frac{d\theta_z}{dz}\simeq\frac{1}{2}B\tan\theta_0.$$

So the change in the angle of propagation is constant as the wave propagates towards the ground.

If we were to trace the wave as it propagates we would see it look like:



The wave refracts because the thermocline.

What would happen if the thermocline were decreasing as opposed to increasing?

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As a final example we look at the ocean and its thermoclines (chapter 15)

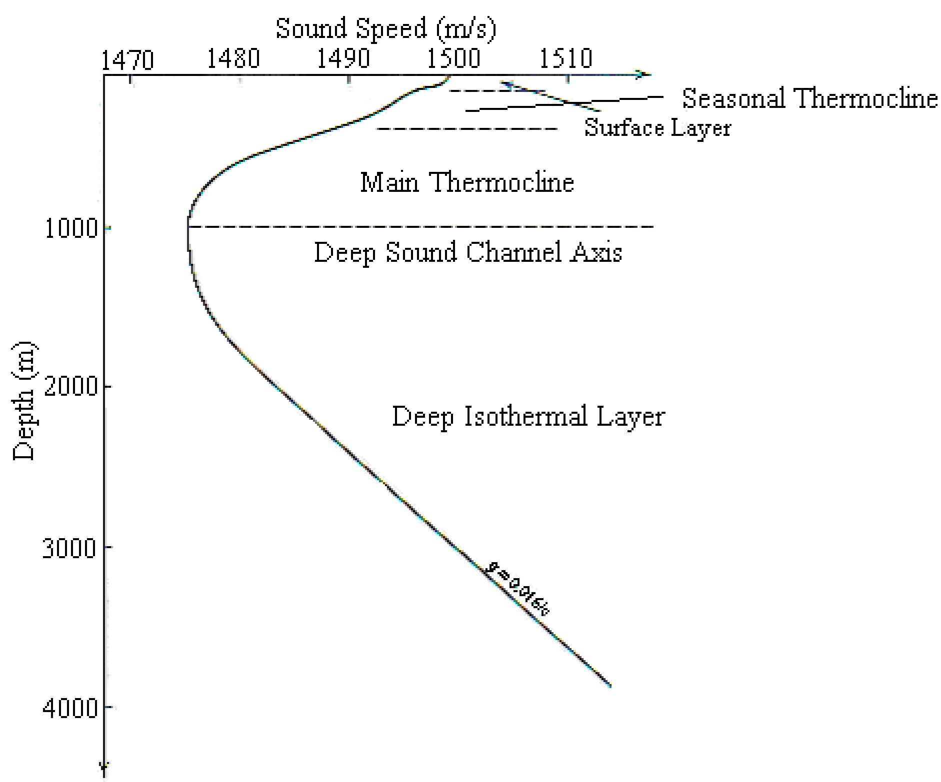

Figure 15.4.1 Representative sound-speed profile for midatlantic deep-ocean water

You can imagine the refractive behavior of sound in the ocean…