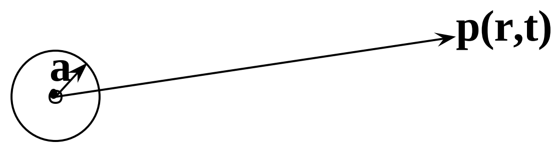
Chapter 7 – Radiation and Reception of Acoustic Waves

(7.1) Will show that a small (compared to λ) source of arbitrary shape and velocity distribution, a simple source, produces the same field as a small spherical source.

Let's consider a radially oscillating sphere of radius *a*, and surface velocity $u_s(t) = U_o e^{jwt}$



Surface displacement = $a + x(t)$ where $a >> |x(t)|$ or $a >> \frac{U_o}{w}$

Remember for spherical sources

$$\tilde{p}(r,t) = \frac{\tilde{A}}{r} e^{(jwt-kr)} \qquad \text{for diverging wave}$$

and $$\tilde{u} = \left(1 - \frac{j}{kr}\right)\frac{p}{r_o c}\hat{r}$$

where $$\tilde{u}(r,t) = \left(1 - \frac{j}{kr}\right)\frac{\tilde{A}}{r r_o c} e^{j(wt-kr)}$$

Let's determine an expression for $\tilde{A}$ at $r = a$

$$\tilde{u}(a,t) = \left(1 - \frac{j}{ka}\right)\frac{\tilde{A}}{a r_o c} e^{j(wt-ka)} = U_o e^{jwt}$$

where

$$U_o = \left(1 - \frac{j}{ka}\right)\frac{\tilde{A}}{a r_o c} e^{-jka}$$

Solving for $\tilde{A}$ yields $$\tilde{A} = \frac{a r_o c U_o e^{jka}}{\left(1 - \frac{j}{ka}\right)} = \frac{jka^2 r_o c U_o e^{jka}}{(jka+1)} = \frac{jwa^2 r_o U_o e^{jka}}{(jka+1)}$$

Therefore, the acoustic pressure at *r* when $u_s(t) = U_o e^{jwt}$ is

$$\tilde{p} = \frac{\tilde{A}}{r} e^{j(wt-kr)} = \frac{jwa^2 r_o U_o e^{jka}}{r(jka+1)} e^{j(wt-kr)} = \frac{jw r_o a^2 U_o}{r(jka+1)} e^{j(wt-k(r-a))}$$

The actual acoustic pressure measured is represented by the real part of $\tilde{p}(r,t)$, $\text{Re}\{p(r,t)\}$.

So, we want to get $\tilde{p}(r,t)$ in a form that allows us to separate out the real part:

$$p(r,t) = \frac{jw r_o a^2 U_o}{r(1+jka)} e^{j(wt-k(r-a))} \times \frac{1-jka}{1-jka}$$

gives

$$p(r,t) = \frac{(j+ka)wr_o a^2 U_o}{r\left[1-(ka)^2\right]}\left[\cos\left(wt-k(r-a)\right)+j\sin\left(wt-k(r-a)\right)\right]$$

so the real part is

$$\mathrm{Re}\left\{p(r,t)\right\} = \frac{wr_o a^2 U_o}{r\left[1-(ka)^2\right]}\left[ka\cos\left(wt-k(r-a)\right)-\sin\left(wt-k(r-a)\right)\right].$$

For $ka \ll 1$ then

$$\mathrm{Re}\left\{p(r,t)\right\} \simeq \frac{-wr_o a^2 U_o}{r}\sin\left(wt-k(r-a)\right)$$

or for the far field when $r \gg a$

$$\mathrm{Re}\left\{p(r,t)\right\} \simeq \frac{-wr_o a^2 U_o}{r}\sin\left(wt-kr\right).$$

Likewise we can define at the surface *r* = *a*, the <u>surface acoustic impedance</u> is

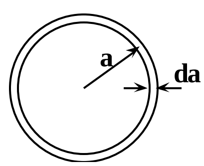
$$\mathrm{z_a} = \frac{\mathrm{j\omega\rho_o a}}{1+\mathrm{jka}}$$

For ka << 1, $z_a \simeq jwr_o a\{1-jka\}$

The text defines a term *Q* which is called “Strength of a Spherical Source” as
Q = Surface Area x Velocity Amplitude, that is,
$Q = \left(4p\,a^2\right)\left(U_o\right)$ (Eq. 7.2.10)

Note unit: $\mathrm{m}^2\left(\frac{\mathrm{m}}{\mathrm{s}}\right) = \frac{\mathrm{m}^3}{\mathrm{s}}$

Let’s look at *Q* in a bit more detail that describes the source in terms of the amount of fluid the source displaces. Take a sphere with radius *a* and volume $\frac{4}{3}p\,a^3$:



If the sphere’s radius is increased by a distance *da*, then the volume of fluid displaced is $dV = 4p\,a^2 da$. Total flow rate (fluid volume/time) is $F = \frac{dV}{dt} = 4p\,a^2\frac{da}{dt}$.

Note that $\frac{da}{dt}$ for a spherical source is $\tilde{u} = \left(1-\frac{j}{kr}\right)\frac{p}{r_o c}\hat{r}$ at *r* = *a*, that is,

$$\frac{da}{dt} = \tilde{u}(a,t) = \left(1-\frac{j}{ka}\right)\frac{\tilde{A}}{ar_o c}e^{j(wt-ka)}$$

$$F = 4p\,a^2\frac{da}{dt} = 4p\,a^2\,\frac{\tilde{A}}{a\,r_o c}\left(1-\frac{j}{ka}\right)e^{j(wt-ka)} = \frac{4p\,a\tilde{A}}{r_o c}\left(1-\frac{j}{ka}\right)e^{j(wt-ka)}$$

Work done by the source on the surrounding fluid is the product of pressure and volume change. Thus, the rate at which total power is supplied by the source is $\langle pF \rangle$, and

$$\langle pF \rangle = \frac{1}{2}\mathrm{Re}\left\{ p^{*}F \right\} = \frac{1}{2}\mathrm{Re}\left\{ \left(\frac{\tilde{A}}{a} e^{j(wt-ka)} \right)^{*} \left(\frac{4p a\tilde{A}}{r_o c}\left(1-\frac{j}{ka}\right) e^{j(wt-ka)} \right) \right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{ \frac{4p\left|\tilde{A}\right|^2}{r_o c}\left(1-\frac{j}{ka}\right) \right\} = \frac{2p\left|\tilde{A}\right|^2}{r_o c}$$

Note that the intensity at a distance *r* from the point source is $I = \dfrac{\left|\tilde{A}\right|^2}{2r^2 r_o c}$ and radiated power is

$4p r^2 I = \dfrac{2p\left|\tilde{A}\right|^2}{r_o c}$.

From volume flow rate $F = \dfrac{4p a\tilde{A}}{r_o c}\left(1-\dfrac{j}{ka}\right)e^{j(wt-ka)}$, and as the radius $a \to 0$ for fixed amplitude,

$$Q = F\left(a=0,t\right) = \frac{4p a\tilde{A}}{r_o c}\left(-\frac{j}{ka}\right)e^{jwt} = \frac{4p\tilde{A}}{jk r_o c}e^{jwt} = Q_o e^{jwt}$$

where $Q_o = \dfrac{4p\tilde{A}}{jk r_o c}$ or $\tilde{A} = \dfrac{jk r_o c Q_o}{4p}$

One reason $Q(t)$ is a good way to characterize a source is that even a source lacking spherical symmetry may still have a well defined value of $Q(t)$. Any such source will produce radiation exactly the same as would an ideal monopole source with the same value of $Q(t)$ provided that $ka \ll 1$.

For a spherical source,

$$\tilde{u}(a,t) = \frac{\tilde{A}}{a r_o c}\left(1-\frac{j}{ka}\right)e^{j(wt-ka)}$$

$$u(a\to 0,t) = \frac{\tilde{A}}{jka^2 r_o c}e^{jwt} = U_o e^{jwt}$$

$$U_o = \frac{\tilde{A}}{jka^2 r_o c} = \frac{\left(\dfrac{jk r_o c Q_o}{4p}\right)}{jka^2 r_o c} = \frac{Q_o}{4p a^2}$$

Therefore, $\tilde{p}(r,t) = \dfrac{jw r_o a^2 U_o}{r}e^{j(wt-kr)}$ becomes, for a simple source

$$\tilde{p}(r,t) = \frac{jw r_o a^2 U_o}{r}e^{j(wt-kr)} = \frac{jw r_o a^2}{r}\left(\frac{Q_o}{4p a^2}\right)e^{j(wt-kr)} = \frac{jw r_o Q_o}{4p r}e^{j(wt-kr)}$$

$$= \frac{j(kc) r_o Q_o}{4p r}e^{j(wt-kr)} = j r_o c\frac{kQ_o}{4p r}e^{j(wt-kr)} \qquad (7.2.13)$$

The intensity at a distance r from a simple source where $ka << 1$ is

$$I(r) = \frac{\left|\tilde{p}(r,t)^2\right|}{2\rho_o c} = \frac{\left(\frac{\omega \rho_o a^2 U_o}{r}\right)^2}{2\rho_o c} = \frac{\left(\rho_o c \frac{kQ_o}{4\pi r}\right)^2}{2\rho_o c} = \frac{1}{8}\rho_o c\left(\frac{Q_o}{r\lambda}\right)^2 \qquad (7.2.15)$$

Since $Q_o = 4\pi a^2 U_o$ then the intensity is proportional to the radius of the source to the fourth power. So, small sources relative to the wavelength are not very efficient and are poor radiators of acoustic energy.

The total average power is

$$\Pi = 4\pi r^2 I = 4\pi r^2\left(\frac{1}{8}\rho_o c\left(\frac{Q_o}{r\lambda}\right)^2\right) = \frac{\pi}{2}\rho_o c\left(\frac{Q_o}{\lambda}\right)^2 \qquad (7.2.16)$$

*************************** Example 7.1 ***************************

For a small source in air at 20°C for which $Q_o = 0.001\ \text{m}^3/\text{s}$ (rms) at 500 Hz, determine $I(5\ \text{m})$ and Π.

Note that λ = 0.686 m. $I(r) = \frac{1}{8}\rho_o c\left(\frac{Q_o}{r\lambda}\right)^2$

$$I(5\ \text{m}) = \frac{1}{8}(415\text{Pa-s/m})\left(\frac{0.01\sqrt{2}\ \text{m}^3/\text{s}}{(5\ \text{m})(0.686\text{m})}\right)^2 = 0.88\ \text{mW}/\text{m}^2$$

$$\Pi = \frac{\pi}{2}\rho_o c\left(\frac{Q_o}{\lambda}\right)^2 = \frac{\pi}{2}(415\ \text{Pa}-\text{s}/\text{m})\left(\frac{0.01\sqrt{2}\ \text{m}^3/\text{s}}{0.686\ \text{m}}\right)^2 = 0.28\ \text{W}$$

Note that at a frequency of 100 Hz, Q_o would have to be 5 times larger in order to produce the same power.

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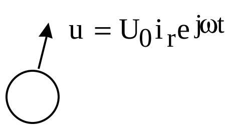
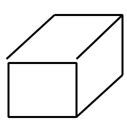
*************************** Example 7.2 ***************************



sphere $\qquad Q = 4\pi a^2 U_0$ (real) for radius a



cube $\qquad Q_{cube} = 6a^2 U_0$ (real) for side length a and for all faces in phase

If 3 sides are 180° out of phase with other 3, then $Q = 0$. All simple sources of the same source strength radiate the same, independent of shape, i.e., the same as a spherical source with ka << 1. So, the cube will radiate as a sphere with source strength Q_{cube}.

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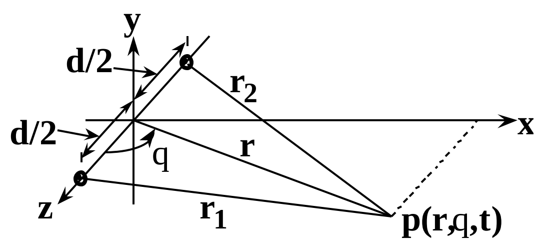
Acoustic Doublet Radiation Pattern for $d << r$



$$u\big|_{z=+d/2} = +U_o e^{j\omega t} \qquad \text{and} \qquad u\big|_{z=-d/2} = -U_o e^{j\omega t}$$

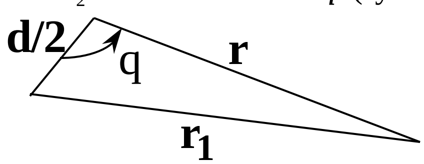
For a simple source with $u_s = U_o e^{j\omega t}$ and $Q_o = 4\pi a^2 U_o$:

$$\tilde{p}(r,t) = \frac{j\omega\rho_o Q_o}{4\pi r} e^{j(\omega t - kr)} = j\rho_o c \frac{Q_o k}{4\pi r} e^{j(\omega t - kr)} = \frac{j\rho_o c k a^2 U_o}{r} e^{j(\omega t - kr)}$$

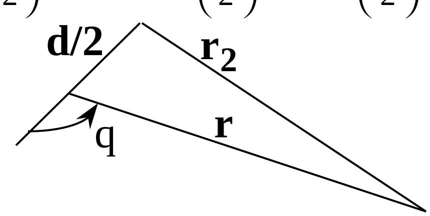
$$p(r,\theta,t) = p(r,t)\big|_{z=+d/2} + p(r,t)\big|_{z=-d/2} = \frac{j\rho_o c k a^2 U_o}{r_1} e^{j(\omega t - kr_1)} - \frac{j\rho_o c k a^2 U_o}{r_2} e^{j(\omega t - kr_2)}$$

$$= j\rho_o c k a^2 U_o e^{j\omega t} \left\{ \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2} \right\}$$

Let's obtain r_1 and r_2 in terms of r and θ (symmetry around azimuth)



$$r_1^2 = \left(\frac{d}{2}\right)^2 + r^2 - 2(r)\left(\frac{d}{2}\right)\cos\theta = \left(\frac{d}{2}\right)^2 + r^2 - rd\cos\theta$$



$$r_2^2 = \left(\frac{d}{2}\right)^2 + r^2 - 2(r)\left(\frac{d}{2}\right)\cos(180^\circ - \theta) = \left(\frac{d}{2}\right)^2 + r^2 + rd\cos\theta \quad r_1 = \sqrt{\left(\frac{d}{2}\right)^2 + r^2 - rd\cos\theta}$$

$$= r\sqrt{1 + \left(\frac{d}{2r}\right)^2 - \left(\frac{d}{r}\right)\cos\theta}$$

$$r_2 = \sqrt{\left(\frac{d}{2}\right)^2 + r^2 + rd\cos\theta} = r\sqrt{1+\left(\frac{d}{2r}\right)^2 + \left(\frac{d}{r}\right)\cos\theta}$$

For $r >> d$, that is, $\dfrac{d}{r} << 1$,

$$r_1 \approx r\sqrt{1-\left(\frac{d}{r}\right)\cos\theta} \approx r\left(1-\left(\frac{d}{2r}\right)\cos\theta\right) = r - \left(\frac{d}{2}\right)\cos\theta$$

$$r_2 \approx r\sqrt{1+\left(\frac{d}{r}\right)\cos\theta} \approx r\left(1+\left(\frac{d}{2r}\right)\cos\theta\right) = r + \left(\frac{d}{2}\right)\cos\theta$$

Therefore, for $p(r,\theta,t) = j\rho_o cka^2 U_o e^{j\omega t}\left\{\dfrac{e^{-jkr_1}}{r_1} - \dfrac{e^{-jkr_2}}{r_2}\right\}$, let's evaluate the bracketed term

$$\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2} = \frac{e^{-jkr\left\{1-\left(\frac{d}{2r}\right)\cos\theta\right\}}}{r\left\{1-\left(\frac{d}{2r}\right)\cos\theta\right\}} - \frac{e^{-jkr\left\{1+\left(\frac{d}{2r}\right)\cos\theta\right\}}}{r\left\{1+\left(\frac{d}{2r}\right)\cos\theta\right\}}$$

$$= \frac{e^{-jkr}}{r}\left\{\frac{e^{j\left(\frac{kd}{2}\right)\cos\theta}}{1-\left(\frac{d}{2r}\right)\cos\theta} - \frac{e^{-j\left(\frac{kd}{2}\right)\cos\theta}}{1+\left(\frac{d}{2r}\right)\cos\theta}\right\}$$

$$= \frac{e^{-jkr}}{r}\left\{\frac{\frac{d\cos\theta}{r}\cos\left(\frac{kd\cos\theta}{2}\right) + j2\sin\left(\frac{kd\cos\theta}{2}\right)}{1-\left(\frac{d}{2r}\right)^2\cos^2\theta}\right\}$$

For $\dfrac{kd\cos\theta}{2} << 1$, because $d << \lambda$:

$$\cos\left(\frac{kd\cos\theta}{2}\right) \to 1$$

$$\sin\left(\frac{kd\cos\theta}{2}\right) \to \frac{kd\cos\theta}{2}$$

$$1-\left(\frac{d}{2r}\right)^2\cos^2\theta \to 1$$

so that

$$\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2} \simeq \frac{e^{-jkr}}{r}\left\{\frac{d\cos\theta}{r} + jkd\cos\theta\right\}$$

and

$$p(r,\theta,t) = j\rho_o cka^2 U_o e^{j\omega t}\left\{\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2}\right\}$$

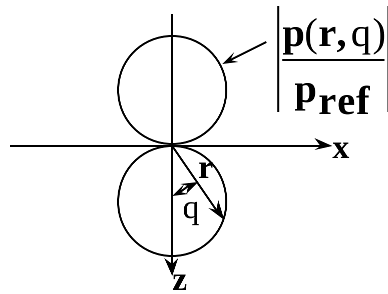
$$= \frac{j\rho_o cka^2 U_o e^{j(\omega t - kr)}}{r}\left\{\frac{d\cos\theta}{r} + jkd\cos\theta\right\}$$

$$= \frac{j\rho_o cka^2 U_o e^{j(\omega t - kr)}}{r}\left\{\frac{d}{r} + jkd\right\}\cos\theta\ .$$

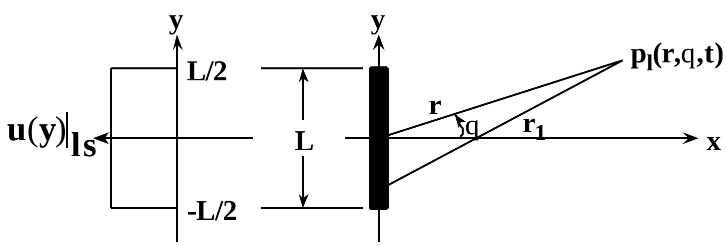
Note that $\left|\frac{p(r,\theta)}{p_{ref}}\right| = \left|\sqrt{\left(\frac{d}{r}\right)^2 + (kd)^2}\cos\theta\right|$

where p_{ref} is the pressure from a single source.

Graphically, when r >> d and λ >> d, the polar representation is:



(7.3) The Continuous Line Source

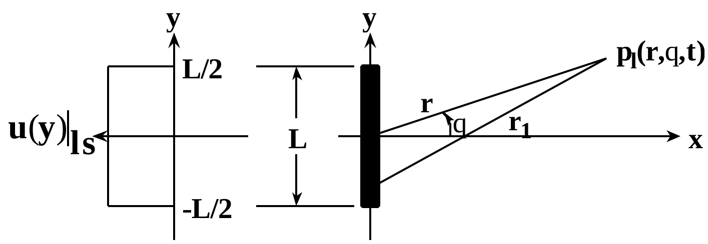


$u(y,t) = U_o e^{j\omega t} = u(y)\big|_{ls}\, e^{j\omega t}$ where $y \le \left|\frac{L}{2}\right|$

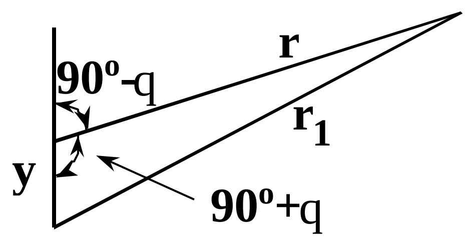
We have a couple of ways to determine pressure field pattern:1. Integrate directly the simple source solution on *y* from -*L*/2 to *L*/2.2. Fourier transformI think the simplest way is to integrate directly the simple source solution on y from -L/2 to L/2.To do that we start out with our pressure for the simple source from its relation to the particle velocity

$p_{ss}\left(r_1,t\right)=\dfrac{j\rho_o ck4\pi a^2U_o}{4\pi r_1}e^{j(\omega t-kr_1)}$ at some point r_1 from the line source (also defined by the angle θ).

We seek r_1 in terms of r, y and θ for



Looking at this more closely



So that we can find r_1 through

$$r_1^2=y^2+r^2-2\left(y\right)\left(r\right)\cos\left(90^o+\theta\right)=y^2+r^2+2yr\sin\theta$$

$$r_1=\sqrt{y^2+r^2+2yr\sin\theta}=r\sqrt{1+\left(\frac{y}{r}\right)^2+2\left(\frac{y}{r}\right)\sin\theta}$$

For $r >> L$ (the length of the source) then $r >> y$ and

$$\mathrm{r}_1\approx \mathrm{r}\sqrt{1+2\left(\frac{\mathrm{y}}{\mathrm{r}}\right)\mathrm{Sin}\,\theta}\approx \mathrm{r}\left\{1+\frac{1}{2}2\left(\frac{\mathrm{y}}{\mathrm{r}}\right)\mathrm{Sin}\,\theta\right\}$$

where $r >> L$ or $\dfrac{L}{r}<<1$ is the far field assumption.

Therefore, $r_1=r+y\sin\theta$

From $p_{ss}\left(r_1,t\right)=\dfrac{j\rho_o ck4\pi a^2U_o}{4\pi r_1}e^{j(\omega t-kr_1)}$

where $r_1=r+y\sin\theta$ then $4\pi a^2\rightarrow 2\pi ady$ (defining a cylinder of length dy instead of a spherical source). Let's sum up all of the dy cylinders from -L/2 to L/2

$$p_l\left(r,q,t\right)=\frac{j\,r_o ck2p\,aU_o e^{jwt}}{4p}\int_{-L/2}^{L/2}\frac{e^{-jk(r+y\sin q)}}{r+y\sin q}dy$$

$$=\frac{j\,r_o ck2p\,aU_o e^{j(wt-kr)}}{4p\,r}\int_{-L/2}^{L/2}\frac{e^{-jky\sin q}}{\left\{1+\left(\frac{y}{r}\right)\sin q\right\}}dy$$

Far field assumption: $\frac{\mathrm{L}}{\mathrm{r}}<<1$

$$p_l\left(r,q,t\right)=\frac{j\,r_o ck2p\,aU_o e^{j(wt-kr)}}{4p\,r}\int_{-L/2}^{L/2}e^{-jky\sin q}dy$$

Evaluating the integral:

$$\int_{-L/2}^{L/2}e^{-jky\sin q}dy=\frac{1}{-jk\sin q}\left\{e^{-j\frac{kL\sin q}{2}}-e^{j\frac{kL\sin q}{2}}\right\}=\frac{1}{jk\sin q}\left\{e^{j\frac{kL\sin q}{2}}-e^{-j\frac{kL\sin q}{2}}\right\}$$

$$=\frac{1}{jk\sin q}\left\{j2\sin\left(\frac{kL\sin q}{2}\right)\right\}=\frac{2}{k\sin q}\left\{\sin\left(\frac{kL\sin q}{2}\right)\right\}$$

$$=L\frac{\sin\left(\frac{kL\sin q}{2}\right)}{\frac{kL\sin q}{2}}=L\mathrm{sinc}\left(\frac{kL\sin q}{2}\right)$$

Therefore, from $p_l\left(r,q,t\right)=\frac{j\,r_o ck2p\,aU_o e^{j(wt-kr)}}{4p\,r}\int_{-L/2}^{L/2}e^{-jky\sin q}dy$,

$$p_l\left(r,q,t\right)=\frac{j\,r_o ck2p\,aU_o e^{j(wt-kr)}}{4p\,r}L\mathrm{sinc}\left(\frac{kL\sin q}{2}\right)=\frac{j\,r_o ckaLU_o e^{j(wt-kr)}}{2r}\mathrm{sinc}\left(\frac{kL\sin q}{2}\right)$$

$$=\frac{1}{2}j\,r_o c\left(\frac{a}{r}\right)kLU_o e^{j(wt-kr)}\mathrm{sinc}\left(\frac{kL\sin q}{2}\right)$$

Note that for a line source $Q_o=2p\,aLU_o$

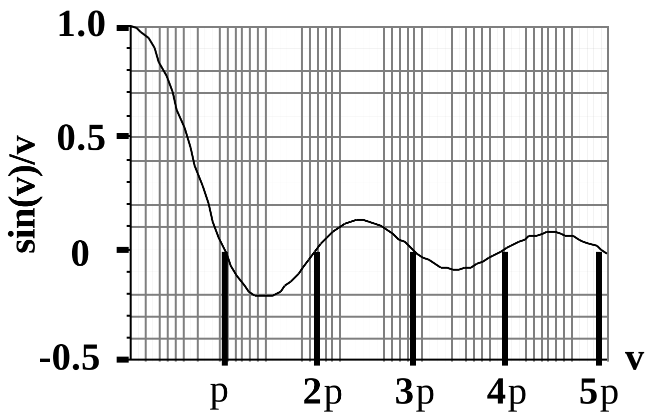
Acoustic pressure amplitude in the far field is: $P_l\left(r,q\right)=P_{ax}\left(r,q=0\right)H\left(q\right)$ where

$$P_{ax}\left(r,q=0\right)=P_{ax}\left(r\right)=\frac{r_o c}{2}\left(\frac{a}{r}\right)kLU_o$$

and

$$H\left(q\right)=\left|\mathrm{sinc}\left(\frac{kL\sin q}{2}\right)\right|$$

Let's evaluate the sinc function $\operatorname{sinc}(v) = \dfrac{\sin(v)}{v}$



SEE WEB PAGE FOR SINC FUNCTIONS

Let's examine $\operatorname{sinc}(v) = \dfrac{\sin(v)}{v}$ in terms of a series representation

For $v = 0$, $\operatorname{sinc}(v) = \dfrac{\sin(v)}{v} = \dfrac{v - \dfrac{v^3}{3!} + \dfrac{v^5}{5!} - \ldots}{v} = 1 - \dfrac{v^2}{3!} + \dfrac{v^4}{5!} - \ldots$

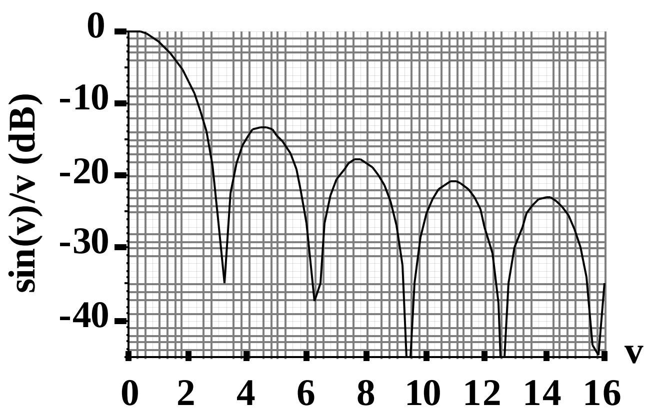
Therefore, $\operatorname{sinc}(0) = 1$.

For $\operatorname{sinc}(v) = \dfrac{\sin(v)}{v} = 0$, $\sin(v) = 0$ when $v = \pi, 2\pi, 3\pi, \ldots$

Peaks in sinc (v) occur when:

- v = 4.5 (-0.217, -13.3 dB),
- v = 7.7 (0.128, -17.8 dB),
- v = 10.9 (-0.091, -20.8 dB),
- v = 14.1 (0.071, -23.0 dB), ...

Observe the graphical representation of sinc (v) when plotted in dB



What do you expect to happen to the directional factor when the value of kL increases?

**************************** Example 7.2 ****************************

Determine the pressure amplitude distribution for a continuous line source for kL = 4 and kL = 24.

First, for kL = 24

$$H(\theta) = \left|\text{sinc}\left(\frac{kL\sin\theta}{2}\right)\right| = \left|\text{sinc}\left(\frac{24\sin\theta}{2}\right)\right| = \left|\text{sinc}(12\sin\theta)\right| = \left|\frac{\sin(12\sin\theta)}{12\sin\theta}\right|$$

The minimums occur when $H(\theta) = 0$ or when $\sin(12\sin\theta) = 0$, that is, when

$$12\sin\theta = n\pi, \quad n = \pm 1, \pm 2, \ldots$$

$$\theta = \sin^{-1}\left(\frac{n\pi}{12}\right), \quad n = \pm 1, \pm 2, \ldots$$

For $n = \pm 1$, $\theta = \sin^{-1}\left(\pm\frac{\pi}{12}\right) = \pm 15.2^o$

For $n = \pm 2$, $\theta = \sin^{-1}\left(\pm\frac{2\pi}{12}\right) = \pm 31.6^o$

For $n = \pm 3$, $\theta = \sin^{-1}\left(\pm\frac{3\pi}{12}\right) = \pm 51.8^o$

For $n = \pm 4$, $\theta = \sin^{-1}\left(\pm\frac{4\pi}{12}\right) = \sin^{-1}(\pm 1.047)$ (no solution)

Null beyond 90° (Thus, no null then)

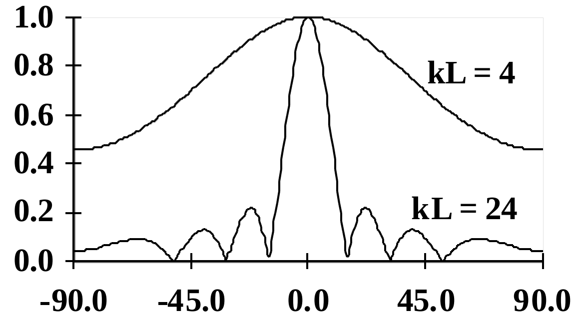
Note for kL = 4, $H(\theta) = \left|\text{sinc}(2\sin\theta)\right|$

$$H(0^o) = \left|\text{sinc}(2\sin 0^o)\right| = 1$$

$$H(45^o) = \left|\text{sinc}(2\sin 45^o)\right| = 0.70 \ (-3.12 \text{ dB})$$

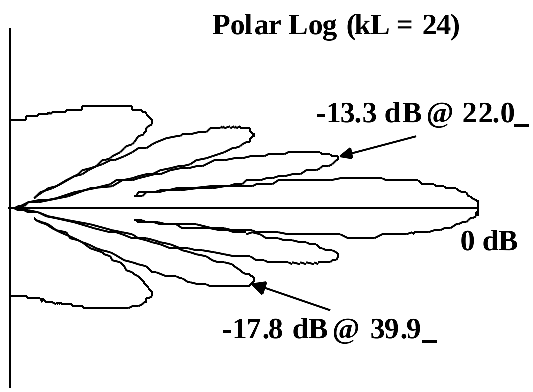
$$H(90^o) = \left|\text{sinc}(2\sin 90^o)\right| = 0.45 \ (-6.85 \text{ dB})$$

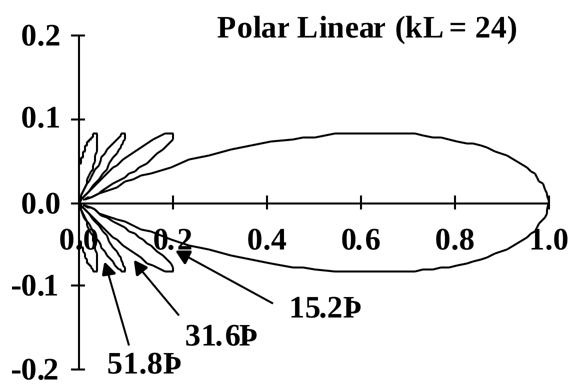
Cartesian representation of $H(q)$ for $kL = 4$ and $kL = 24$



What would you expect to happen to the directional factor if $kL << 1$?

Polar linear representation of $H(q)$ and Polar log representation of $H(q)$





Note that the peaks of $\text{sinc}(v)$ occur when:

v = 4.5 (-0.217, -13.3 dB),
v = 7.7 (0.128, -17.8 dB),
v = 10.9 (-0.091, -20.8 dB),
v = 14.1 (0.071, -23.0 dB), ...

For kL = 24 for $H(q) = |\text{sinc}(12\sin q)|$, maximum at v = 4.5 (-13.3 dB) or

$$12\sin q = 4.5$$

$$q = \sin^{-1}\left(\frac{4.5}{12}\right) = 22.0^{o}$$

For kL = 24 for $H(q) = |\text{sinc}(12\sin q)|$, maximum at v = 7.7 (-17.8 dB)

$$12\sin q = 7.7$$

$$q = \sin^{-1}\left(\frac{7.7}{12}\right) = 39.9^{o}$$

At least two means are used to define the width of the main lobe. Two of them are:

1. Between zeros
2. Between -3 dB points

For kL = 24, zeros, for $n = \pm 1$

$$q = \sin^{-1}\left(\pm\frac{p}{12}\right) = \pm 15.2^{o}$$

Therefore, width of main lobe (main beam) between zeros = 30.4°

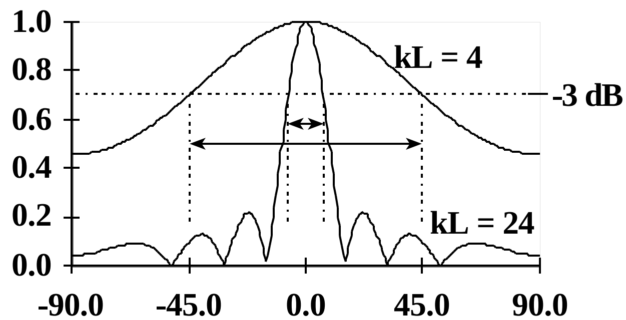
For -3 dB beam width, observe that the magnitude of $H(q)$ is - 3 dB when $H(q) = 0.7071$.

$$H(q) = |\text{sinc}(v)| = 0.7071 \text{ when } v = 1.3915$$

Therefore, for kL = 24

v = 12 sin$q$ = 1.3915, q = 6.66° and the -3 dB is 13.3°

For kL = 4, the -3 dB beamwidth is 88.2°



Typically, the -3-dB beamwidth is the standard used.