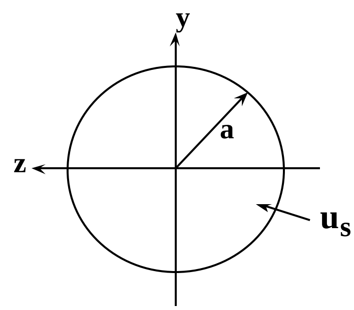
(7.4) Radiation from a Plane Circular Piston

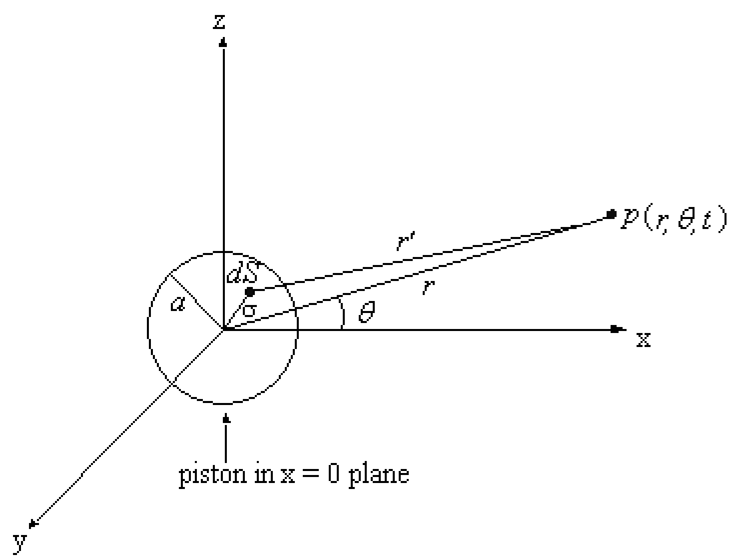
The plane circular piston is of particular interest in acoustics because it is a model for a number of sources, i.e. loudspeakers, open ended organ pipes, ventilation ducts, and many types of single element transducers.



Let's consider a piston of radius *a* mounted on a flat rigid baffle of infinite extent. We want to look at the radiating surface of the piston assuming that it moves with a uniform speed

$$u_s = U_0 e^{jwt}$$

normal to the baffle when $\sqrt{y^2 + z^2} < a$.



If we consider a simple source of area *dS* then we have for the source strength (surface area of source x velocity amplitude)

$$dQ = U_0 dS$$

So that the contribution from one baffled simple source is

$$dp = j r_0 c \frac{dQk}{2p r'} e^{j(wt - kr')}$$

The total pressure in the disk is found by summing up all of the “simple” sources on the disk

$$p(r,q,t) = j\frac{r_0 c U_0 k}{2p}\int_S \frac{e^{j(wt-kr')}}{r'}dS \qquad \text{for } 0 \le \sigma \le a.$$

Typically, we divide the field from the piston source into the <u>near field</u> and the <u>far field</u>.

Near Field – complicated variation in the pressure with position due to the complex interference from simple sources (Huygens sources).

An analytic solution is possible only along the axis. The extent of the near field is defined in terms of this result.

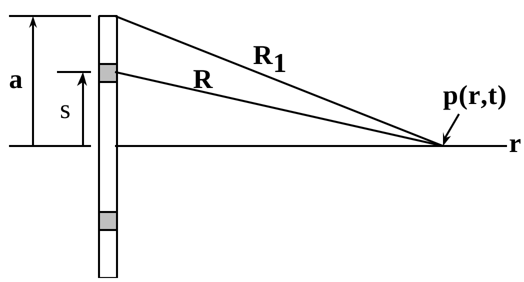
Far Field – Can get an approximate analytic solution everywhere in far field.

I. The Near Field approximations

We’ll consider the case where r is <u>not</u> much greater that a, that is, in the near field (Fresnel Zone). Here, we can obtain a closed-form solution only for the on-axis (axial) case. We start with

$$dp(r,t) = \frac{jw r_o U_o}{2p R} e^{j(wt-kR)} dS$$

where $dp(r,t)$ is the axial pressure at an axial distance r due to an annular ring on the source surface.



You can see from the diagram that there is no dependence on q when looking at pressure on the axis. Summing up the annular rings over the radius a

$$p(r,t) = \int_S \frac{jw r_o U_o}{2p R} e^{j(wt-kR)} dS$$

where $dS = 2p s\, ds$ for an incremental annulus and $R = \sqrt{s^2 + r^2}$.

Therefore,

$$p(r,t) = \frac{jwr_o U_o e^{jwt}}{2p} \int_0^a \frac{e^{-jkR}}{R} 2p s\, ds \;\; = jwr_o U_o e^{jwt} \int_0^a \frac{e^{-jkR}}{R} s\, ds$$

If we note that

$$-\frac{d}{ds} \frac{e^{-jk\sqrt{r^2+s^2}}}{jk} = \frac{s\, e^{-jk\sqrt{r^2+s^2}}}{\sqrt{r^2+s^2}} = \frac{s\, e^{-jkR}}{R}$$

Then

$$p(r,0,t) = r_0 c U_0 \left\{1 - e^{-jk\left(\sqrt{r^2+a^2}-r\right)}\right\} e^{j(wt-kr)} .$$

Pulling $e^{-j\frac{1}{2}k\left(\sqrt{r^2+a^2}-r\right)}$ out of the bracket gives

$$p(r,0,t) = r_0 c U_0 \left\{ e^{j\frac{1}{2}k\left(\sqrt{r^2+a^2}-r\right)} - e^{-j\frac{1}{2}k\left(\sqrt{r^2+a^2}-r\right)} \right\} e^{-j\frac{1}{2}k\left(\sqrt{r^2+a^2}-r\right)} e^{j(wt-kr)} .$$

But that just gives us

$$p(r,0,t) = r_0 c U_0 \sin\left[\frac{1}{2}k\left(\sqrt{r^2+a^2}-r\right)\right] e^{-j\frac{1}{2}k\left(\sqrt{r^2+a^2}-r\right)} e^{j(wt-kr)} .$$

The pressure magnitude on axis is

$$p(r,0) = 2 r_0 c U_0 \left| \sin\left\{ \frac{1}{2} kr \left[\sqrt{1+\left(\frac{a}{r}\right)^2} - 1 \right] \right\} \right|$$

So, what does this equation mean physically about the acoustic pressure along the axis in the near field?

$$\frac{1}{2} kr \left[\sqrt{1+\left(\frac{a}{r}\right)^2} - 1 \right] = mp/2$$

maxima for m odd
minima for m even

If we solve for *r* in terms of *m* then

$$\boxed{\frac{r_m}{a} = \frac{a}{ml} - \frac{ml}{4a}} \qquad \text{for } m = 0, 1, 2, \ldots$$

$m = 0$ corresponds to $r \to \infty$

$m = 1 \;\Rightarrow\;$ last axial maximum – This is where the far field equations begin to be valid for the plane piston source.

$$\frac{r_1}{a} = \frac{a}{l} - \frac{l}{4a} \quad \text{or} \quad \Rightarrow \quad r_1 \approx \frac{a^2}{l} \quad \text{for a} > l$$

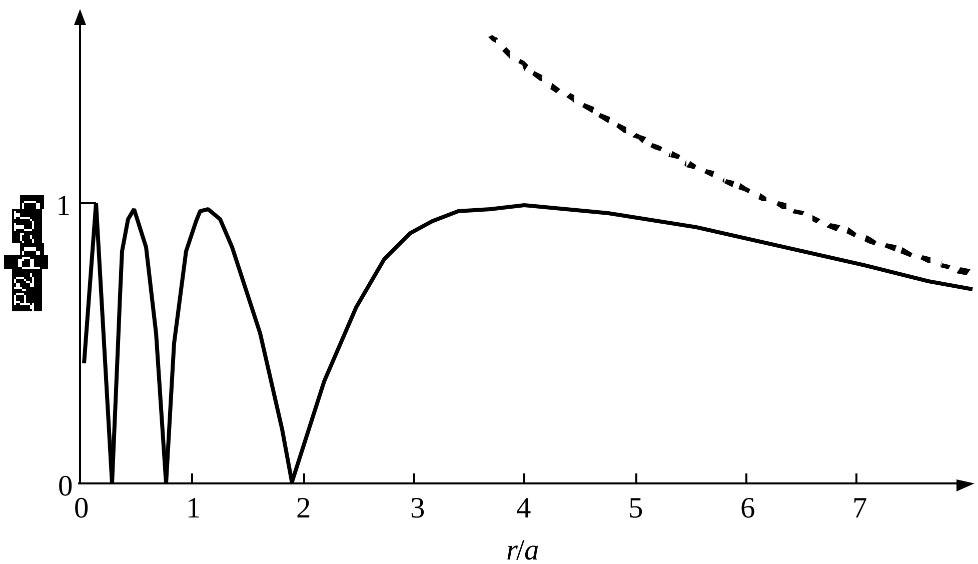


Figure 7.4.2 Axial pressure amplitude for a baffled circular plane piston of radius a radiating sound of wave number *k* with $ka = 8p$. Solid line is calculated from the exact theory. Dashed line is the far field approximation extrapolated into the near field. For this case, the far field approximation is accurate only for distances beyond about seven piston radii.

$$r_1 \cong \frac{a^2}{l}$$

$$\frac{r_1}{a} \cong \frac{a}{l} \cong 4 \qquad \text{for } ka = 8p$$

Note that for $r > \approx 2r_1$ the pressure decreases as $\frac{1}{r}$.

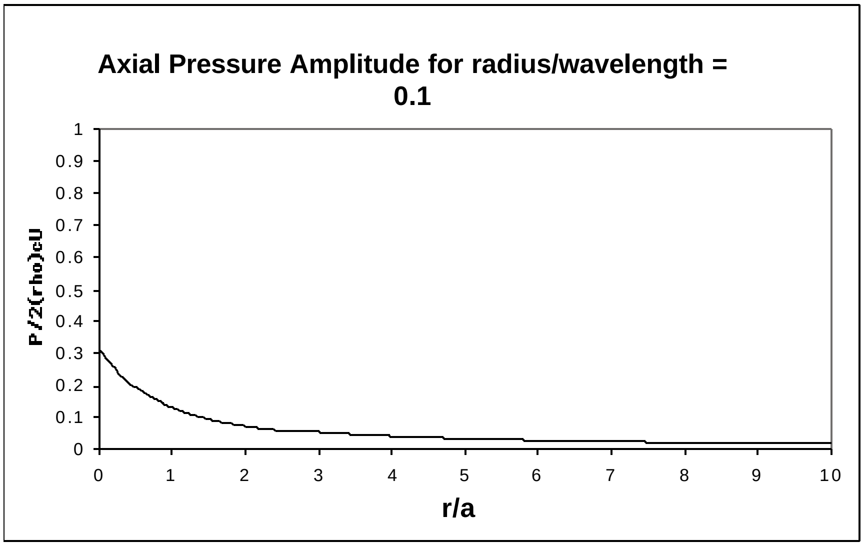
****************************** Example 7.2 ******************************

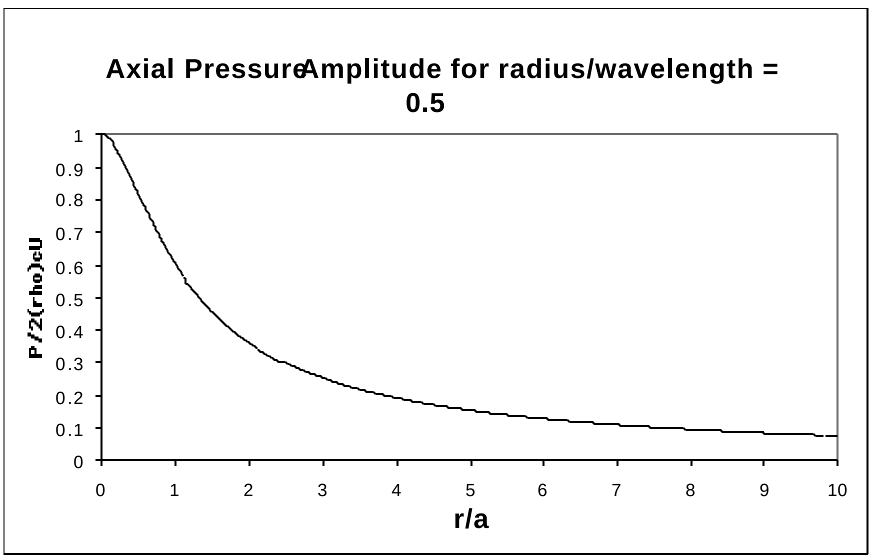
Examples of Axial Pressure Amplitude versus *r/a*

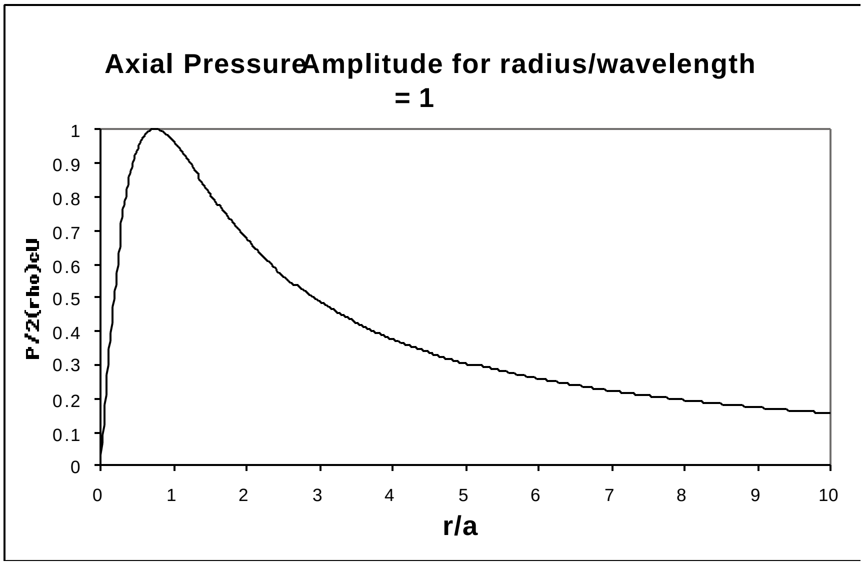
Using for the baffle source:

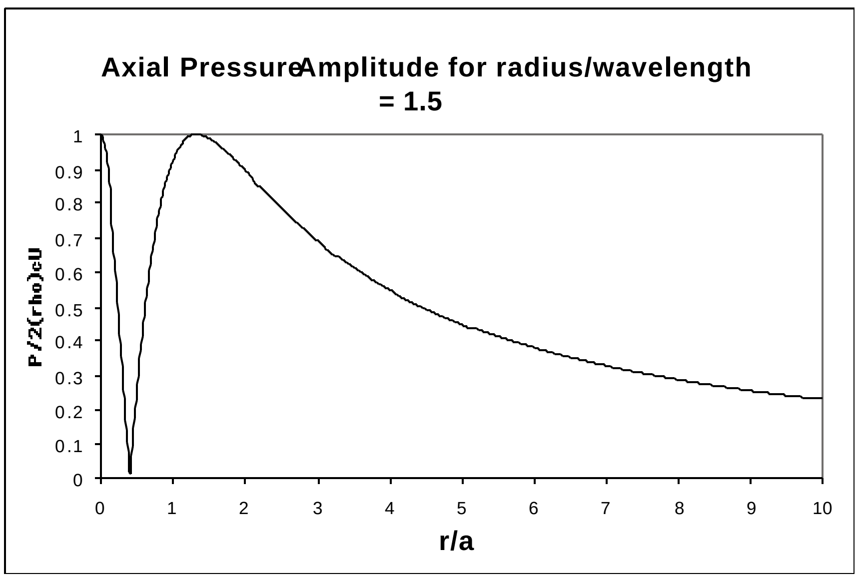
$$p(r,0) = 2\,r_0 c U_0 \left| \sin\left\{ \frac{1}{2} kr \left[\sqrt{1 + \left(\frac{a}{r}\right)^2} - 1 \right] \right\} \right|$$

The following are examples of $\frac{\left|p\left(r,0\right)\right|}{2r_o cU_o} = \left|\sin\left\{\frac{kr}{2}\left(\sqrt{1+\left(\frac{a^2}{r^2}\right)}-1\right)\right\}\right|$ versus *r/a* for different values of a/l (recall that $ka = 2p(a/l)$).






Axial Pressure Amplitude for radius/wavelength = 1
P/2(rho)cU
r/a
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6
7
8
9
10


Axial Pressure Amplitude for radius/wavelength = 1.5
P/2(rho)cU
r/a
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
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7
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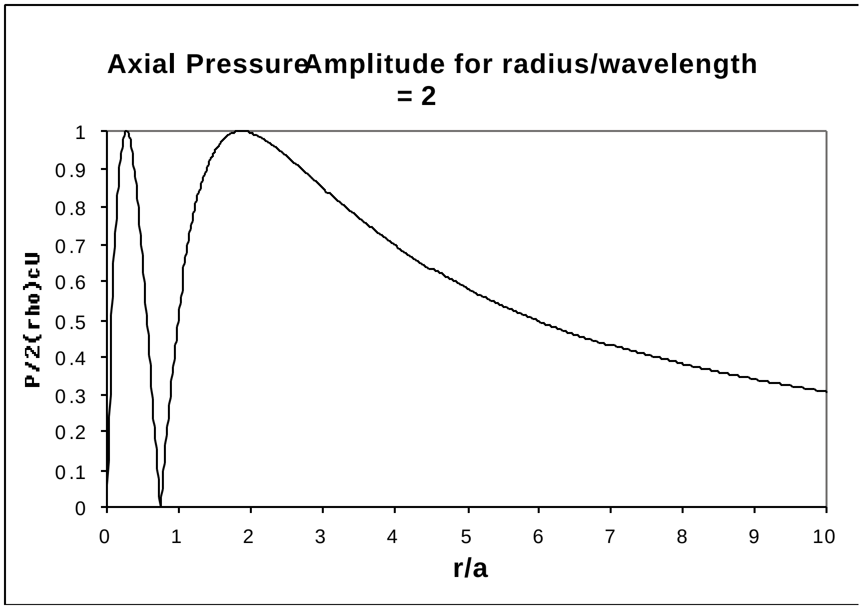

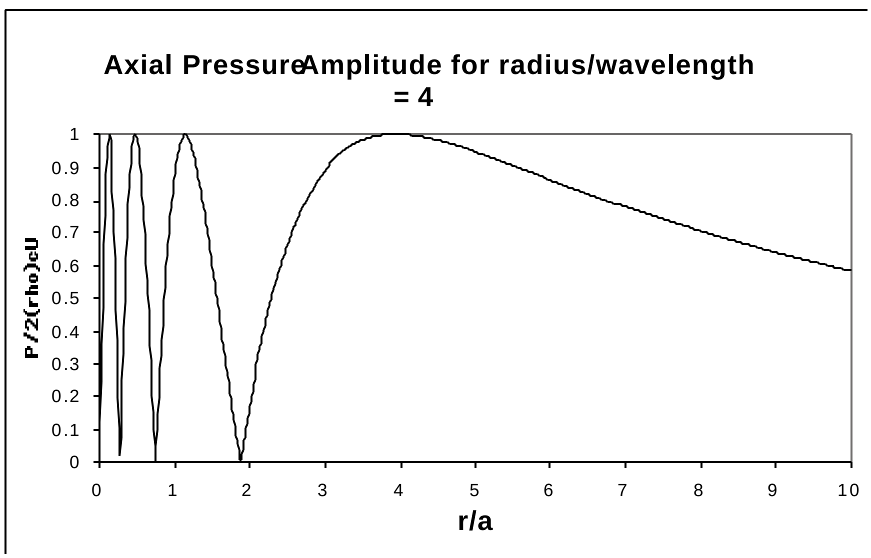

From these pictures what do you think happens to the far field as a/l gets larger?
How do the number of peaks and the number of multiples a is above the wavelength match up?

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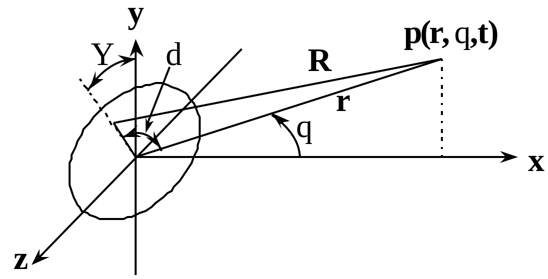
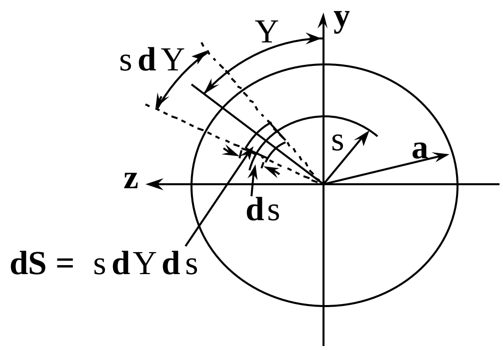
II. The Far Field approximations

The book uses a line source approximation to derive the Far field approximation for the baffled piston source. We will look at a more traditional way of deriving the far field approximation. Let's define the coordinate system we will be working with:





Recall that the contribution from one baffled simple source is

$$dp(r,\theta,t) = j\rho_0 c\frac{dQk}{2\pi r'}e^{j(\omega t-kR)}.$$

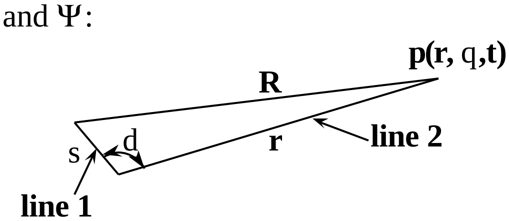
where $dQ = U_0 dS$. The incremental pressure, dp, at the field position (r, θ, t) from the incremental surface dS is:

$$dp(r,\theta,t) = j\rho_0 c\frac{U_0 k}{2\pi r'}e^{j(\omega t-kR)}dS = \frac{j\omega\rho_o U_o}{2\pi R}e^{j(\omega t-kR)}dS$$

or

$$p(r,\theta,t) = \frac{j\omega\rho_o U_o}{2\pi}e^{j\omega t}\int_{S:\sigma\le a}\frac{e^{-jkR}}{R}dS$$

To determine R in terms of σ and Ψ:



Law of Cosines: $R^2 = \sigma^2 + r^2 - 2r\sigma\cos\delta$ where $\cos\delta$ is determined from the Direction Cosines of lines 1 and 2, that is, $\cos\delta = l_1 l_2 + m_1 m_2 + n_1 n_2$ where the Direction Cosines are determined by the particular axial component length of a vector divided by the total length of the vector.

$$l_1 = 0 \qquad l_2 = \cos q$$
$$m_1 = \cos\Psi \qquad m_2 = \sin q$$
$$n_1 = \sin\Psi \qquad n_2 = 0$$

which yields: $\cos d = \cos\Psi \sin q$.

$$R^2 = s^2 + r^2 - 2rs\ \cos d = s^2 + r^2 - 2rs\ \cos\Psi\sin q = r^2\left(1+\frac{s^2}{r^2} - 2\left(\frac{s}{r}\right)\cos\Psi\sin q\right)$$

Because: $a<<r$, $\frac{s^2}{r^2} << \frac{s}{r} << 1$

$$R^2 = r^2\left(1+\frac{s^2}{r^2} - 2\left(\frac{s}{r}\right)\cos\Psi\sin q\right) = r^2\left(1-2\left(\frac{s}{r}\right)\cos\Psi\sin q\right)$$

$$R = r\sqrt{1-2\left(\frac{s}{r}\right)\cos\Psi\sin q} \approx r\left(1-\left(\frac{s}{r}\right)\cos\Psi\sin q\right) = r - s\ \cos\Psi\sin q\ .$$

So,

$$p(r,q,t) = \frac{jwr_oU_o}{2p}e^{jwt}\int_{S:s\le a}\frac{e^{-jkR}}{R}dS = \frac{jwr_oU_o}{2p}e^{jwt}\int_{S:s\le a}\frac{e^{-jk(r-s\ \cos\Psi\sin q)}}{r-s\ \cos\Psi\sin q}dS$$
$$= \frac{jwr_oU_o}{2p}e^{j(wt-kr)}\int_{S:s\le a}\frac{e^{jks\ \cos\Psi\sin q}}{r-s\ \cos\Psi\sin q}dS$$

For the far-field approximation: $s\ \cos\Psi\sin q \ll r$ in denominator because change in amplitude is assumed to be negligible; remember: $\frac{s^2}{r^2} << \frac{s}{r} << 1$. So,

$$p(r,q,t) = \frac{jwr_oU_o}{2p\, r}e^{j(wt-kr)}\int_{S:s\le a} e^{jks\ \cos\Psi\sin q}dS$$

Using for our incremental surface area, $dS = s\, ds\, d\Psi$, yields

$$p(r,q,t) = \frac{jwr_oU_o}{2p\, r}e^{j(wt-kr)}\int_{S:s\le a} e^{jks\ \cos\Psi\sin q}s\, ds\, d\Psi$$

Limits of integration:

$s\ :\ 0 \rightarrow a$

$\Psi :\ 0 \rightarrow 2p$

giving

$$\int_{S:s\le a} e^{jks\ \cos\Psi cin q}s\, ds\, d\Psi = \int_{s=0}^{a}\left\{\int_{\Psi=0}^{2p} e^{jks\ \cos\Psi\sin q}d\Psi\right\}s\, ds$$

Therefore, let’s first look at $\int_{\Psi=0}^{2p} e^{jks\ \cos\Psi\sin q}d\Psi$

Using the general relationship for the Bessel's function of the first kind of order *n*

$$J_n(x) = \frac{(-j)^n}{2p}\int_0^{2p} e^{jx\cos\Psi}\cos(n\Psi)\,d\Psi$$

Comparing with the above integral we see that we have a Bessel function of order *n* = 0, so

$$J_o(k\,s\,sinq) = \frac{1}{2p}\int_0^{2p} e^{jks\,sinq\,sos\Psi}d\Psi$$

where $x = ks\,sinq$. Our pressure then is

$$p(r,q,t) = \frac{jwr_oU_o}{2pr}e^{j(wt-kr)}\int_{s=0}^{a}\{2p\,J_o(ks\,\sin q)\}s\,ds$$
$$= \frac{jwr_oU_o}{r}e^{j(wt-kr)}\int_{s=0}^{a}s\,J_o(ks\,\sin q)ds$$

To evaluate $\int_{s=0}^{a}s\,J_o(ks\,\sin q)ds$, we observe that $\int xJ_o(x)\,dx = xJ_1(x)$. For $\int xJ_o(x)\,dx = xJ_1(x)$,

we let $x = ks\,\sin q \quad\Rightarrow\quad s = \dfrac{x}{k\sin q}$ and $dx = k\sin q\,ds \quad\Rightarrow\quad ds = \dfrac{dx}{k\sin q}$. Our limits of integration become:

$$\sigma = 0 \quad\Rightarrow\quad x = 0$$
$$\sigma = a \quad\Rightarrow\quad x = ka\text{Sin}\,\theta$$

giving,

$$\int_{s=0}^{a}s\,J_o(ks\,\sin q)ds = \int_0^{ka\sin q}\left(\frac{x}{k\sin q}\right)J_o(x)\left(\frac{dx}{k\sin q}\right) = \frac{1}{(k\sin q)^2}xJ_1(x)\Big|_0^{ka\sin q}$$
$$= \frac{a}{k\sin q}J_1(ka\sin q).$$

Finally our pressure is given by

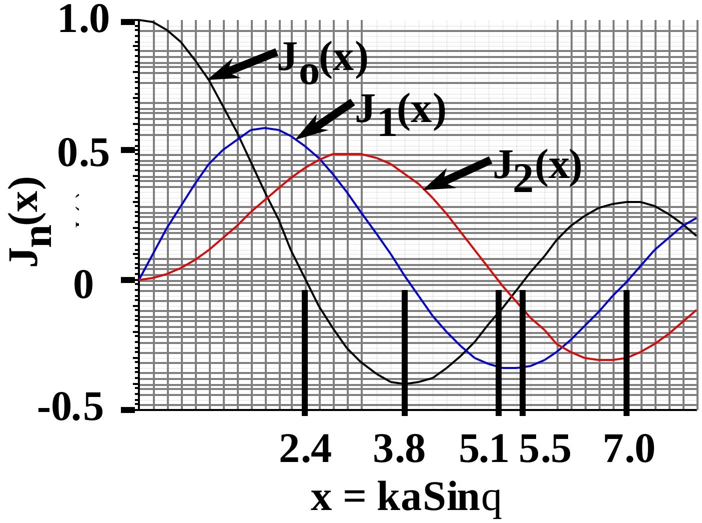
$$p(r,q,t) = \frac{jwr_oa^2U_o}{2r}e^{j(wt-kr)}\left[\frac{2J_1(ka\sin q)}{ka\sin q}\right]$$

The pressure amplitude will be directional depending on q . We define the <u>Directional Factor</u> (or also called the beam pattern) of a source, $H(q)$ by

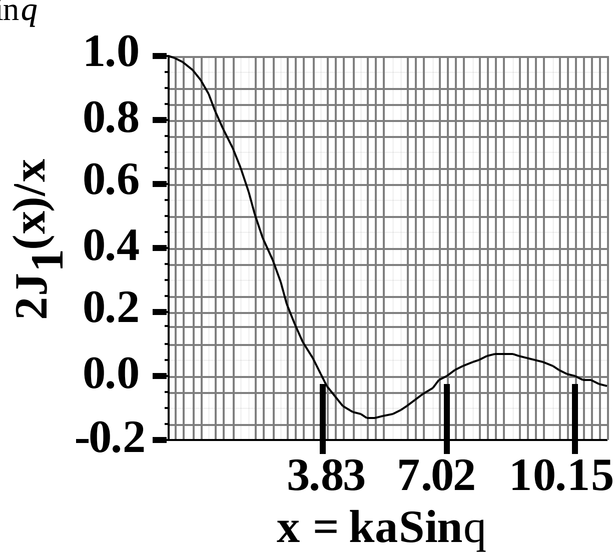
$$H(q)^2 = \frac{I}{I_{ref}} = \left[\frac{2J_1(ka\sin q)}{ka\sin q}\right]^2 \qquad\rightarrow\qquad H(q) = \left|\frac{2J_1(ka\sin q)}{ka\sin q}\right|$$

Values for the directional factor are listed in Appendix 6 for a baffled piston.

Let's look briefly at what the Bessel function (*n* = 0,1,2) looks like

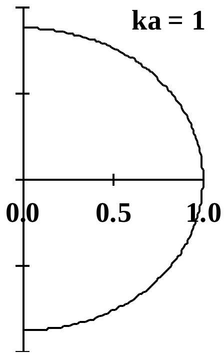


Looking at the Directivity Factor, $H(\theta)=\left|\frac{2J_1(x)}{x}\right|$ (pressure) and $\left[H(\theta)\right]^2=\left[\frac{2J_1(x)}{x}\right]^2$ (intensity) where $x = ka\sin\theta$



We can see that the size of ka is going to determine the number of peaks x can cycle through for different values of θ. For ka = 1, no zeros will be reached as we cycle through θ.
At $\theta = 90°$, $ka\sin\theta = 1$,

$$H\left(90^o\right)=\frac{2J_1(1)}{1}=0.8801 \text{ (-1.11 dB)}$$

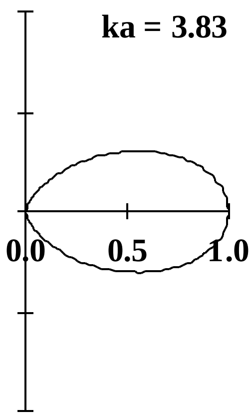


For *ka* = 3.83:

$$q_z = \sin^{-1}\left(\frac{3.83}{3.83}\right) = 90^o$$

$$q_{-3dB} = 2\sin^{-1}\left(\frac{1.613741}{3.83}\right) = 49.8^o$$

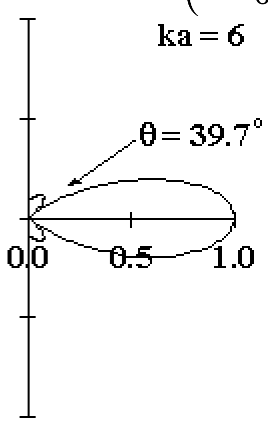
Why –3 dB?



For ka = 6:

$$q_z = \sin^{-1}\left(\frac{3.83}{6}\right) = 39.7^o$$

$$q_{-3dB} = 2\sin^{-1}\left(\frac{1.613741}{6}\right) = 31.2^o$$



The book has a really nice figure for the beam pattern for ka = 10, $10\log_{10}\left|H(q)\right|^2$,

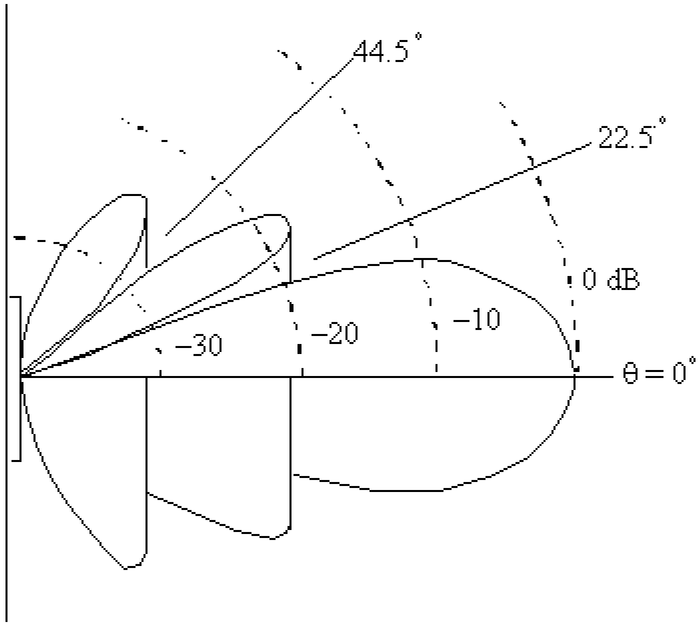


Figure 7.4.5 Beam pattern $H(q)$ for a circular plane piston of radius a radiating sound with ka = 10.

What do you expect to happen to the beam pattern as $ka \rightarrow 0$?