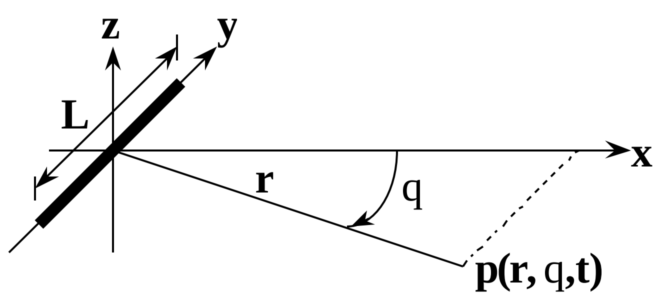
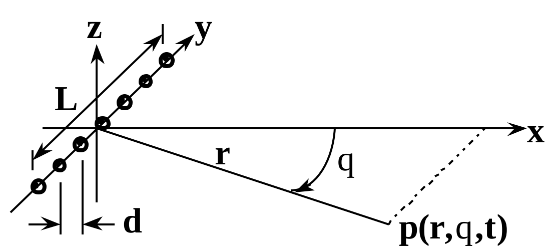
(7.8) Simple Line Array (N sources)

Recall for an in-phase line source, $H(q) = \left|\mathrm{sinc}\left(\frac{kL}{2}\sin q\right)\right|$



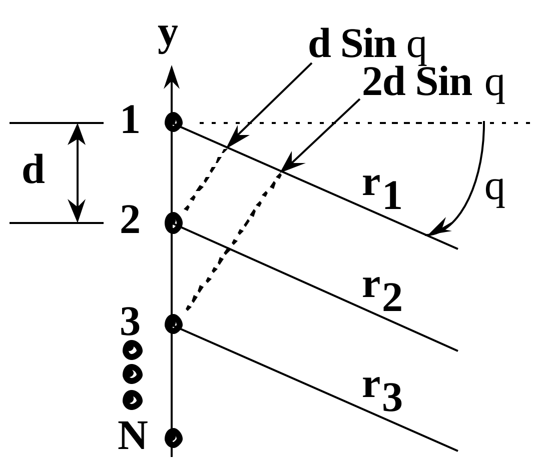
Now, consider *N* point (simple) sources, each with same source strength and phase, and separated by a center-to-center distance *d*, where $L = Nd - d = (N-1)\,d$.



For a simple source with $u_s = U_o e^{jwt}$, $p(r,t) = \frac{j\,r_o cka^2 U_o}{r} e^{j(wt-kr)} = \frac{\tilde{A}}{r} e^{j(wt-kr)}$

For the *N* simple sources,

$$p(r,q,t) = \frac{\tilde{A}}{r_1} e^{j(wt-kr_1)} + \frac{\tilde{A}}{r_2} e^{j(wt-kr_2)} + \frac{\tilde{A}}{r_3} e^{j(wt-kr_3)} + \ldots = \sum_{i=1}^{N} \frac{\tilde{A}}{r_i} e^{j(wt-kr_i)}$$



For the far field approximation, all of the r_i can be approximated as parallel. To determine a general expression for r_i, note that for $p = f(r,q,t)$

source	distance to $p(r,q,t)$
1	r_1
2	$r_2 = r_1 - d\sin q$
3	$r_3 = r_1 - 2d\sin q$
4	$r_4 = r_1 - 3d\sin q$
...	...
i	$r_i = r_1 - (i-1)d\sin q$
...	...

$$p(r,q,t) = \sum_{i=1}^{N} \frac{\tilde{A}}{r_i} e^{j(wt - kr_i)} = \sum_{i=1}^{N} \frac{\tilde{A} e^{j(wt - k(r_1 - (i-1)d\sin q))}}{r_1 - (i-1)d\sin q}$$

$$= \tilde{A} e^{j(wt - kr_1)} \sum_{i=1}^{N} \frac{e^{jk(i-1)d\sin q}}{r_1 - (i-1)d\sin q}$$

We define the array center to be: $\left(\frac{N-1}{2}\right)d$

Define r as distance between array center and $p(r,q,t)$ field location.

From $r_i = r_1 - (i-1)d\sin q$, $r = r_1 - \left(\frac{N-1}{2}\right)d\sin q$

From $L = (N-1)d \quad \Rightarrow \quad (N-1) = \frac{L}{d}$,

$$r = r_1 - \left(\frac{L/d}{2}\right)d\sin q = r_1 - \frac{L}{2d}d\sin q$$

From $r = r_1 - \frac{L}{2d}d\sin q$,

$$r_1 = r + \frac{L}{2d}d\sin q = r + \frac{1}{2}\left(\frac{L}{d}\right)d\sin q = r + \frac{1}{2}\left(\frac{L}{d}\right)\Delta r \qquad \text{where } \Delta r = d\sin q$$

From $r_1 = r + \frac{1}{2}\left(\frac{L}{d}\right)\Delta r$,

$$p(r,q,t) = \tilde{A} e^{j(wt - kr_1)} \sum_{i=1}^{N} \frac{e^{jk(i-1)d\sin q}}{r_1 - (i-1)d\sin q} = \tilde{A} e^{j\left(wt - k\left(r + \frac{1}{2}\left(\frac{L}{d}\right)\Delta r\right)\right)} \sum_{i=1}^{N} \frac{e^{jk(i-1)d\sin q}}{r_1 - (i-1)d\sin q}$$

$$= \tilde{A} e^{j(wt - kr)} e^{-jk\frac{1}{2}\left(\frac{L}{d}\right)\Delta r} \sum_{i=1}^{N} \frac{e^{jk(i-1)d\sin q}}{r_1 - (i-1)d\sin q}$$

From $r_1 = r + \frac{1}{2}\left(\frac{L}{d}\right)\Delta r$ and $\Delta r = d\sin q$, the denominator term $r_1 - (i-1)\,d\sin q$ becomes:

$$r_1 - (i-1)\,d\sin q = r + \frac{1}{2}\left(\frac{L}{d}\right)\Delta r - (i-1)\,d\sin q = r + \frac{1}{2}\left(\frac{L}{d}\right)d\sin q - (i-1)\,d\sin q$$
$$= L\left\{\frac{r}{L} - \left(\frac{-1}{2} + \frac{d\,(i-1)}{L}\right)\sin q\right\}$$

Thus, the denominator term is: $L\left\{\frac{r}{L} - \left(\frac{-1}{2} + \frac{d\,(i-1)}{L}\right)\sin q\right\}$

For the far-field assumption, that is, $r >> L$, which suggests that $\frac{r}{L} >> 1$, the denominator term is approximated to: $L\left\{\frac{r}{L} - \left(\frac{-1}{2} + \frac{d\,(i-1)}{L}\right)\sin q\right\} \approx r$

Thus, for the far-field assumption:

$$p(r,q,t) = \tilde{A}e^{j(wt-kr)}e^{-jk\frac{1}{2}\left(\frac{L}{d}\right)\Delta r}\sum_{i=1}^{N}\frac{e^{jk(i-1)d\sin q}}{r_1 - (i-1)\,d\sin q} = \tilde{A}e^{j(wt-kr)}e^{-jk\frac{1}{2}\left(\frac{L}{d}\right)\Delta r}\sum_{i=1}^{N}\frac{e^{jk(i-1)d\sin q}}{r}$$
$$= \frac{\tilde{A}}{r}e^{j(wt-kr)}e^{-jk\frac{1}{2}\left(\frac{L}{d}\right)\Delta r}\sum_{i=1}^{N}e^{jk(i-1)\Delta r}$$

To evaluate $\sum_{i=1}^{N}e^{jk(i-1)\Delta r}$, use the geometric series: $\sum_{m=1}^{N}r^m = \frac{r^{N+1}-r}{r-1}, r \neq 1$

$$\sum_{i=1}^{N}e^{jk(i-1)\Delta r} = e^{-jk\Delta r}\sum_{i=1}^{N}e^{jki\Delta r} = e^{-jk\Delta r}\left\{\frac{e^{jk\Delta r(N+1)} - e^{jk\Delta r}}{e^{jk\Delta r}-1}\right\} = \frac{e^{jk\Delta rN}-1}{e^{jk\Delta r}-1}$$

Now,

$$p(r,q,t) = \frac{\tilde{A}}{r}e^{j(wt-kr)}e^{-jk\frac{1}{2}\left(\frac{L}{d}\right)\Delta r}\sum_{i=1}^{N}e^{jk(i-1)\Delta r} = \frac{\tilde{A}}{r}e^{j(wt-kr)}e^{-jk\frac{1}{2}\left(\frac{L}{d}\right)\Delta r}\left(\frac{e^{jk\Delta rN}-1}{e^{jk\Delta r}-1}\right)$$
$$= \frac{\tilde{A}}{r}e^{j(wt-kr)}e^{-jk\left(\frac{N-1}{2}\right)\Delta r}\left(\frac{e^{jk\Delta rN}-1}{e^{jk\Delta r}-1}\right)$$
$$= \frac{\tilde{A}}{r}e^{j(wt-kr)}\left(\frac{e^{jk\left(\frac{N+1}{2}\right)\Delta r} - e^{-jk\left(\frac{N-1}{2}\right)\Delta r}}{e^{jk\Delta r}-1}\right)$$

$$= \frac{\tilde{A}}{r} e^{j(wt-kr)} e^{j\frac{k\Delta r}{2}} \left(\frac{e^{jk\left(\frac{N}{2}\right)\Delta r} - e^{-jk\left(\frac{N}{2}\right)\Delta r}}{e^{jk\Delta r}-1} \right)$$

$$= \frac{\tilde{A}}{r} e^{j(wt-kr)} \left(\frac{e^{jk\left(\frac{N}{2}\right)\Delta r} - e^{-jk\left(\frac{N}{2}\right)\Delta r}}{e^{j\frac{k}{2}\Delta r} - e^{-j\frac{k}{2}\Delta r}} \right)$$

$$= N \frac{\tilde{A}}{r} e^{j(wt-kr)} \left[\frac{1}{N} \frac{\sin\left(\frac{N}{2} k\Delta r\right)}{\sin\left(\frac{1}{2} k\Delta r\right)} \right]$$

Therefore, $p(r,q,t) = N \frac{\tilde{A}}{r} e^{j(wt-kr)} \left[\frac{1}{N} \frac{\sin\left(\frac{N}{2} k\Delta r\right)}{\sin\left(\frac{1}{2} k\Delta r\right)} \right]$ where $\Delta r = d \sin q$

Directional factor: $H(q) = \left| \frac{1}{N} \frac{\sin\left(\frac{N}{2} k\Delta r\right)}{\sin\left(\frac{1}{2} k\Delta r\right)} \right| = \left| \frac{1}{N} \frac{\sin\left(\frac{N}{2} kd \sin q\right)}{\sin\left(\frac{1}{2} kd \sin q\right)} \right|$ (Eq 7.8.6)

For q = 0° ($\Delta r = d \sin q = 0$)

$$H(q) = \left| \frac{1}{N} \frac{\sin\left(\frac{N}{2} k\Delta r\right)}{\sin\left(\frac{1}{2} k\Delta r\right)} \right| = \left| \frac{1}{N} \frac{\sin(Nx)}{\sin(x)} \right| = \left| \frac{1}{N} \frac{Nx - \frac{(Nx)^3}{3!} + \frac{(Nx)^5}{5!} - \ldots}{x - \frac{(x)^3}{3!} + \frac{(x)^5}{5!} - \ldots} \right| \xrightarrow[x\to 0]{} \left| \frac{1}{N} \frac{Nx}{x} \right| = 1$$

Both numerator and denominator <u>each</u> have zeros (and not when q = 0).

$$H(q) = \left| \frac{1}{N} \frac{\sin\left(\frac{N}{2} k\Delta r\right)}{\sin\left(\frac{1}{2} k\Delta r\right)} \right| = \left| \frac{1}{N} \frac{\sin\left(\frac{N}{2} kd \sin q\right)}{\sin\left(\frac{1}{2} kd \sin q\right)} \right|$$

- ❖ Numerator zeros: $\mathrm{Sin}\left(\frac{\mathrm{N}}{2} \mathrm{k}\Delta \mathrm{r}\right) = 0$

when $\frac{N}{2} k\Delta r = \frac{N}{2} kd \left|\sin q_n\right| = np$, n = 1, 2, ...

$$\left|\sin q_n\right| = \frac{2np}{Nkd} = \frac{n}{N} \frac{l}{d}$$

❖ Denominator zeros: $\sin\left(\frac{1}{2}k\Delta r\right)=0$

when $\frac{1}{2}k\Delta r=\frac{1}{2}kd\left|\sin q_m\right|=mp$, m = 1, 2, ...

$$\left|\sin q_m\right|=\frac{2mp}{kd}=m\frac{l}{d}$$

At q_m (the denominator zeros), the numerator of $H(q)=\left|\frac{1}{N}\frac{\sin\left(\frac{N}{2}kd\sin q\right)}{\sin\left(\frac{1}{2}kd\sin q\right)}\right|$ is:

$$\sin\left(\frac{N}{2}kd\sin q\right)\underset{q\to q_m}{\longrightarrow}\sin\left(\frac{N}{2}kd\sin q_m\right)=\sin\left(\frac{N}{2}kd\left(m\frac{l}{d}\right)\right)=\sin(mNp)\to 0$$

Thus, for $q=q_m$ where $\left|\sin q_m\right|=\frac{2mp}{kd}=m\frac{l}{d}$,

$$H(q_m)=\left|\frac{1}{N}\frac{\sin\left(\frac{N}{2}kd\sin q_m\right)}{\sin\left(\frac{1}{2}kd\sin q_m\right)}\right|=\left|\frac{1}{N}\frac{\sin\left(\frac{N}{2}kd\left(m\frac{l}{d}\right)\right)}{\sin\left(\frac{1}{2}kd\left(m\frac{l}{d}\right)\right)}\right|=\left|\frac{1}{N}\frac{\sin(mNp)}{\sin(mp)}\right|$$

Let $m=e+i$ where i is an integer and e is small

$$H(q_m)=\lim_{e\to 0}\left|\frac{1}{N}\frac{\sin([e+i]Np)}{\sin([e+i]p)}\right|=\lim_{e\to 0}\left|\frac{1}{N}\frac{\sin(eNp+iNp)}{\sin(ep+ip)}\right|$$

$$=\lim_{e\to 0}\left|\frac{1}{N}\frac{\sin(eNp)}{\sin(ep)}\right|\simeq\lim_{e\to 0}\left|\frac{1}{N}\frac{eNp}{ep}\right|=1$$

Therefore, there are multiple <u>major lobes</u> at $q_m=\sin^{-1}\left(m\frac{l}{d}\right)=\sin^{-1}\left(m\frac{2p}{kd}\right)$, m = 0, 1, 2, ...

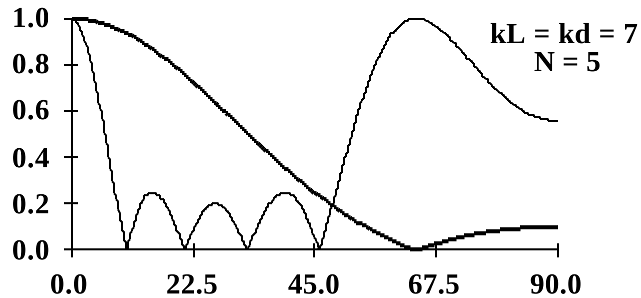
For kd = 7

$$q_o=\sin^{-1}\left(0\frac{2p}{7}\right)=0^o$$

$$q_1=\sin^{-1}\left(1\frac{2p}{7}\right)=63.8^o$$

$$q_2=\sin^{-1}\left(2\frac{2p}{7}\right)>90^o$$

Thus, two (or three) major lobes (comparing line source and array)



For kd = 15

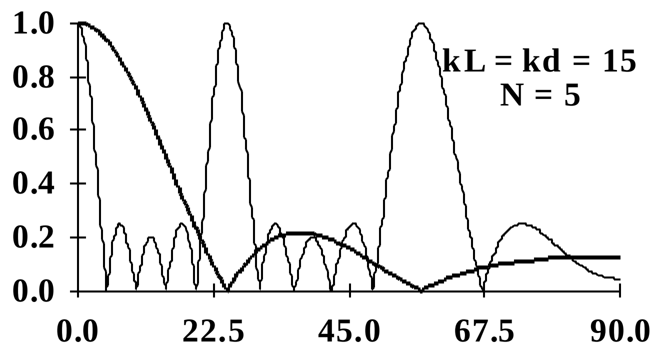
$$q_o = \sin^{-1}\left(0\,\frac{2p}{15}\right) = 0^o$$

$$q_1 = \sin^{-1}\left(1\frac{2p}{15}\right) = 24.8^o$$

$$q_2 = \sin^{-1}\left(2\,\frac{2p}{15}\right) = 56.9^o$$

$$q_3 = \sin^{-1}\left(3\,\frac{2p}{15}\right) > 90^o$$

Thus, three (or five) major lobes



In addition to the major lobes, there are minor lobes. The zeros between minor lobes can be determined from: $q_n = \sin^{-1}\left(\frac{n}{N}\frac{l}{d}\right) = \sin^{-1}\left(\frac{n}{N}\frac{2p}{kd}\right)$ n = 1, 2, ...

For kd = 7 and N = 5

$$q_1 = \sin^{-1}\left(\frac{1}{5}\frac{2p}{7}\right) = 10.3^o$$

$$q_2 = \sin^{-1}\left(\frac{2}{5}\frac{2p}{7}\right) = 21.0^o$$

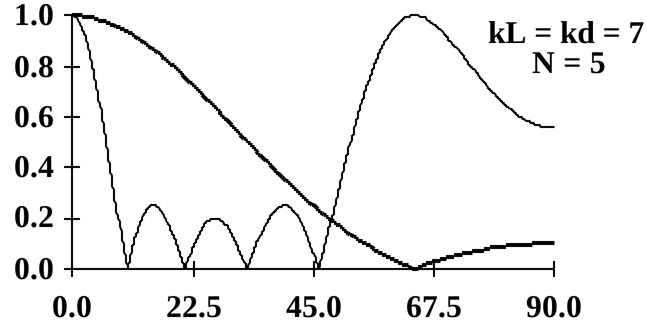
$$q_3 = \sin^{-1}\left(\frac{3}{5}\frac{2p}{7}\right) = 32.6^o$$

$$q_4 = \sin^{-1}\left(\frac{4}{5}\frac{2p}{7}\right) = 45.9^o$$

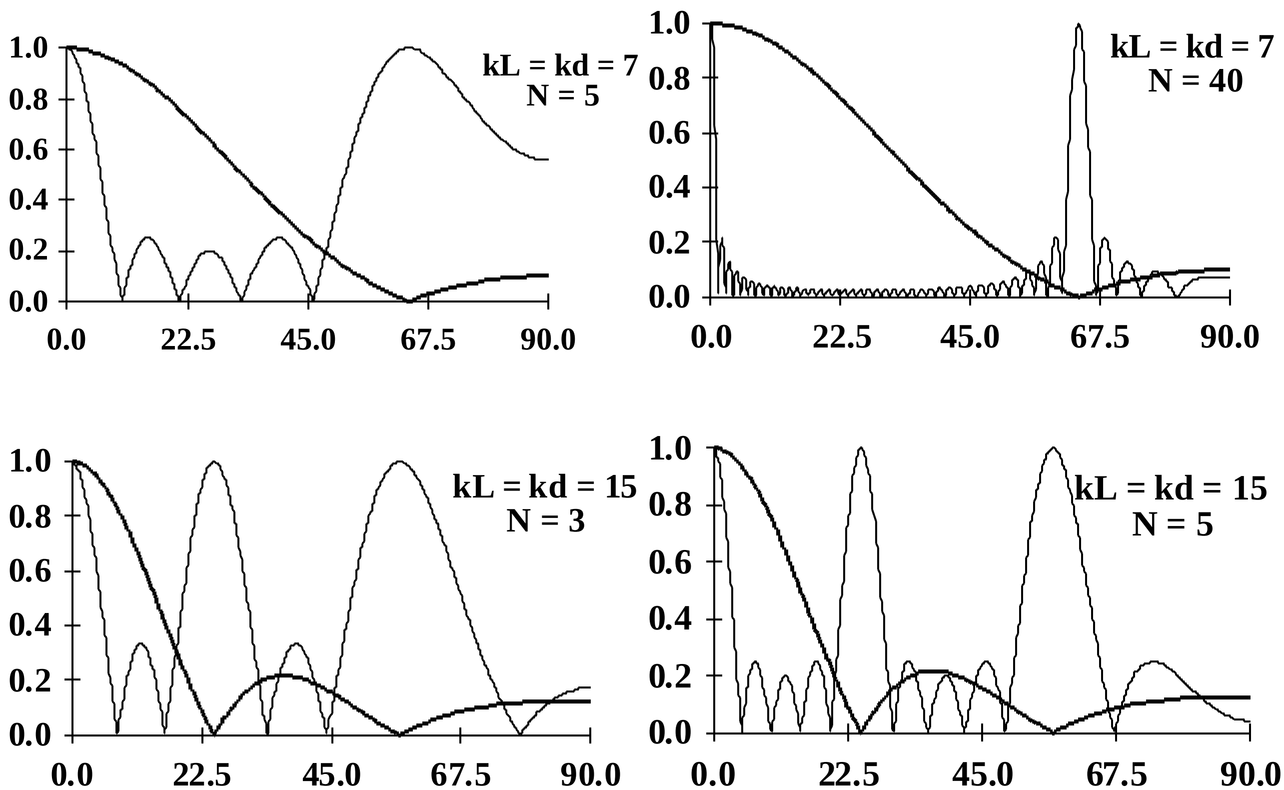
$$q_5 = \sin^{-1}\left(\frac{5}{5}\frac{2p}{7}\right) = 63.8^o$$

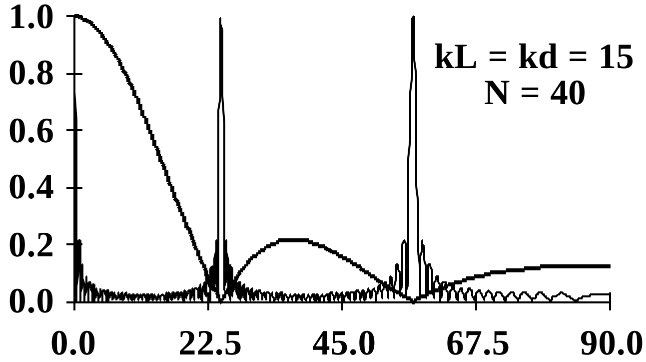
Note that q_5 is the location of one of the major lobes, that is, when $\frac{n}{N} = m$.

$$q_6 = \sin^{-1}\left(\frac{6}{5}\frac{2p}{7}\right) > 90^o$$



In summary,





What advantage is there to using an array as opposed to a line source? What disadvantages?