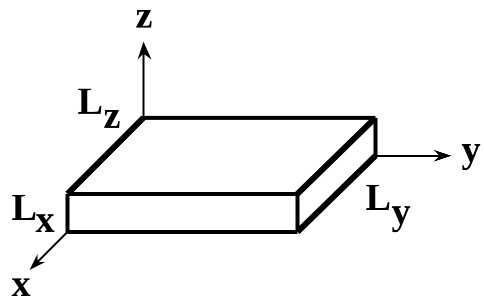
Chapter 9 – Cavities and Waveguides

(9.2) Rectangular Cavity

Consider a rectangular cavity



This cavity (or room) has perfectly smooth, rigid walls. This box could approximate a living room, an auditorium or approximate a concert hall.
The acoustic boundary conditions are such that the normal components of the particle velocity = 0, that is, $\hat{n}\cdot\vec{u}=0$ at all boundaries (rigid wall). From Equation of Force (Eq. 5.4.11)

$$r_o\frac{\partial\vec{u}}{\partial t}=-\nabla p$$

$$\frac{\partial\vec{u}}{\partial t}=-\frac{1}{r_o}\nabla p$$

or for harmonic waves

$$jw\vec{u}=-\frac{1}{r_o}\nabla p$$

$$\vec{u}=-\frac{1}{jwr_o}\nabla p$$

$$\hat{n}\cdot\vec{u}=-\frac{1}{jwr_o}\hat{n}\cdot\nabla p=0$$

Therefore, because $\hat{n}\cdot\vec{u}=0$, $\hat{n}\cdot\nabla p=0$. So for the three orthogonal directions, the boundary conditions are:

$$\left(\frac{\partial p}{\partial x}\right)\bigg|_{x=0}=\left(\frac{\partial p}{\partial x}\right)\bigg|_{x=L_x}=0$$

$$\left(\frac{\partial p}{\partial y}\right)\bigg|_{y=0}=\left(\frac{\partial p}{\partial y}\right)\bigg|_{y=L_y}=0 \qquad \text{(Eq 9.2.1)}$$

$$\left(\frac{\partial p}{\partial z}\right)\bigg|_{z=0}=\left(\frac{\partial p}{\partial z}\right)\bigg|_{z=L_z}=0.$$

The 3D wave equation, $\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$, with the Cartesian coordinates representation gives

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

We can use the separation of variables technique to solve the 3D wave equation and obtain the Helmholtz equation once again. Assume $p(x,y,z,t) = X(x)Y(y)Z(z)e^{jwt}$ and substitute into

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}.$$

If we follow through with the process (see Chapter 5 notes for 3D harmonic plane waves) we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+k_x^2+k_y^2+k_z^2=0$$

Where $\frac{w^2}{c^2}=k^2=k_x^2+k_y^2+k_z^2$ and k_x^2, k_y^2, and k_z^2 are separation constants.
Thus,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+k_x^2=0 \qquad \text{which implies} \qquad X(x)=A\cos(k_x x)+B\sin(k_x x)$$
$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+k_y^2=0 \qquad \text{which implies} \qquad Y(y)=C\cos(k_y y)+D\sin(k_y y)$$
$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+k_z^2=0 \qquad \text{which implies} \qquad Z(z)=E\cos(k_z z)+F\sin(k_z z)$$

Therefore,

$$p(x,y,z,t)=X(x)Y(y)Z(z)e^{jwt}$$
$$=\{A\cos(k_x x)+B\sin(k_x x)\}\ \{C\cos(k_y y)+D\sin(k_y y)\}\ \{E\cos(k_z z)+F\sin(k_z z)\}e^{jwt}$$

Applying boundary conditions $\hat{n}\cdot\nabla p=0$:

1st, 'x' Boundary conditions:

$$\left.\left(\frac{\partial \text{p}}{\partial \text{x}}\right)\right|_{\text{x}=0}=\left.\left(\frac{\partial \text{p}}{\partial \text{x}}\right)\right|_{\text{x}=\text{L}_\text{x}}=0$$
$$\frac{\partial \text{p}}{\partial \text{x}}=\{-Ak_x\sin(k_x x)+Bk_x\cos(k_x x)\}\ Y(y)\ Z(z)e^{jwt}$$

From the B.C.s then

$$\left(\frac{\partial p}{\partial x}\right)\Bigg|_{x=0} = 0 \quad \Rightarrow \quad B = 0$$

$$\left(\frac{\partial p}{\partial x}\right)\Bigg|_{x=L_x} = 0 \quad \Rightarrow \quad \sin\left(k_x L_x\right) = 0$$

which means

$$k_x L_x = l\pi \text{ , l = 0, 1, 2, ...}$$

$$k_x = \frac{l\pi}{L_x} \quad \Rightarrow \quad k_{xl} = \frac{l\pi}{L_x}$$

so that

$$p\left(x,y,z,t\right) = \left\{A\cos\left(k_{xl}x\right)\right\} \, Y\left(y\right) \, Z\left(z\right)e^{j\omega t}$$

What does this look like to you?

2nd, ‘*y*’ Boundary conditions:

$$\left(\frac{\partial p}{\partial y}\right)\Bigg|_{y=0} = \left(\frac{\partial p}{\partial y}\right)\Bigg|_{y=L_y} = 0$$

$$\frac{\partial p}{\partial y} = \left\{A\cos\left(k_{xl}x\right)\right\} \left\{-Ck_y \sin\left(k_y y\right) + Dk_y \cos\left(k_y y\right)\right\} \, Z\left(z\right)e^{j\omega t}$$

From the B.C.s then

$$\left(\frac{\partial p}{\partial y}\right)\Bigg|_{y=0} = 0 \quad \Rightarrow \quad D = 0$$

$$\left(\frac{\partial p}{\partial y}\right)\Bigg|_{y=L_y} = 0 \quad \Rightarrow \quad \sin\left(k_y L_y\right) = 0$$

so that

$$k_y L_y = m\pi \text{ , } m = 0, 1, 2, ...$$

$$k_y = \frac{m\pi}{L_y} \quad \Rightarrow \quad k_{ym} = \frac{m\pi}{L_y}$$

and

$$p\left(x,y,z,t\right) = \left\{A\cos\left(k_{xl}x\right)\right\} \left\{C\cos\left(k_{ym}y\right)\right\} \, Z\left(z\right)e^{j\omega t}$$

3rd, ‘*z*’ Boundary conditions:

$$\left(\frac{\partial p}{\partial z}\right)\Bigg|_{z=0} = \left(\frac{\partial p}{\partial z}\right)\Bigg|_{z=L_z} = 0$$

Which similarly gives us

$$k_z L_z = np \text{ , } n = 0, 1, 2, ...$$

$$k_z = \frac{np}{L_z} \quad \Rightarrow \quad k_{zn} = \frac{np}{L_z}$$

and our total solution is

$$p(x,y,z,t) = \{A\cos(k_{xl}x)\} \{C\cos(k_{ym}y)\} \{E\cos(k_{zn}z)\} e^{jwt}$$

$$p_{lmn}(x,y,z,t) = A_{lmn}\cos(k_{xl}x)\cos(k_{ym}y)\cos(k_{zn}z)e^{jwt}$$

So, what sort of solution is this? What kind of acoustic disturbance does this describe?

From $\frac{w^2}{c^2} = k^2 = k_x^2 + k_y^2 + k_z^2$, $\frac{w^2}{c^2} = k^2 = k_{xl}^2 + k_{ym}^2 + k_{zn}^2 = \frac{w_{lmn}^2}{c^2}$

Therefore, $p_{lmn}(x,y,z,t) = A_{lmn}\cos(k_{xl}x)\cos(k_{ym}y)\cos(k_{zn}z)e^{jw_{lmn}t}$

These are eigenfunctions (Eq. 9.2.5) with the allowed angular frequencies of vibrations quantized as

$$w_{lmn} = c\sqrt{k_{xl}^2 + k_{ym}^2 + k_{zn}^2} = c\sqrt{\left(\frac{lp}{L_x}\right)^2 + \left(\frac{mp}{L_y}\right)^2 + \left(\frac{np}{L_z}\right)^2}$$

These are eigenfrequencies (Eq. 9.2.7) $f_{lmn} = \frac{c}{2}\sqrt{\left(\frac{l}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2 + \left(\frac{n}{L_z}\right)^2}$

which suggests that each of the natural frequencies may be considered as a vector in *frequency space* with components $f_x = \frac{lc}{2L_x}$, $f_y = \frac{mc}{2L_y}$ and $f_z = \frac{nc}{2L_z}$.

How does a room (rectangular cavity) respond to a driving source of frequency *f*?

****************************** Example 9.1 ******************************

How many modes with frequencies f_{lmn} exist which are less than some specified frequency *f*? (see sec 12.9a & sec 12.9e)

$$\frac{c}{2}\sqrt{\left(\frac{l}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2 + \left(\frac{n}{L_z}\right)^2} = f_{lmn} < f$$

$$\left(\frac{l}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2 + \left(\frac{n}{L_z}\right)^2 < \left(\frac{2f}{c}\right)^2$$

Note that this equation is an ellipsoid (if we think of *l*, *m* and *n* as Cartesian coordinates):

$$\left(\frac{l}{L_x}\right)^2+\left(\frac{m}{L_y}\right)^2+\left(\frac{n}{L_z}\right)^2=\left(\frac{2f}{c}\right)^2$$

Principal axes of ellipsoid extend to

$$x_l = l = \frac{2fL_x}{c}$$

$$y_m = m = \frac{2fL_y}{c}$$

$$z_n = n = \frac{2fL_z}{c}$$

Ellipsoidal volume (in terms of the number of modes) is $\frac{4}{3}p\,lmn$. The total number of modes is estimated by the volume of $\frac{1}{8}=\frac{1}{2^3}$ of the ellipsoid:

$$\mathrm{N}(<\mathrm{f}) \approx \frac{1}{8}\left(\frac{4}{3}\pi\mathrm{lmn}\right)=\frac{1}{8}\left(\frac{4}{3}\pi\left(\frac{2\mathrm{fL_x}}{\mathrm{c}}\right)\left(\frac{2\mathrm{fL_y}}{\mathrm{c}}\right)\left(\frac{2\mathrm{fL_z}}{\mathrm{c}}\right)\right)$$

$$=\frac{4p}{3c^3}L_xL_yL_z\,f^3=\frac{4p}{3c^3}Vf^3$$

where $V = L_xL_yL_z$.

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The average number of modes per unit frequency range is $\frac{\Delta N}{\Delta f} : \frac{\Delta N}{\Delta f} \approx \frac{dN}{df} \approx \frac{4p}{c^3}Vf^2$

The reciprocal of $\frac{\Delta N}{\Delta f}$ is the average frequency range per mode, that is, the typical frequency difference between each mode and its nearest neighbor in frequency: $\frac{\Delta f}{\Delta N} \approx \left(\frac{c^3}{4pV}\right)f^{-2}$

***************************** Example 9.1 ******************************

Eigenfrequency Calculations

Consider a room with L_x = 3 m, L_y = 4 m, and L_z = 5 m that is filled with air at 20 $^{\circ}$C. The following are the lowest four Eigenfrequencies.

$$w_{001}=c\sqrt{\left(\frac{p}{5}\right)^2}=343\left(\frac{p}{5}\right)=215.5\,\mathrm{rad/s}\quad\text{and}\quad f_{001}=\frac{w_{001}}{2p}=34.3\mathrm{Hz}$$

$$w_{010}=c\sqrt{\left(\frac{p}{4}\right)^2}=343\left(\frac{p}{4}\right)=269.4\,\mathrm{rad/s}\quad\text{and}\quad f_{010}=\frac{w_{010}}{2p}=42.9\mathrm{Hz}$$

$$w_{011}=c\sqrt{\left(\frac{p}{4}\right)^2+\left(\frac{p}{5}\right)^2}=343(0.3202)p=345.0\,\mathrm{rad/s}\quad\text{and}\quad f_{011}=\frac{w_{001}}{2p}=54.9\mathrm{Hz}$$

$$w_{100} = c\sqrt{\left(\frac{p}{3}\right)^2} = 343\left(\frac{p}{3}\right) = 359.2\,\text{rad}/\text{s} \quad \text{and} \quad f_{100} = \frac{w_{100}}{2p} = 57.2\text{Hz}$$

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*************************** Example 9.2 ****************************

A room (air at 20°C) has dimensions $L_x = 5$ m, $L_y = 8$ m, and $L_z = 3$ m (V = 120 m^3). (a) What is the lowest natural mode frequency corresponding to air motion in the y direction? (b) What is the eigenfrequency for a mode which fits 5 half-wavelengths into the width, 3 into the length, and 6 into the height of the room? (c) What is the approximate total number of modes below 1 kHz? (d) What is the approximate mode density near 1 kHz? (e) What is the average frequency for each increment near 1 kHz?

ANSWERS: (a) $f_{010} = \frac{c}{2L_y} = \frac{343\text{m/s}}{2\times 8\text{m}} = 21.4\text{Hz}$

(b) $f_{lmn} = \frac{c}{2}\sqrt{\left(\frac{l}{L_x}\right)^2 + \left(\frac{m}{L_y}\right)^2 + \left(\frac{n}{L_z}\right)^2}$

$$f_{538} = \frac{343}{2}\sqrt{\left(\frac{5}{5}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{6}{3}\right)^2} = 388.8\text{ Hz}$$

(c) $N\left(<1\text{kHz}\right) \approx \frac{4p}{3c^3}Vf^3 = \frac{4p}{3\left(343\text{m/s}\right)^3}\left(120\text{m}\right)\left(1\text{kHz}\right)^3 = 12{,}456\text{modes}$

(d) $\frac{\Delta N}{\Delta f} \approx \frac{4p}{c^3}Vf^2 = \frac{4p}{\left(343\text{m/s}\right)^3}\left(120\text{m}\right)\left(1\text{kHz}\right)^2 = 37\text{modes/Hz}$

(e) $\frac{\Delta f}{\Delta N} \approx \left(\frac{c^3}{4pV}\right)f^{-2} = 0.27\text{Hz/mode}$

Note: near 100 Hz, the spacing is about 2.7 Hz/mode.

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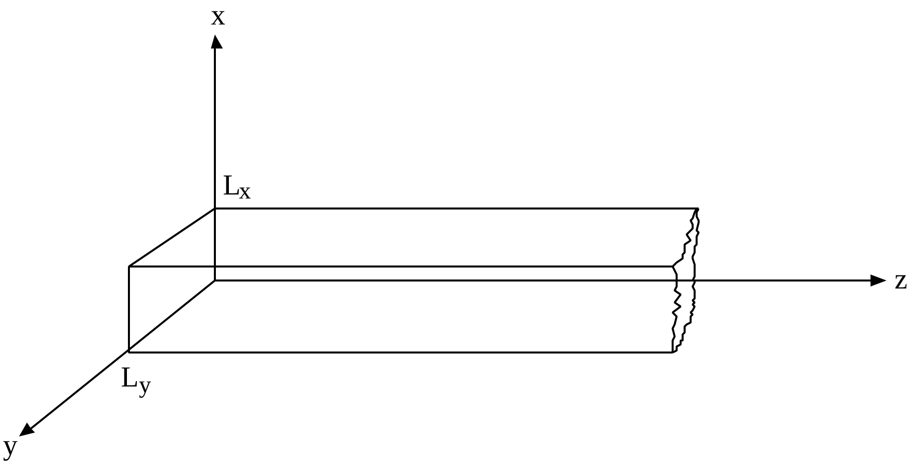
(9.5) Waveguide of Constant Cross Section

We want to look at what happens when you have definite walls in three directions but one direction has dimensions much larger than the dimensions of the other two walls. Enough so that the dimension in one direction could be considered as near infinite compared to the dimensions in the other two directions or we could have an acoustic source at one end of z and the other end open. The figure below illustrates what we are discussing.



This sort of construction means that we will have standing waves in the transverse directions and a traveling wave in the other direction. The solution would be of the form

$$p_{lm} = A_{lm}\cos\left(k_{xl}x\right)\cos\left(k_{ym}y\right)e^{j(wt-k_z z)}$$

with eigenfunctions

$$k^2 = \left(\frac{w}{c}\right)^2 = k_{xl}^2 + k_{ym}^2 + k_z^2$$

$$= \left(\frac{lp}{L_x}\right)^2 + \left(\frac{mp}{L_y}\right)^2 + k_z^2 \quad \text{for } l, m = 0,1,2,\ldots$$

$$\therefore k_z = \sqrt{\left(\frac{w}{c}\right)^2 - k_{xl}^2 - k_{ym}^2}$$

Values for all w (not quantized)

What happens when $k_{xl}^2 + k_{ym}^2 > \left(\frac{w}{c}\right)^2$?

k_z imaginary $\Rightarrow$ no propagation, evanescent mode which exhibits rapid attenuation…

for k_z real $\Rightarrow$ propagation mode

Can determine cutoff frequency for a given mode (value of l and m)

$$\underline{\underline{\begin{pmatrix} cutoff \\ freq \end{pmatrix}}} w_{lm} = c\,k_{lm} = c\sqrt{k_{xl}^{2} + k_{ym}^{2}}$$

$$\underline{\underline{\text{phase speed}}}\quad c_p = \frac{w}{k_z} = \frac{c}{\sqrt{1-\left(\frac{k_{lm}}{k}\right)^2}} = \frac{c}{\cos q}$$

where q is angle that component traveling waves made with the z axis.

Also $l_z = \dfrac{l}{\cos q}$

<u>group speed</u>

$$c_g = c \cos q$$

Note: $c_p \geq c$
$c_g \leq c$

For a spectrum of frequencies

$$c_p = \frac{w}{k_z} \qquad\qquad c_g = \frac{dw}{dk_z}$$

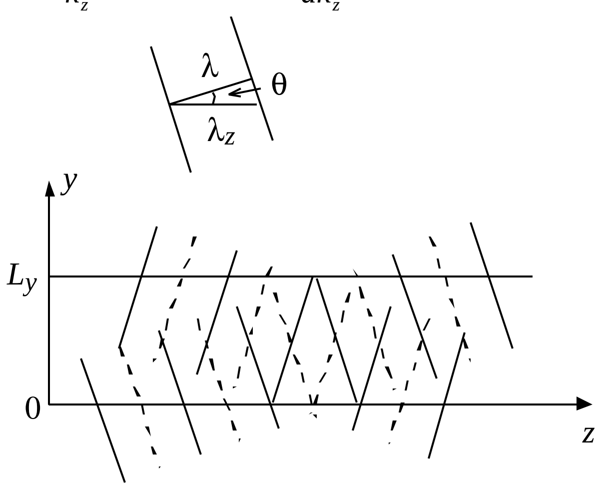


Figure 9.5.2 Component plane waves for the (0, 1) mode in a rigid-walled, rectangular cavity. These waves travel with speed c in directions that make angles $\pm q$ with the z axis of the waveguide.

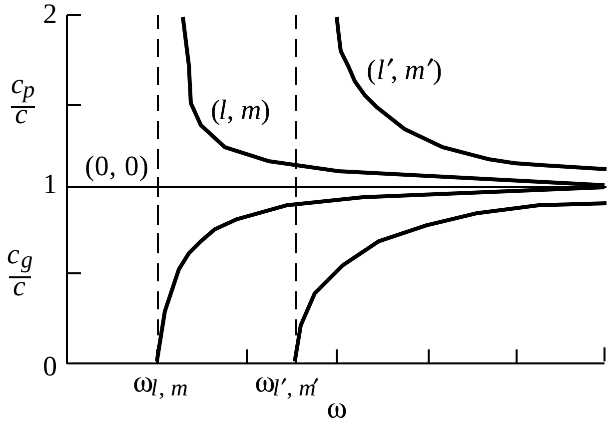


Figure 9.5.3 Group and phase speeds for the lowest three normal modes in a rigid-walled waveguide.