CHAPTER 1

Types of Fields

Our task in this book is to discuss the mathematical techniques which are useful in the calculation and analysis of the various types of fields occurring in modern physical theory. Emphasis will be given primarily to the exposition of the interrelation between the equations and the physical properties of the fields, and at times details of mathematical rigor will be sacrificed when it might interfere with the clarification of the physical background. Mathematical rigor is important and cannot be neglected, but the theoretical physicist must first gain a thorough understanding of the physical implications of the symbolic tools he is using before formal rigor can be of help. Other volumes are available which provide the rigor; this book will have fulfilled its purpose if it provides physical insight into the manifold field equations which occur in modern theory, together with a comprehension of the physical meaning behind the various mathematical techniques employed for their solution.

This first chapter will discuss the general properties of various fields and how these fields can be expressed in terms of various coordinate systems. The second chapter discusses the various types of partial differential equations which govern fields, and the third chapter treats of the relation between these equations and the fundamental variational principles developed by Hamilton and others for classic dynamics. The following few chapters will discuss the general mathematical tools which are needed to solve these equations, and the remainder of the work will be concerned with the detailed solution of individual equations.

Practically all of modern physics deals with fields: potential fields, probability fields, electromagnetic fields, tensor fields, and spinor fields.

Mathematically speaking, a field is a set of functions of the coordinates of a point in space. From the point of view of this book a field is some convenient mathematical idealization of a physical situation in which extension is an essential element, i.e., which cannot be analyzed in terms of the positions of a finite number of particles. The transverse displacement from equilibrium of a string under static forces is a very simple example of a one-dimensional field; the displacement $y$ is different for
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different parts of the string, so that $y$ can be considered as a function of the distance $x$ along the string. The density, temperature, and pressure in a fluid transmitting sound waves can be considered as functions of the three coordinates and of time. Fields of this sort are obviously only approximate idealizations of the physical situation, for they take no account of the atomic properties of matter. We might call them material fields.

Other fields are constructs to enable us to analyze the problem of action at a distance, in which the relative motion and position of one body affects that of another. Potential and force fields, electromagnetic and gravitational fields are examples of this type. They are considered as being "caused" by some piece of matter, and their value at some point in space is a measure of the effect of this piece of matter on some test body at the point in question. It has recently become apparent that many of these fields are also only approximate idealizations of the actual physical situation, since they take no account of various quantum rules associated with matter. In some cases the theory can be modified so as to take the quantum rules into account in a more or less satisfactory way.

Finally, fields can be constructed to "explain" the quantum rules. Examples of these are the Schroedinger wave function and the spinor fields associated with the Dirac electron. In many cases the value of such a field at a point is closely related to a probability. For instance the square of the Schroedinger wave function is a measure of the probability that the elementary particle is present. Present quantum field theories suffer from many fundamental difficulties and so constitute one of the frontiers of theoretical physics.

In most cases considered in this book fields are solutions of partial differential equations, usually second-order, linear equations, either homogeneous or inhomogeneous. The actual physical situation has often to be simplified for this to be so, and the simplification can be justified on various pragmatic grounds. For instance, only the "smoothed-out density" of a gas is a solution of the wave equation, but this is usually sufficient for a study of sound waves, and the much more tedious calculation of the actual motions of the gas molecules would not add much to our knowledge of sound.

This Procrustean tendency to force the physical situation to fit the requirements of a partial differential equation results in a field which is both more regular and more irregular than the "actual" conditions. A solution of a differential equation is more smoothly continuous over most of space and time than is the corresponding physical situation, but it usually is also provided with a finite number of mathematical discontinuities which are considerably more "sharp" than the "actual" conditions exhibit. If the simplification has not been too drastic, most
of the quantities which can be computed from the field will correspond fairly closely to the measured values. In each case, however, certain discrepancies between calculated and measured values will turn up, due either to the "oversmooth" behavior of the field over most of its extent or to the presence of mathematical discontinuities and infinities in the computed field, which are not present in "real life." Sometimes these discrepancies are trivial, in that the inclusion of additional complexities in the computation of the field to obtain a better correlation with experiment will involve no conceptual modification in the theory; sometimes the discrepancies are far from trivial, and a modification of the theory to improve the correlation involves fundamental changes in concept and definitions. An important task of the theoretical physicist lies in distinguishing between trivial and nontrivial discrepancies between theory and experiment.

One indication that fields are often simplifications of physical reality is that fields often can be defined in terms of a limiting ratio of some sort. The density field of a fluid which is transmitting a sound wave is defined in terms of the "density at a given point," which is really the limiting ratio between the mass of fluid in a volume surrounding the given point and the magnitude of the volume, as this volume is reduced to "zero." The electric intensity "at a point" is the limiting ratio between the force on a test charge at the point and the magnitude of the test charge as this magnitude goes to "zero." The square of the Schrödinger wave function is the limiting ratio between the probability that the particle is in a given volume surrounding a point and the magnitude of the volume as the volume is shrunk to "zero," and so on. A careful definition of the displacement of a "point" of a vibrating string would also utilize the notion of limiting ratio.

These mathematical platitudes are stressed here because the technique of the limiting ratio must be used with caution when defining and calculating fields. In other words, the terms "zero" in the previous paragraph must be carefully defined in order to achieve results which correspond to "reality." For instance the volume which serves to define the density field for a fluid must be reduced several orders of magnitude smaller than the cube of the shortest wavelength of transmitted sound in order to arrive at a ratio which is a reasonably accurate solution of the wave equation. On the other hand, this volume must not be reduced to a size commensurate with atomic dimensions, or the resulting ratio will lose its desirable properties of smooth continuity and would not be a useful construct. As soon as this limitation is realized, it is not difficult to understand that the description of a sound wave in terms of a field which is a solution of the wave equation is likely to become inadequate if the "wavelength" becomes shorter than interatomic distances.

In a similar manner we define the electric field in terms of a test
charge which is made small enough so that it will not affect the distribution of the charges "causing" the field. But if the magnitude of the test charge is reduced until it is the same order of magnitude as the electronic charge, we might expect the essential atomicity of charge to involve us in difficulties (although this is not necessarily so).

In some cases the limiting ratio can be carried to magnitudes as small as we wish. The probability fields of wave mechanics are as "fine-grained" as we can imagine at present.

1.1 Scalar Fields

When the field under consideration turns out to be a simple number, a single function of space and time, we call it a scalar field. The displacement of a string or a membrane from equilibrium is a scalar field. The density, pressure, and temperature of a fluid, given in terms of the sort of limiting ratio discussed previously, are scalar fields. As mentioned earlier, the limiting volume cannot be allowed to become as small as atomic dimensions in computing the ratio, for the concepts of density, pressure, etc., have little meaning for individual molecules. The ratios which define these fields must approach a "macroscopic limit" when the volume is small compared with the gross dimensions of the fluid but is still large compared with atomic size; otherwise there can be no physical meaning to the concept of scalar field here.

All these scalar fields have the property of invariance under a transformation of space coordinates (we shall discuss invariance under a space-time transformation later in this chapter). The numerical value of the field at a point is the same no matter how the coordinates of the point are expressed. The form of mathematical expression for the field may vary with the coordinates. For instance, a field expressed in rectangular coordinates may have the form \( \psi = y \); in spherical coordinates it has the different form \( \psi = r \sin \theta \cos \varphi \), but in either coordinate system, at the point \( x = 10, y = 10, z = 0 \) \( (r = \sqrt{200}, \theta = 45^\circ, \varphi = 0) \) it has the value \( \psi = 10 \). This is to be contrasted to the behavior of the \( x \) component of the velocity of a fluid, where the direction of the \( x \) axis may change as the coordinates are changed. Therefore, the numerical value of the \( x \) component at a given point will change as the direction of the \( x \) axis is rotated.

This property of invariance of a scalar will be important in later discussions and is to be contrasted alternatively to the invariance of form of certain equations under certain coordinate transformations. For some of the scalar fields mentioned here, such as density or temperature or electric potential, the definition of the field has been such as to make the property of invariance almost tautological. This is not always the
case with less familiar fields, however. In some cases, in fact, the property of invariance must be used as a touchstone to find the proper expression for the field.

**Isotimic Surfaces.** The surfaces defined by the equation $\psi = \text{constant}$, where $\psi$ is the scalar field, may be called *isotimic surfaces* (from Greek *isotimos*, of equal worth). Isotimic surfaces are the obvious generalizations of the contour lines on a topographic map. In potential theory they are referred to as equipotentials; in heat conduction they are isothermals; etc. They form a family of nonintersecting surfaces,

\[ e \cosh \mu = \frac{1}{2} \sqrt{(r + c)^2 + z^2} + \frac{1}{2} \sqrt{(r - c)^2 + z^2}, \]

of $\theta = \text{constant}$, where

\[ e \cos \theta = \frac{1}{2} \sqrt{(r + c)^2 + z^2} - \frac{1}{2} \sqrt{(r - c)^2 + z^2}, \]

and of $\phi = C$, where $\tan \phi = y/x$.

which are often useful in forming part of a system of coordinates naturally suitable for the problem. For instance if the field is the well-known potential $\psi = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, the isotimic surfaces (in this case surfaces of constant potential) are concentric spheres of radius $r = \sqrt{x^2 + y^2 + z^2} = \text{constant}$; and the natural coordinates for the problem are the spherical ones, $r, \theta, \phi$. Another set of surfaces is shown in Fig. 1.1, together with the corresponding coordinate system. The surfaces $\mu = \text{constant}$ correspond to the equipotential surfaces about a circular charged disk of radius $c$, lying in the $x,y$ plane ($\mu = 0$).

The derivatives of the scalar $\psi$ with respect to the rectangular coordinates $x$, $y$, $z$ measure the rapidity with which the field changes as we change position. For instance the change in $\psi$ from the point $(x,y,z)$ to the point $(x + dx, y + dy, z + dz)$ is

\[ d\psi = \left( \frac{\partial \psi}{\partial x} \right) dx + \left( \frac{\partial \psi}{\partial y} \right) dy + \left( \frac{\partial \psi}{\partial z} \right) dz \quad (1.1.1) \]

If the two points are in the same isotimic surface $d\psi = 0$; in fact the
differential equation for these surfaces is

$$\left(\frac{\partial \psi}{\partial x}\right)dx + \left(\frac{\partial \psi}{\partial y}\right)dy + \left(\frac{\partial \psi}{\partial z}\right)dz = 0 \tag{1.1.2}$$

The displacement \((dx,dy,dz)\) is perpendicular to the surface if the component displacements satisfy the equation,

$$\frac{dx}{\partial \psi/\partial x} = \frac{dy}{\partial \psi/\partial y} = \frac{dz}{\partial \psi/\partial z} \tag{1.1.3}$$

These relations are the differential equations for a family of lines, called the normal lines, which are everywhere perpendicular to the isotimic surfaces. Together with the isotimic surfaces they can be used to define the natural coordinate system for the field. For instance, for the field \(\psi = (x^2 + y^2 + z^2)^{-1/2}\) the surfaces are spheres (as we noted before) and the normal lines are the radii, suggesting (but not completely defining) the spherical coordinates \((r, \theta, \varphi)\).

The normal lines are pointed along the direction of most rapid change of \(\psi\). A little manipulation of Eqs. (1.1.1) and (1.1.3) will show that the change of \(\psi\) as one goes a distance \(ds\) along a normal line is

$$\sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2} \, ds$$

The square-root factor is called the magnitude of the gradient of \(\psi\). Its properties will be discussed in more detail later in the chapter.

The Laplacian. An extremely important property of a scalar field is expressed in terms of its second derivatives. In the simple one-dimensional case where \(\psi\) is the transverse displacement of a string from its straight-line equilibrium position, the second derivative \(d^2\psi/dx^2\) is closely related to the difference between the value of \(\psi\) at \(x\) and its average values at neighboring points. Using the fundamental definition of the derivative

$$\lim \left[\psi(x) - \frac{1}{2}[\psi(x - dx) + \psi(x + dx)]\right] = -\frac{1}{2} \lim \left[\{\psi(x + dx) - \psi(x)\} - \{\psi(x) - \psi(x - dx)\}\right] = -\frac{1}{2}(d^2\psi/dx^2)(dx)^2$$

Consequently if the second derivative is negative, \(\psi\) at \(x\) is larger than the average of \(\psi\) at \(x + dx\) and \(x - dx\) and the plot of \(\psi\) against \(x\) will have a downward curvature at that point. If the second derivative is zero, \(\psi\) will have no curvature.

It is not difficult to see that the equation for the shape of a stretched, flexible string acted on by a transverse force \(F(x)\) per unit length of string, distributed along the string, must be expressed in terms of this second derivative. For the greater the transverse force at a point, the greater must be the curvature of the string there, in order that the tension \(T\) along the string may have greater transverse components to equalize
the force. The equation turns out to be

\[ T\left(\frac{d^2\psi}{dx^2}\right) = -F(x) \]

as a detailed analysis of the problem will show later.

We now seek a three-dimensional counterpart to this measure of curvature of \( \psi \). The limiting value of the difference between \( \psi \) at \( x \) and the average value of \( \psi \) at neighboring points is

\[-\frac{1}{4}(dx\ dy\ dz)^2 \nabla^2 \psi,\]

where

\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \quad \text{(1.1.4)} \]

is the obvious generalization of the one-dimensional second-derivative operator. The mathematical operation given in Eq. (1.1.4) is labeled by the symbol \( \nabla^2 \) (read del squared), which is called the Laplace operator. The result of the operation on the function \( \psi \) is called the Laplacian of \( \psi \). If \( \nabla^2 \psi \) is negative at some point there is a tendency for \( \psi \) to concentrate at that point.

One immediate consequence of this definition is that a scalar function \( \psi(x,y,z) \) can have no maxima or minima in a region where \( \nabla^2 \psi = 0 \). This is a result of considerable importance.

The equation \( \nabla^2 \psi = 0 \), called Laplace's equation, occurs in so many parts of physical theory that it is well to have a clear picture of its meaning. Accordingly we shall quote a number of facts concerning the solutions of Laplace's equation which will not be proved till later in the chapter.

Suppose a perfectly elastic membrane to be in equilibrium under a uniform tension applied around its boundary edge. If the edge lies entirely in a plane, then the membrane will lie entirely in the plane. If the boundary support is distorted so that it no longer lies all in a plane, the membrane will be distorted also. This distortion can be represented by \( \psi(x,y) \), the displacement, normal to the plane, of the point \( (x,y) \) on the membrane. It turns out that this displacement satisfies Laplace's equation in two dimensions, \( \nabla^2 \psi = 0 \). The equation simply corresponds to the statement that the tension straightens out all the "bulges" in the membrane, that the displacement at any point equals the average value of the displacement for neighboring points. It also corresponds (as we shall prove later) to the statement that the membrane has arranged itself so that its average slope is as small as it can be. Since the total stretching of the membrane is proportional to the average square of the slope, we see that the Laplace equation for the membrane corresponds to the requirement that the membrane assumes a shape involving the least stretching possible.

An additional loading force on the membrane, perpendicular to the equilibrium plane \( \psi = 0 \), produces a "bulging" of the membrane. As will be proved later, the Laplacian of \( \psi \) at a point on a loaded membrane
is proportional to the load per unit area at that point. One can say that the two-dimensional Laplacian operator measures the "bulgingness" of the shape function $\psi$.

The generalization of this discussion to three dimensions is somewhat harder to picture but not more complicated in principle. We might picture the scalar function $\psi$ as corresponding to the concentration of a solute in a solvent. The three-dimensional analogue of "bulgingness" might be termed "lumpiness"; if there is a tendency for the solute to "lump together" at any point, the Laplacian of the concentration will be negative there. In a case where $\nabla^2 \psi = 0$, the solute has no "lumpiness" at all, its density arranging itself so as to average out as smoothly as possible the differences imposed by the boundary conditions. As in the two-dimensional case, the Laplace equation corresponds to the requirement that $\psi$ at every point be equal to the average value of $\psi$ at neighboring points.

The presence of electric charge density $\rho$ causes a (negative) concentration of the electric potential $\psi$, so that $\nabla^2 \psi = -\rho/\epsilon$, where $\epsilon$ is a constant. The presence of a distributed source of heat $Q$ in a solid causes a concentration of temperature $T$, so that $\nabla^2 T = KQ$, where $K$ is a constant. In a great many cases the scalar field is affected by a source function $q(x,y,z)$ (which is itself another scalar field, obeying some other equations) according to the equation

$$\nabla^2 \varphi = -q$$

(1.1.5)

This equation is called Poisson's equation. We shall discuss it in more detail later in this chapter and shall devote a considerable amount of space later in the book to its solution.

1.2 Vector Fields

We have discussed in a preliminary sort of way a number of fields which were characterized by a single magnitude at each point. These
were called scalar fields. Many other fields require a magnitude and direction to be given at each point for complete characterization. These will be called vector fields. These fields also can often be defined in terms of limiting ratios, though the definitions are usually more complex than those for scalar fields. The force on a portion of fluid in a gravitational or electric field is a vector, having magnitude and direction. The limit of the ratio between this force and the volume occupied by the portion of fluid, as this volume is decreased in size, defines a vector at each point in space, which is the force field. As with scalar fields it is sometimes important not to let the volume decrease to atomic dimensions.

Sometimes the vector field is most easily defined by a scalar ratio, which has a direction inherent in it. For instance, for an electric conductor carrying current we can imagine an instrument which would measure the amount of current crossing an element of area \(dA\) centered at some point in the conductor. We should find that the amount of current measured would depend not only on the size of \(dA\) but also on the orientation of the element \(dA\). In fact the measurement would turn out to correspond to the expression \(J \, dA \cos \vartheta\), where \(\vartheta\) is the angle between the normal to \(dA\) and some direction which is characteristic of the current distribution. The magnitude of the vector field at the point measured would then be \(J\), and the direction would be that which defines the angle \(\vartheta\).

Vector fields in three dimensions are specified by giving three quantities at each point, the magnitude and two angles giving the direction, or three components of the vector along the three coordinate axes. Four-vectors will be considered later.

Boldface Latin capital letters \((\mathbf{A}, \mathbf{F}, \mathbf{X})\) will be used in this book to denote vectors; the corresponding normal-weight letters \((A, F, X)\) will denote the corresponding magnitudes (normal-weight letters will usually denote scalar quantities). The components of \(\mathbf{A}\) along the three coordinate axes will be denoted \(A_x, A_y,\) and \(A_z\). A vector of unit length in the direction of \(\mathbf{A}\) is denoted by \(\mathbf{a}\); and in conformity with usual practice the unit vectors along the \(x, y,\) and \(z\) directions will be labeled \(\mathbf{i}, \mathbf{j},\) and \(\mathbf{k}\), respectively. Unit vectors along curvilinear coordinate axes will usually be labeled \(\mathbf{a}_r,\) with a subscript indicating the axis in question (for instance, in polar coordinates, the unit vector along \(\varphi\) is \(\mathbf{a}_\varphi\)). Unless otherwise noted the coordinate systems used will be right-handed ones: if a right-hand screw is placed along the \(z\) axis, then a rotation from \(x\) into \(y\) will move the screw in the positive \(z\) direction; or if the \(x, y\) plane is on a blackboard which is faced by the observer, the \(x\) direction can point to the observer's right, the \(y\) axis can point upward, and the \(z\) axis will then point toward the observer.

In the notation given above vectors \(\mathbf{A}\) and \(\mathbf{B}\) obey the following general equations:
\[ \mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \]
\[ \mathbf{A} + \mathbf{B} = (A_x + B_x) \mathbf{i} + (A_y + B_y) \mathbf{j} + (A_z + B_z) \mathbf{k} \] (1.2.1)

which constitute a definition of vector components and of vector addition.

Vectors are not invariant to change of coordinates in the same sense as scalars are; for their components change as the coordinate directions change. The transformation properties of vectors will be discussed later.

**Multiplication of Vectors.** Two vectors can be multiplied together in two different ways: one resulting in a scalar and the other in vector. The *scalar* or *dot product* of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is equal to the product of the magnitude of one by the component of the other in the direction of the first:

\[ \mathbf{A} \cdot \mathbf{B} = AB \cos \vartheta = A_x B_x + A_y B_y + A_z B_z \] (1.2.2)

where \( \vartheta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \). The expression \( AB \cos \vartheta \) is independent of the coordinate system used to compute the components \( A_x, \) etc., so that the value of the dot product is independent of the coordinate system chosen. The dot product is therefore a true scalar, the simplest invariant which can be formed from two vectors.

The dot product is a useful form for expressing many physical quantities: The work done in moving a body equals the dot product of the force and the displacement; the electrical energy density in space is proportional to the dot product of the electric intensity and the electric displacement; and so on. The dot product of two unit vectors is equal to the direction cosine relating the two directions. The maximum value of the dot product of two vectors is obtained when the two vectors are parallel; it is zero when the two are perpendicular. In a sense the dot product is a measure of the coalignment of the two vectors.

The *vector* or *cross product* \( \mathbf{A} \times \mathbf{B} \) of two vectors is a vector with magnitude equal to the area of the parallelogram defined by the two vectors and with direction perpendicular to this parallelogram. The choice as to which end of the perpendicular to put the arrowhead is decided arbitrarily by making the trio \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{A} \times \mathbf{B} \) a right-handed system: if a right-hand screw is placed perpendicular to \( \mathbf{A} \) and \( \mathbf{B}, \) a rotation of \( \mathbf{A} \) into \( \mathbf{B} \) will move the screw in the direction of \( \mathbf{A} \times \mathbf{B}. \) In terms of right-hand rectangular coordinates

\[ \mathbf{A} \times \mathbf{B} = (A_y B_z - B_y A_z) \mathbf{i} + (A_z B_x - B_z A_x) \mathbf{j} + (A_x B_y - B_x A_y) \mathbf{k} \]

Magnitude of \( \mathbf{A} \times \mathbf{B} = AB \sin \vartheta \) (1.2.3)

We note that vector multiplication is not commutative, for \( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \)

**Axial Vectors.** Although the cross product of two vectors is a vector with most of the transformation properties of a "true" vector (as we shall see later), there is one difference which is of importance. The cross product, as defined in Eq. (1.2.3), changes sign if we change from a right-
handed to a left-handed system of coordinates. This is another aspect of the fact that the cross product has the directional properties of an element of area rather than of an arrow. The direction connected with an area element is uniquely determined, the direction normal to the element, except that there is no unequivocal rule as to which side of the element is the positive direction of the vector. The area fixes the shank of the arrow, so to speak, but does not say which end should bear the point. Which end does carry the point must be decided by some purely arbitrary rule such as the right-hand rule, mentioned above, which we shall use in this book.

Arealike vectors, with a fixed shank but interchangeable head, are called axal vectors (they are also sometimes called pseudovectors). We shall see later that the three components of an axial vector are actually the three components of a second-order antisymmetric tensor in three dimensions. Indeed only in three dimensions is it possible to represent an antisymmetric tensor by an axial vector.

As indicated above, the axial vector associated with an element of area $\mathbf{dA}$ can be written:

$$\mathbf{dA} = n \, dA = \mathbf{dx} \times \mathbf{dy}$$

where $n$ is the unit vector normal to the element and where $\mathbf{dx}$ and $\mathbf{dy}$ are the vectors associated with the component elements $dx$ and $dy$. If the first two notations are used, it is necessary in addition to indicate which side of the element is positive; if the last notation is used, our arbitrary right-hand rule will automatically decide the question.

Other axial vectors can be represented in terms of cross products: the angular momentum of a particle about some point is equal to the cross product of the vector representing the particle’s momentum and the radius vector from the origin to the particle; the torque equals the cross product of the force vector and the vector representing the lever arm; and so on. Rotation fixes a plane and an axis normal to the plane, the characteristics of an axial vector. According to our convention the direction of a rotational vector follows the motion of a right-handed screw turned by the rotation.

A useful example of a product of three vectors is the scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(1.2.4)

The magnitude of this quantity is equal to the volume (or to minus the volume) enclosed by the parallelepiped defined by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Such a scalar, a dot product of an axial vector and a true vector, which changes sign if we change from left- to right-handed coordinates or if we interchange the order of the vectors, is sometimes called a pseudoscalar.
We note that the dot product of two axial vectors (or of two true vectors) is a "true" scalar, with no uncertainty in sign.

Incidentally, the rules governing the products of the unit vectors are

\[ i^2 = i \cdot i = j^2 = k^2 = 1; \quad i \cdot j = i \cdot k = j \cdot k = 0 \]
\[ i \times j = \cdots = 0; \quad i \times j = k; \quad j \times k = i; \quad k \times i = j \] (1.2.5)

**Lines of Flow.** As we have said earlier, a vector field is defined by specifying a vector at each point in space, in other words, by specifying a vector which is a function of \( x, y, \) and \( z \): \( \mathbf{F}(x, y, z) \). In most cases of interest this vector is a continuous function of \( x, y, \) and \( z \), except at isolated points, or singularities, or along isolated lines, or singular lines. Where the vector is continuous, we can define lines of flow of the field, which are lines at every point tangent to the vector at that point. The differential equation for the lines is obtained by requiring that the components \( dx, dy, dz \) of displacement along the line be proportional to the components \( F_x, F_y, \) and \( F_z \) of the vector field at that point:

\[
\frac{dx}{F_x} = \frac{dy}{F_y} = \frac{dz}{F_z}\]  (1.2.6)

Compare this with Eq. (1.1.3).

In certain simple cases these equations can be integrated to obtain the algebraic equations for the family of lines of flow. For instance, if

![Diagram showing helical lines of flow](image)

**Fig. 1.3** Helical lines of flow \( \theta, \phi = \) constants, together with pseudo-potential surfaces \( \psi = \) constant, as given on page 15.

\[
F_x = -ay, \quad F_y = ax, \quad F_z = b(x^2 + y^2), \] the lines of flow are helical. The equation \( dx/F_x = dy/F_y \) becomes \( x \, dx = -y \, dy \), which integrates to the equation for the circular cylinder, \( x^2 + y^2 = \varphi^2 \), with \( \varphi \) an arbitrary constant, denoting the particular line of flow chosen. The equation \( dy/F_y = dz/F_z \) becomes \( \varphi^2 \, dy/\sqrt{\varphi^2 - y^2} = (a/b) \, dz \), after substitution for \( x \) from the equation relating \( x \) and \( y \). This integrates to

\[
z = \left( b\varphi^2/a \right) \sin^{-1} \left( \frac{y}{\varphi} \right) + \vartheta = \left( b/a \right) \left( x^2 + y^2 \right) \tan^{-1} \left( \frac{y}{x} \right) + \vartheta\]
where $\theta$ is the other constant of integration which is needed to specify completely a particular line of flow. The two equations

$$
\varphi = \sqrt{x^2 + y^2}, \quad \theta = z - \frac{(b/a)(x^2 + y^2)}{\tan^{-1}(y/x)}
$$

define a doubly infinite family of flow lines, one line for each pair of values chosen for $\varphi$ and $\theta$.

Another example is the case $F_x = x/r^2$, $F_y = y/r^2$, $F_z = z/r^2$, where $r^2 = (x^2 + y^2 + z^2)$. The equations for the flow lines reduce to $dx/x = dy/y = dz/z$. The first equation gives $\ln(x) = \ln(y) + \text{constant}$, or $x/y = \text{constant}$. Similarly we have either $x/z = \text{constant}$ or, combining the two, $(x^2 + y^2)/z^2 = \text{constant}$. The most suitable forms for the constants of integration, analogous to the forms given in the previous paragraph, are

$$
\varphi = \tan^{-1}(y/x); \quad \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z)
$$

Again, a choice of values for $\varphi$ and $\theta$ picks out a particular flow line; in this case a straight line radiating from the origin.

From another point of view $\varphi$ and $\theta$ can be considered as functions of $x$, $y$, and $z$ and are called flow functions. The values of $\varphi$ and $\theta$ at some point label the flow lines at that point. From still another point of view the two families of surfaces $\varphi = \text{constant}$ and $\theta = \text{constant}$ can be considered as two families of coordinate surfaces for a generalized system of coordinates. The intersection of two such surfaces, $\varphi = \varphi_0$ and $\theta = \theta_0$, is the flow line corresponding to the pair $(\varphi_0, \theta_0)$ and is a coordinate line in the new system.

**Potential Surfaces.** The lines of flow may also determine another family of surfaces, perpendicular to the lines (unless the lines are so
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“curled” that no such set of surfaces can exist). By analogy with Eq. (1.1.2) the equation for such a surface is

\[ \mathbf{F} \cdot d\mathbf{s} = F_x \, dx + F_y \, dy + F_z \, dz = 0 \quad (1.2.7) \]

corresponding to the fact that any displacement vector in the surface must be perpendicular to \( \mathbf{F} \).

In certain cases this equation is integrable. If there is a function \( \psi \) such that \( \mu (\partial \psi / \partial x) = F_x, \mu (\partial \psi / \partial y) = F_y, \mu (\partial \psi / \partial z) = F_z \), then the equation for the family of surfaces becomes \( \psi = \text{constant} \). The quantity \( \mu \) may be a function of \( x, y, \) and \( z \) and is called an integrating factor. The criterion for whether an equation for the surfaces exists in integral form is developed as follows: Assuming that there is a \( \psi \) as defined above, we substitute in the expression

\[
F_x \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + F_y \left( \frac{\partial F_z}{\partial z} - \frac{\partial F_z}{\partial x} \right) + F_z \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_y}{\partial y} \right) \\
= \mu \frac{\partial \psi}{\partial x} \left[ \frac{\partial \psi}{\partial y} \left( \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} \right) - \frac{\partial \psi}{\partial z} \left( \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \right) \right] + \mu \frac{\partial \psi}{\partial z} \left[ \frac{\partial \psi}{\partial x} \left( \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \right) - \frac{\partial \psi}{\partial y} \left( \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right) \right] = 0 \quad (1.2.8)
\]

If there is a function \( \psi \), this last expression is zero. Therefore, if the expression involving the components of \( \mathbf{F} \) turns out to be zero, then it is possible to integrate the differential equation for the surfaces perpendicular to the flow lines. In other words, if the vector whose \( x, y, \) and \( z \) components are \( \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right), \left( \frac{\partial F_z}{\partial z} - \frac{\partial F_z}{\partial x} \right), \) and \( \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_y}{\partial y} \right) \) is perpendicular to the vector \( \mathbf{F} \) at every point, then it is possible to obtain the equation for the normal surfaces in integral form, as \( \psi (x, y, z) = \text{constant} \). More will be said concerning this vector later. The function \( \psi \) is called a pseudopotential function.

In certain special cases \( \mu \) is a constant and can be set equal to \((-1)\), so that \( F_x = -(\partial \psi / \partial x), F_y = -(\partial \psi / \partial y), F_z = -(\partial \psi / \partial z) \). The excuse for the negative sign will appear later. In these cases the scalar function is called the potential function for the vector field \( \mathbf{F} \), and the surfaces \( \psi = \text{constant} \) are called equipotential surfaces. For this to be true, the three quantities \( \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right), \left( \frac{\partial F_z}{\partial z} - \frac{\partial F_z}{\partial x} \right), \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_y}{\partial y} \right) \) must each be zero, as can be seen by substituting the expressions assumed for the components of \( \mathbf{F} \) in terms of \( \psi \).

In other cases the equation for the surface is not integrable, either with or without the use of the integrating factor; then it is impossible to find a sensibly behaved family of surfaces everywhere perpendicular to the lines of flow. We shall return to this discussion on page 18.